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UNIMODULAR FUNCTIONS AND INTERPOLATING BLASCHKE PRODUCTS

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Abstract. The result by Bourgain that every unimodular function ψ on the unit circle can be factored as $\psi = e^{i\tilde{v}} B_1 \overline{B}_2$ with B_1 and B_2 Blaschke products can be improved. We show that the same result holds with B_1 and B_2 interpolating Blaschke products. This will at the same time be a refinement of Jones's result that every unimodular function can be approximated in the H^∞-norm by a ratio of interpolating Blaschke products.

1. Introduction

A Blaschke product is an H^∞-function on the open unit disk $\mathbb{D}$ of the form

$$B(z) = z^m \prod_{|z_n| \neq 0} \frac{-z_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z},$$

where $\{z_n\}$ is a set of points in $\mathbb{D}$ such that

$$\sum \bigl(1 - |z_n|\bigr) < \infty$$

and m is the number of z_n's equal to 0. The set $\{z_n\}$ is called the zero set of the Blaschke product, as the zeros of $B(z)$ are precisely the points z_n counted with multiplicity. We have $|B(z)| \leq 1$ in $\mathbb{D}$ and non-tangential limits $|B(z)| = 1$ almost everywhere on the unit circle $\mathbb{T}$. See [4, pp. 53–57] for further information on Blaschke products. The Blaschke product is called interpolating if the zero set is an interpolating sequence. That is, if every interpolation problem

$$f(z_j) = a_j, \qquad j \in \mathbb{N},\ (a_j) \in \ell^\infty,$$

has a solution $f \in H^\infty$. A famous result by Carleson shows that, equivalently, a Blaschke product with zero set $\{z_n\}$ is an interpolating Blaschke product if and only if the following two conditions hold [2]:

i) $\inf_{n \neq m} d(z_n, z_m) > 0$.
ii) For all Carleson squares $Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \ell(Q), 1 - \ell(Q) < r < 1\}$,

$$\sum_{z_n \in Q} \bigl(1 - |z_n|\bigr) < C\ell(Q)$$

for some constant C.

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Here $\ell(Q)$ is the base length of Q, while d denotes the pseudo-hyperbolic distance

$$d(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|.$$

In 1969 Douglas and Rudin asked whether $\int_{\mathbb{T}} \log|f| > -\infty$ was both a sufficient and necessary condition on $f \not\equiv 0$ for f to be of the form $f = g\bar{h}$ with $g, h \in H^\infty$ [3]. The question was answered affirmatively by Bourgain in 1986 [1]. He proved the following result, from which the answer to Douglas's and Rudin's problem follows easily.

Theorem 1. *Suppose $a \in L^\infty(\mathbb{T})$ with $\|a\|_\infty \leq \pi$. Then there are Blaschke products B_1 and B_2 such that*

$$\|(a - \operatorname{Arg} \tfrac{B_1}{B_2})^\sim\|_\infty < c$$

where c is a constant.

We use $f \mapsto \tilde{f}$ for the conjugation operator. By examining Bourgain's proof carefully, we may strengthen the theorem. In fact,

$$\|a - \operatorname{Arg} \tfrac{B_1}{B_2}\|_\infty + \|(a - \operatorname{Arg} \tfrac{B_1}{B_2})^\sim\|_\infty < \varepsilon$$

for every $\varepsilon > 0$. Define $v = -(a - \operatorname{Arg} \frac{B_1}{B_2})^\sim$. Then $v \in L^\infty$, and because $|B_j| = 1$ almost everywhere on $\mathbb{T}$, Theorem 1 can be reformulated as follows:

Theorem 1′. *Suppose ψ is a unimodular function on $\mathbb{T}$. Then for every $\varepsilon > 0$ there exist Blaschke products B_1 and B_2 such that*

$$\psi = e^{i\tilde{v}} \tfrac{B_1}{B_2} = e^{i\tilde{v}} B_1 \overline{B_2} \tag{1}$$

for some $v \in L^\infty$ with $\|v\|_\infty < \varepsilon$ and $\|\tilde{v}\|_\infty < \varepsilon$.

