PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 1, Pages 75–80
S 0002-9939(05)07952-9
Article electronically published on June 14, 2005

EIGENVALUE INEQUALITIES IN AN EMBEDDABLE FACTOR

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(Communicated by Joseph A. Ball)

ABSTRACT. We provide a characterization of the possible eigenvalues of the sum of two selfadjoint elements of a II_1 factor which can be embedded in the ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1 factor.

1. INTRODUCTION

Consider a II_1 factor $\mathcal{M}$ with normal faithful trace τ such that $\tau(1) = 1$. Each selfadjoint element $a \in \mathcal{M}$ can be written as

$$a = \int_0^1 u(t)\, de(t)$$

where u is a nonincreasing function on $[0, 1)$, and e is a spectral measure on $[0, 1)$ such that $\tau(e([0, t])) = t$ for $t \in [0, 1)$. The function u is uniquely determined if it is also assumed right-continuous; in this case we will use the notation λ_a for u, and call it the eigenvalue function of a.

We want to address the following question: given nonincreasing right-continuous functions u, v, w on $[0, 1)$, under what conditions do there exist elements $a, b \in \mathcal{M}$ such that $\lambda_a = u$, $\lambda_b = v$, and $\lambda_{a+b} = w$?

The finite-dimensional analogue of this question was completely solved; the solution will be described in some detail later (see [9] and [6] for a detailed discussion). In our context, some necessary conditions on u, v, w were found in [7], [4], and [5]. These conditions are subsumed by an analogue of a result of Freede and Thompson [10] which we proved in [1].

In this paper we focus on the case in which $\mathcal{M}$ is an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1 factor $\mathcal{R}$. In this case we produce necessary and sufficient conditions on u, v, w for the existence of $a, b \in \mathcal{M}$ with the required properties. Our conditions are derived from those found in the finite-dimensional situation. Indeed, our methods can be applied to more general questions of this type (such as, e.g., conditions on λ_a, λ_b, λ_c, and λ_{a+b+c}). We do not know whether our conditions on (u, v, w) are satisfied by $(\lambda_a, \lambda_b, \lambda_{a+b})$ if a, b are elements in a factor which does not embed in $\mathcal{R}^\omega$ (but note that the results in [7, 4, 1] are proved in complete generality). The

Received by the editors January 8, 2004 and, in revised form, September 2, 2004.
2000 *Mathematics Subject Classification.* Primary 15A42; Secondary 46L10.
The authors were supported in part by grants from the National Science Foundation. The second author expresses her gratitude to the Department of Mathematics of Indiana University for its kind hospitality while this paper was written.

answer would of course be affirmative if every II_1 factor could be embedded in $\mathcal{R}^\omega$, a problem first formulated by Connes [3].

2. Finite-dimensional algebras

In this section we describe the solution to the eigenvalue problem for sums of finite selfadjoint matrices. The solution involves sums of eigenvalues indexed by some subsets of $\{1, 2, \ldots, n\}$. Fix a natural number n and, for a subset $I \subset \{1, 2, \ldots, n\}$, write $\Sigma_I = \sum_{i \in I} i$. Following the notation in [6], we consider for every $r \leq n$ the following collection of triples (I, J, K) of subsets of cardinality r in $\{1, 2, \ldots, n\}$:

$$U_r^n = \{(I, J, K) : \Sigma_I + \Sigma_J = \Sigma_K + \frac{r(r+1)}{2}\}.$$

It will also be convenient to view a subset I of r elements in $\{1, 2, \ldots, n\}$ as an increasing function $I : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, n\}$, i.e.,

$$I = \{I(1), I(2), \ldots, I(r)\}.$$

We then define inductively on r a subset T_r^n of U_r^n as follows: $T_1^n = U_1^n$ and, for $1 < r \leq n$,

$$\begin{aligned} T_r^n = \quad & \{(I, J, K) \in U_r^n : \Sigma_{I \circ I'} + \Sigma_{J \circ J'} \leq \Sigma_{K \circ K'} + \tfrac{p(p+1)}{2} \\ & \text{for every } \ p < r \ \text{ and } \ (I', J', K') \in T_p^r\}. \end{aligned}$$

It is easily verified that $T_r^n \subset T_r^N$ if $N > n$.

The following result was conjectured by Horn [8], and proved as a consequence of work by Klyachko, Totaro, Knudson and Tao (cf. [6] for an exposition).