The question of whether an arbitrary Blaschke product (and thus an arbitrary inner function) can be approximated by an interpolating Blaschke product has been investigated for some time. The problem, which is still open, was posed by Peter Jones who in 1981 approximated an arbitrary unimodular function by a ratio of interpolating Blaschke products in the H^∞-norm [6]. In the present paper we do something similar. We approximate a ratio of Blaschke products B_1/B_2 by a ratio of interpolating Blaschke products B_1^*/B_2^* in the sense that

$$\tfrac{B_1}{B_2} = e^{i\tilde{v}} \tfrac{B_1^*}{B_2^*},$$

with $v \in L^\infty$ and where $\|v\|_\infty$ and $\|\tilde{v}\|_\infty$ are small. By doing so, we show that Bourgain's result also holds for interpolating Blaschke products.

Theorem 2. *Suppose ψ is a unimodular function on $\mathbb{T}$. Then for every $\varepsilon > 0$ there exist interpolating Blaschke products B_1^* and B_2^* such that*

$$\psi = e^{i\tilde{v}} \tfrac{B_1^*}{B_2^*} = e^{i\tilde{v}} B_1^* \overline{B_2^*}$$

for some $v \in L^\infty$ with $\|v\|_\infty < \varepsilon$ and $\|\tilde{v}\|_\infty < \varepsilon$.

This result can also be viewed as a strengthened version of Jones's theorem.

2. Unimodular functions on $\mathbb{T}$

To prove Theorem 2 we use a result of Marshall and Stray [7] concerning the product of interpolating Blaschke products, and a result of Garnett and Nicolau [5] showing how a Blaschke product can be approximated by a ratio of interpolating Blaschke products.

Lemma 3. *Let B_1^* and B_2^* be interpolating Blaschke products. Then for every $\varepsilon > 0$ there is an interpolating Blaschke product, B^*, such that*

$$B_1^* B_2^* = B^* e^{i\tilde{v}}$$

on $\mathbb{T}$ for some $v \in L^\infty$ with $\|v\|_\infty < \varepsilon$ and $\|\tilde{v}\|_\infty < \varepsilon$.

Proof. This lemma can, although not stated explicitly, be inferred from [7]. The main part of the argument can also be found in [8, pp. 101–103], so we only sketch it here. Denote the zero sets of B_1^* and B_2^* by $\{a_n\}$ and $\{b_n\}$ respectively. Choose δ so small that

$$d(a_n, a_m) \geq 2\delta \quad \text{and} \quad d(b_n, b_m) \geq 2\delta \quad \text{for all } n \neq m.$$

By moving zeros from B_1^* to B_2^* if necessary, the closed disks

$$\Delta_j = \{z : \ d(z, a_j) \leq c\delta\}, \qquad c \text{ small},$$

are disjoint and contain exactly one b_n each. See Figure 1. We must show that also the zeros left in B_1^* can be separated from those of B_2^* such that condition i) holds. By Frostman's theorem [4, p. 79] there is an $\varepsilon_0 \in (c\delta, \frac{4}{3}c\delta)$ such that

$$\widehat{B}_1^*(z) = \frac{B_1^*(z) - \varepsilon_0}{1 - \varepsilon_0 B_1^*(z)}$$

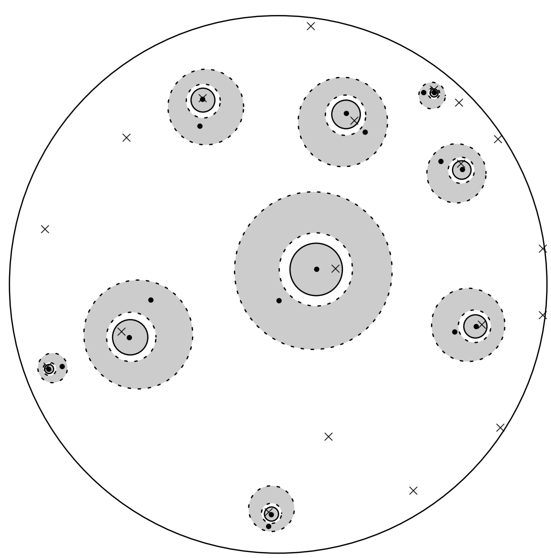

Figure 1. The closed disks Δ_j and the corresponding pseudohyperbolic annuli A_j. The zeros of B_1^* and $\widehat{B}_1^*$ are marked $\bullet$, while the zeros of B_2^* are marked $\times$.

is a Blaschke product. It can then be shown that the zeros of $\widehat{B}_1^*$ lie in the pseudo-hyperbolic annuli

$$A_j = \{z : \ \varepsilon_0 < d(z, a_j) < \delta\}.$$

Thus they are separated from the zeros of B_2^*. It follows that $B^* = \widehat{B}_1^* B_2^*$ is an interpolating Blaschke product approximating $B_1^* B_2^*$. On $\mathbb{T}$, $B_1^* \overline{B_1^*} = |B_1^*|^2 = 1$ almost everywhere. Hence we can write

$$\widehat{B}_1^* = \frac{B_1^* - \varepsilon_0}{1 - \varepsilon_0 B_1^*} = \frac{B_1^*\left(1 - \varepsilon_0 \overline{B_1^*}\right)}{1 - \varepsilon_0 B_1^*} = B_1^* \frac{\overline{h}}{h}$$

where $h = 1 - \varepsilon_0 B_1^*$ is an outer function. So $h = e^{u + i\tilde{u}}$ with $u \in L^\infty$. This gives $\widehat{B}_1^* = e^{-2i\tilde{u}} B_1^*$ and by taking ε_0 small enough, we can ensure that $\|u\|_\infty$ and $\|\tilde{u}\|_\infty$ are less than $\frac{\varepsilon}{2}$. □

Observe that this lemma is easily extended to finite products of interpolating Blaschke products.

The main step in the proof of Theorem 2 is the approximation of Blaschke products by ratios of interpolating Blaschke products. This is accomplished through the following lemma, which is a slight modification of a result by Garnett and Nicolau [5].

Lemma 4. *Let B be a Blaschke product. Then for every $\varepsilon > 0$ there exist interpolating Blaschke products B_1^* and B_2^* such that*

$$B = e^{i\tilde{v}} \tfrac{B_1^*}{B_2^*} = e^{i\tilde{v}} B_1^* \overline{B_2^*}$$

on $\mathbb{T}$ for some $v \in L^\infty$ with $\|v\|_\infty < \varepsilon$ and $\|\tilde{v}\|_\infty < \varepsilon$.

Proof. We follow the same construction as Garnett and Nicolau. Let $0 < \alpha < \beta < 1$, $M = 2^K > 1$ and $\delta < 1$ be constants whose values will be determined later. Note that by applying a preliminary conformal mapping we may assume $|B(0)| > \beta$. We will consider dyadic Carleson squares of the form

$$Q_{n,j} = \{re^{i\theta} : \ 2\pi j 2^{-n} \leq \theta < 2\pi(j+1)2^{-n}, 1 - 2^{-n} \leq r < 1\}$$

and their top-halves $T(Q_{n,j})$. Let $G_1 = \{Q_1^{(1)}, Q_2^{(1)}, \ldots\}$ be the set of maximal $Q_{n,j}$ with

$$\inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

Write $S_{p,k}^{(1)}$, $1 \leq p \leq M = 2^K$, for the M different $Q_{n+K,j} \subset Q_{n,j} = Q_k^{(1)}$, and let $H_1 = \{V_1^{(1)}, V_2^{(1)}, \ldots\}$ be the set of maximal $Q_{n,j}$ for which $V_l^{(1)} \subset Q_k^{(1)}$ for some $Q_k^{(1)}$ and

$$\inf_{T(V_l^{(1)})} |B(z)| > \beta.$$

The function $|B|$ has non-tangential limit 1 almost everywhere, so

$$\sum_{V_l^{(1)} \subset Q_k^{(1)}} \ell(V_l^{(1)}) = \ell(Q_k^{(1)}). \tag{2}$$