2.1. **Theorem.** *Consider nonincreasing sequences $\alpha, \beta, \gamma \in \mathbb{R}^n$. The following are equivalent:*

(1) *there exist selfadjoint $n \times n$ matrices A, B such that the eigenvalues of A (resp. B, resp. $A+B$) repeated according to their multiplicities are the components of α (resp. β, resp. γ).*

(2) $\sum_{i=1}^n \alpha_i + \sum_{j=1}^n \beta_j = \sum_{k=1}^n \gamma_k$ *and for all $r < n$ and all $(I, J, K) \in T_r^n$, we have*

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geq \sum_{k \in K} \gamma_k.$$

It will be convenient to introduce the eigenvalue function of a selfadjoint $n \times n$ matrix A. Thus, if the eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, we will write

$$\lambda_A(t) = \lambda_i \ \text{ for } \ t \in \left[\frac{i-1}{n}, \frac{i}{n}\right).$$

Let us note that $\lambda_{A \otimes I} = \lambda_A$, where I denotes an identity matrix of arbitrary size.

Now, given a subset $I \subset \{1, 2, \ldots, n\}$, we can construct a subset $\sigma_I^n \subset [0, 1)$ as follows:

$$\sigma_I^n = \bigcup_{i \in I} \left[\frac{i-1}{n}, \frac{i}{n}\right).$$

The conditions in Theorem 2.1 can now be expressed as

$$\int_0^1 \lambda_A(t)\, dt + \int_0^1 \lambda_B(t)\, dt = \int_0^1 \lambda_{A+B}(t)\, dt$$

and

$$\int_{\sigma_I^n} \lambda_A(t)\,dt + \int_{\sigma_J^n} \lambda_B(t)\,dt \geq \int_{\sigma_K^n} \lambda_{A+B}(t)\,dt$$

whenever $(I, J, K) \in T_r^n$ and $r < n$. Note that under this form, one can ask whether these inequalities hold for $(I, J, K) \in T_r^N$ with arbitrary N and $r < N$. This is in fact true. Indeed, let us observe that, given a subset I with r elements of $\{1, 2, \ldots, N\}$ and an integer n, there exists a subset I_n with nr elements of $\{1, 2, \ldots, nN\}$ such that $\sigma_I^N = \sigma_{I_n}^{nN}$. In fact

$$I_n = \bigcup_{i \in I} \{n(i-1)+1, n(i-1)+2, \ldots, ni\}.$$

Moreover, properties of the Littlewood-Richardson coefficients (cf., for instance, [6] or [2]) show that (I_n, J_n, K_n) belongs to T_{nr}^{nN} whenever $(I, J, K) \in T_r^N$. This leads immediately to the following result.

2.2. **Proposition.** *Consider selfadjoint matrices $A, B \in \mathcal{M}_n$, and natural numbers N, r such that $r < N$. Then*

$$\int_{\sigma_I^N} \lambda_A(t)\,dt + \int_{\sigma_J^N} \lambda_B(t)\,dt \geq \int_{\sigma_K^N} \lambda_{A+B}(t)\,dt$$

whenever $(I, J, K) \in T_r^N$.

Proof. As noted above, replacing A and B by $A \otimes I_N, B \otimes I_N \in \mathcal{M}_{nN}$ does not change the eigenvalue functions of A, B, or $A + B$. The result follows now because $\sigma_I^N = \sigma_{I_n}^{nN}$, $\sigma_K^N = \sigma_{K_n}^{nN}$, and $(I_n, J_n, K_n) \in T_{nr}^{nN}$. □

Let us use the notation

$$\mathcal{T} = \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{n-1} \{(\sigma_I^n, \sigma_J^n, \sigma_K^n) : (I, J, K) \in T_r^n\}.$$

The preceding observation allows us to reformulate Theorem 2.1 as follows:

2.3. **Theorem.** *Consider nonincreasing sequences $\alpha, \beta, \gamma \in \mathbb{R}^n$, and form nonincreasing functions u, v, w on $[0, 1)$ such that $u(t) = \alpha_i, v(t) = \beta_i, w(t) = \gamma_i$ for $t \in [\frac{i-1}{n}, \frac{i}{n})$. The following are equivalent:*

(1) *there exist selfadjoint $n \times n$ matrices A, B such that the eigenvalues of A (resp. B, resp. $A + B$) repeated according to their multiplicities are the components of α (resp. β, resp. γ).*

(2) $\int_0^1 u(t)dt + \int_0^1 v(t)dt = \int_0^1 w(t)dt$ *and* $\int_{\omega_1} u(t)dt + \int_{\omega_2} v(t)dt \geq \int_{\omega_3} w(t)dt$ *for every triple* $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$.