Let

$$f(z) = \frac{B(z) - w_0}{1 - \overline{w_0} B(z)}$$

with $w_0 = B(z_0)$, $z_0 \in Q_k^{(1)}$ and $|w_0| = \alpha$. If $1-\beta$ is small, then Schwarz's Lemma applied to f implies that

$$\sup_{T(S_{p,k}^{(1)})} |B(z)| < \beta.$$

We may also deduce that $V_l^{(1)} \subset S_{p,k}^{(1)}$ for some p, k.

Next we iterate the construction. Let $G_2 = \{Q_1^{(2)}, Q_2^{(2)}, \dots\}$ be the set of maximal $Q_{n,j}$ such that

$$Q_{n,j} \subset V_l^{(1)} \in H_1 \qquad \text{and} \qquad \inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

The $Q_k^{(2)}$ are relatively few. In fact, by [4, p. 334], given $\varepsilon_0 > 0$ we can take $(1-\beta)/(1-\alpha)$ so small that

$$\sum_{Q_k^{(2)} \subset V_l^{(1)}} \ell(Q_k^{(2)}) < \varepsilon_0 \ell(V_l^{(1)}). \tag{3}$$

The sets $\{S_{p,k}^{(2)}\}$ and $H_2 = \{V_l^{(2)}\}$ are constructed in the same manner as above. By repeating the argument we obtain

$$Q_k^{(m)} \supset S_{p,k}^{(m)} \supset V_l^{(m)} \supset Q_k^{(m+1)}.$$

Define

$$R_{p,k}^{(m)} = S_{p,k}^{(m)} \setminus \bigcup_{V_l^{(m)} \subset S_{p,k}^{(m)}} V_l^{(m)}$$

and observe that the zeros of $B(z)$ are in

$$\bigcup_{k,m} (Q_k^{(m)} \setminus \bigcup_{V_l^{(m)} \subset Q_k^{(m)}} V_l^{(m)}).$$

By taking $1-\alpha$ small we can make all zeros from $Q_k^{(m)} \setminus \bigcup V_l^{(m)}$ fall into $\bigcup_{p=1}^M R_{p,k}^{(m)}$.

Factor B as $B = B_1 \cdots B_M$ where B_p has zeros only in $\bigcup_{k,m} R_{p,k}^{(m)}$. Fix p and set

$$\Gamma_{p,k}^{(m)} = \partial R_{p,k}^{(m)} \setminus \partial S_{p,k}^{(m)}.$$

See Figure 2. Mark points $z_\nu^* = z_\nu^*(k,m,p)$ on $\Gamma_{p,k}^{(m)}$ such that

$$d(z_\nu^*, z_{\nu+1}^*) = \delta. \tag{4}$$

Let B_p^* be the Blaschke product with zeros $\bigcup_{k,m} z_\nu^*(k,m,p)$. Then condition i) holds by (4). From the definition of the z_ν^*'s there is a constant c dependent on δ such that

$$\sum_{z_\nu^* \in Q_k^{(m)}} \big(1 - |z_\nu^*|\big) \le c \sum_{V_l^{(n)} \subset Q_k^{(m)}} \ell(V_k^{(n)}).$$