3. Embeddable factors

Let us denote by τ_n the normalized trace on the algebra $\mathcal{M}_n$ of complex $n \times n$ matrices, i.e., $\tau_n(T) = \frac{1}{n}Tr(T)$ for $T \in \mathcal{M}_n$. The characteristic property of families of selfadjoint elements in the ultrapower $\mathcal{R}^\omega$ is the existence of finite-dimensional matrix approximants. More precisely, given selfadjoint elements $x_1, x_2, \ldots, x_k \in \mathcal{R}^\omega$, there exist integers $n_1 < n_2 < \cdots$ and selfadjoint matrices $X_1^{(m)}, X_2^{(m)}, \ldots, X_k^{(m)} \in \mathcal{M}_{n_m}$ such that

$$\tau(p(x_1, x_2, \ldots x_k)) = \lim_{m \to \infty} \tau_{n_m}(p(X_1^{(m)}, X_2^{(m)}, \ldots, X_k^{(m)}))$$

for every polynomial p in k noncommuting variables. We will call such a sequence of k-tuples $(X_1^{(m)}, X_2^{(m)}, \ldots, X_k^{(m)})$ a sequence of matricial approximants of $(x_1, x_2, \ldots, x_k)$. The matrices $X_k^{(m)}$ can always be taken to have uniformly bounded norm. In this case we will speak of bounded matricial approximants.

3.1. **Lemma.** *Consider a selfadjoint element x of a II_1 factor $\mathcal{M}$, and let $(X_m)_{m=1}^{\infty}$ be a bounded sequence of matricial approximants of x. Them $\lim_{m\to\infty} \lambda_{X_m}(t) = \lambda_x(t)$ for all but at most countably many values $t \in [0,1)$.*

Proof. Since we have

$$\tau(x^n) = \int_0^1 (\lambda_x(t))^n \, dt,$$

we deduce that

$$\lim_{m\to\infty} \int_0^1 (\lambda_{X_m}(t))^n \, dt = \int_0^1 (\lambda_x(t))^n \, dt$$

for all positive integers n. Now, the functions λ_{X_m} are uniformly bounded, hence Helly's selection theorem insures the existence of subsequences of $(\lambda_{X_m})_{m=1}^{\infty}$ which converge at all but countably many points $t \in [0,1)$. To conclude the proof, we must show that any such pointwise limit coincides with λ_x at all but countably many points. Assume therefore that the limit $\lambda(t) = \lim_{m\to\infty} \lambda_{X_m}(t)$ exists for all but countably many points t, and λ is right continuous. We then have $\int_0^1 \lambda(t)^n \, dt = \int_0^1 \lambda_x(t)^n \, dt$, $n = 1, 2, \ldots$. By the Stone-Weierstrass theorem, $\int_0^1 u(\lambda(t)) \, dt = \int_0^1 u(\lambda_x(t)) \, dt$ for every continuous function u on $\mathbb{R}$, and this equality can be extended by the monotone convergence theorem to all lower semicontinuous functions u. The conclusion $\lambda = \lambda_x$ is now easily reached by taking $u = \chi_{(\alpha,\infty)}$, $\alpha \in \mathbb{R}$. □

We can now prove the main result of this paper.

3.2. **Theorem.** *Consider bounded nonincreasing right-continuous functions u, v, w defined on $[0,1)$. The following are equivalent:*

(1) *there exist $a, b \in \mathcal{R}^{\omega}$ such that $u = \lambda_a$, $v = \lambda_b$, and $w = \lambda_{a+b}$;*

(2) *$\int_0^1 u(t)\,dt + \int_0^1 v(t)\,dt = \int_0^1 w(t)\,dt$ and $\int_{\omega_1} u(t)\,dt + \int_{\omega_2} v(t)\,dt \geq \int_{\omega_3} w(t)\,dt$ for every triple $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$.*