Hence by (2) and (3),

$$\sum_{z_\nu^* \in Q_k^{(m)}} (1 - |z_\nu^*|) < \frac{c}{1-\varepsilon_0} \ell(Q_k^{(m)}),$$

so condition ii) holds for all dyadic Carleson squares, and therefore for all Carleson squares. It follows that B_p^* is an interpolating Blaschke product. By Lemma 3 there is then an interpolating Blaschke product $B^* = e^{i\tilde{v}} B_1^* \cdots B_M^*$ on $\mathbb{T}$. To finish

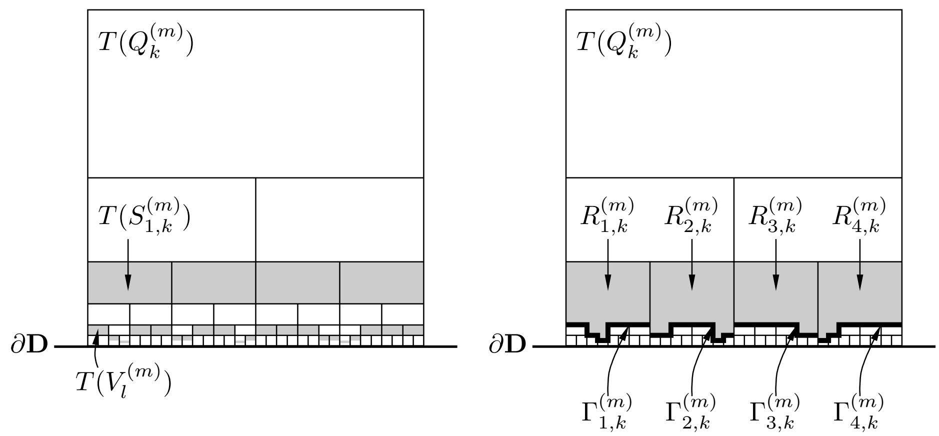

FIGURE 2. Example of a $Q_k^{(m)}$ with $S_{p,k}^{(m)}$, $V_l^{(m)}$, $R_{p,k}^{(m)}$ and $\Gamma_{p,k}^{(m)}$

the proof we will need to show the existence of yet another interpolating Blaschke product C^* such that $C^* = e^{i\tilde{u}}BB^*$.

Before doing so we state three lemmas from [5] which help us with this last part.

Lemma 5. *Let B be a Blaschke product and let $\{z_\nu\}$ be its zeros, counted with their multiplicities. Then B is a finite product of interpolating Blaschke products if and only if there exist positive constants d_0, δ_0 such that for each z_ν there is w_ν with*

$$d(z_\nu, w_\nu) \le d_0$$

and

$$(1 - |w_\nu|^2)|B'(w_\nu)| \ge \delta_0.$$

Lemma 6. $|B_p^*| \le \delta^{1/4}$ *on* $\bigcup_{k,m} R_{p,k}^{(m)}$.

Lemma 7. *There exist $A = A(\alpha, \beta, \delta, M)$ and $\eta = \eta(\alpha, \beta, \delta, M) > 0$ so that if*

$$\inf_{\xi \in \cup_{k,m} R_{p,k}^{(m)}} d(z, \xi) > A \tag{5}$$

and if

$$|B_p B_p^*(z)| = \delta^{1/8},$$

then

$$\big(1 - |z|^2\big)\big|(B_p B_p^*)'(z)\big| \ge \eta.$$

Proof of Lemma 4 *continued.* By Frostman's theorem there is a constant γ with $|\gamma| = \delta^{1/8}$ so that

$$C_p = \frac{B_p B_p^* - \gamma}{1 - \overline{\gamma} B_p B_p^*}$$

is a Blaschke product. Let z_0 be such that $C_p(z_0) = 0$. Then $|(B_p B_p^*)(z_0)| = \delta^{1/8}$ and

$$\big(1 - |z_0|^2\big)\big|C_p'(z_0)\big| \ge \frac{1 - |z_0|^2}{1 - |\gamma|^2}\big|(B_p B_p^*)'(z_0)\big|.$$

If (5) holds, Lemma 7 implies that

$$\big(1-|z_0|^2\big)\big|C_p'(z_0)\big| \geq \frac{\eta}{1-|\gamma|^2} > 0.$$

If, on the other hand, (5) does not hold, there is a $\xi \in \bigcup_{k,m} R_{p,k}^{(m)}$ with $d(z_0,\xi) \leq A$. From Lemma 6 we have $|(B_pB_p^*)(\xi)| \leq \delta^{1/4}$, so somewhere along the hyperbolic geodesic from z_0 to ξ there is a point w with

$$\big(1-|w|^2\big)\big|(B_pB_p^*)'(w)\big| > \hat{\eta} > 0 \qquad \text{and} \qquad d(z,w) < A.$$

Then also

$$\big(1-|w|^2\big)\big|C_p'(w)\big| > 0.$$

So either way Lemma 5 tells us that C_p is a finite product of interpolating Blaschke products.

Lemma 3 then gives us the existence of $u_p \in L^\infty$ such that $C_p^* = e^{-i\tilde{u}_p}C_p$ are interpolating Blaschke products on $\mathbb{T}$. Furthermore,

$$C_p^* = \frac{B_pB_p^*(1-\gamma\overline{B_pB_p^*})}{1-\overline{\gamma}B_pB_p^*}e^{-i\tilde{u}_p} = e^{-i\tilde{v}_p}B_pB_p^* \qquad \text{or} \qquad B_p = e^{i\tilde{v}_p}\frac{C_p^*}{B_p^*},$$

and

$$B = B_1\cdots B_M = e^{i\tilde{v}}\frac{C^*}{B^*} = e^{i\tilde{v}}C^*\overline{B^*}$$

where $B^* = B_1^*\cdots B_M^*$ and $C^* = C_1^*\cdots C_M^*$ are interpolating Blaschke products. Also $v = v_1+\cdots+v_M \in L^\infty$ with $\|v\|_\infty < \varepsilon$ and $\|\tilde{v}\|_\infty < \varepsilon$. □

Proof of Theorem 2. From (1) we have that $\psi = e^{i\tilde{u}}\frac{B_1}{B_2}$ for some $u \in L^\infty$ with $\|u\|_\infty < \frac{\varepsilon}{5}$ and B_1 and B_2 Blaschke products. Lemma 4 aids us in approximating B_1 and B_2 by interpolating Blaschke products $B_{i,j}^*$,

$$\psi = e^{i\tilde{u}}\frac{e^{i\tilde{u}_1}B_{1,1}^*/B_{1,2}^*}{e^{i\tilde{u}_2}B_{2,1}^*/B_{2,2}^*} = e^{i(\tilde{u}+\tilde{u}_1-\tilde{u}_2)}\frac{B_{1,1}^*B_{2,2}^*}{B_{1,2}^*B_{2,1}^*}.$$

By Lemma 3, these products may again be approximated by interpolating Blaschke products. Thus,

$$\psi = e^{i(\tilde{u}+\tilde{u}_1-\tilde{u}_2)}\frac{B_1^*/e^{i\tilde{v}_1}}{B_2^*/e^{i\tilde{v}_2}} = e^{i(\tilde{u}+\tilde{u}_1-\tilde{u}_2-\tilde{v}_1+\tilde{v}_2)}\frac{B_1^*}{B_2^*} = e^{i\tilde{v}}B_1^*\overline{B_2^*},$$

where B_1^* and B_2^* are interpolating Blaschke products and $v = u+u_1-u_2-v_1+v_2 \in L^\infty$. The norms of u_1, u_2, v_1 and v_2 can all be taken less than $\frac{\varepsilon}{5}$; so also can the norms of $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{v}_1$ and $\tilde{v}_2$. Thus $\|v\|_\infty < \varepsilon$ and $\|\tilde{v}\|_\infty < \varepsilon$. □

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