Proof. Assume first that $u = \lambda_a$, $v = \lambda_b$, and $w = \lambda_{a+b}$ for some $a, b \in \mathcal{R}^{\omega}$. As noted before, there exists a bounded sequence $(A_m, B_m)_{m=1}^{\infty}$ of matricial approximants of (a, b), and Lemma 3.1 shows that, with countably many exceptions, $\lim_{m\to\infty} \lambda_{A_m}(t) = u(t)$, $\lim_{m\to\infty} \lambda_{B_m}(t) = v(t)$, and $\lim_{m\to\infty} \lambda_{A_m+B_m}(t) = w(t)$. The relations in (2) now follow from the corresponding relations for λ_{A_m}, λ_{B_m}, $\lambda_{A_m+B_m}$ via the dominated convergence theorem. Conversely, assume that u, v, w satisfy (2). For each integer n define n-tuples $\alpha^{(n)}, \beta^{(n)}, \gamma^{(n)} \in \mathbb{R}^n$ by

$$\alpha_i^{(n)} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} u(t)\,dt, \quad \beta_i^{(n)} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} v(t)\,dt, \quad \gamma_i^{(n)} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} w(t)\,dt,$$

for $i = 1, 2, \ldots, n$. The hypothesis (2) implies that $\alpha^{(n)}, \beta^{(n)}, \gamma^{(n)}$ satisfy the conditions of Theorem 2.1(2), and therefore there exist selfadjoint matrices $A_n, B_n \in \mathcal{M}_n$ such that the eigenvalues of $A_n, B_n, A_n + B_n$ are the components of $\alpha^{(n)}, \beta^{(n)}$, $\gamma^{(n)}$, respectively. Clearly, the norms of A_n and B_n are uniformly bounded, and therefore we can find integers $n_1 < n_2 < \cdots$ such that $\lim_{m\to\infty} p(A_{n_m}, B_{n_m})$ exists for every polynomial p in two noncommuting variables. Since $\mathcal{M}_n$ has a trace-preserving embedding into $\mathcal{R}$, we can find elements $a_m, b_m \in \mathcal{R}$ such that $\lambda_{a_m} = \lambda_{A_{n_m}}$, $\lambda_{b_m} = \lambda_{B_{n_m}}$, $\lambda_{a_m+b_m} = \lambda_{A_{n_m}+B_{n_m}}$. Denote by a and b the elements of $\mathcal{R}^\omega$ determined by the sequences $(a_m)_{m=1}^\infty$ and $(b_m)_{m=1}^\infty$. Clearly then (A_{n_m}, B_{n_m}) are matrix approximants for a and b, so that $\lim_{m\to\infty} \lambda_{A_{n_m}}(t) = \lambda_a(t)$, $\lim_{m\to\infty} \lambda_{B_{n_m}}(t) = \lambda_b(t)$, and $\lim_{m\to\infty} \lambda_{A_{n_m}+B_{n_m}}(t) = \lambda_{a+b}(t)$, for all but countably many values of t. On the other hand, $\lim_{n\to\infty} \lambda_{A_n}(t) = u(t)$ at all points of continuity for u, so that $\lambda_a(t) = u(t)$ at all but countably many points. Thus $\lambda_a = u$ by right continuity. Analogously, $\lambda_b = v$ and $\lambda_{a+b} = w$, thus proving (1). □

4. Concluding remarks

As mentioned in the Introduction, we do not have a proof of the implication (1)⇒(2) for arbitrary II_1 factors. Any counterexample would solve in the negative Connes' question on the embeddability of II_1 factors into $\mathcal{R}^\omega$.

At the other extreme, it may be interesting to see whether the implication (2)⇒(1) holds for a given embeddable factor, such as $\mathcal{R}$ itself.

The inequalities in Theorem 3.2 are not the only ones of the form

$$\int_{\omega_1} u(t)\,dt + \int_{\omega_2} v(t)\,dt \geq \int_{\omega_3} w(t)\,dt$$

which are true for $u = \lambda_a$, $v = \lambda_b$, $w = \lambda_{a+b}$. Thus, for instance, Grothendieck's inequality

$$\int_0^\alpha u(t)\,dt + \int_0^\alpha v(t)\,dt \geq \int_0^\alpha w(t)\,dt$$

only follows from Theorem 3.2 for rational values of α. Let us also note that each triple $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$ can be written in infinitely many ways as $(\sigma_I^n, \sigma_J^n, \sigma_K^n)$ for some $(I, J, K) \in T_r^n$. More precisely, $(\sigma_I^n, \sigma_J^n, \sigma_K^n) = (\sigma_{I_N}^{nN}, \sigma_{J_N}^{nN}, \sigma_{K_N}^{nN})$, as seen earlier.

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