PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 1, Pages 303–310
S 0002-9939(05)07971-2
Article electronically published on June 13, 2005

THE POWER OF THE TANGENT BUNDLE OF THE REAL PROJECTIVE SPACE, ITS COMPLEXIFICATION AND EXTENDIBILITY

TEIICHI KOBAYASHI, HIRONORI YAMASAKI, AND TOSHIO YOSHIDA

(Communicated by Paul Goerss)

Abstract. We establish the formulas on the power τ^k of the tangent bundle $\tau = \tau(RP^n)$ of the real projective n-space RP^n and its complexification $c\tau^k$, and apply the formulas to the problem of extendibility and stable extendiblity of τ^k and $c\tau^k$.

1. Introduction

Let F denote either the real number field R or the complex number field C, and let X be a space and A its subspace. A t-dimensional F-vector bundle ζ over A is said to be *extendible* (respectively *stably extendible*) to X, if and only if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ as F-vector bundles (cf. [9] and [3]). For simplicity, we use the same letter for a vector bundle and its equivalence class.

Let RP^n denote the real projective n-space RP^n and let $\tau = \tau(RP^n)$ stand for the tangent bundle of RP^n. We study the question: Determine the dimension n for which an F-vector bundle over RP^n is extendible (or stably extendible) to RP^m for every $m \geq n$. We have obtained the complete answer for the tangent bundle $\tau = \tau(RP^n)$ in [6] and [8], for the complexification $c\tau$ of τ in [5], for the square $\tau^2 = \tau(RP^n) \otimes \tau(RP^n)$ in [4] and for the complexification $c\tau^2$ of τ^2 in [4], where $\otimes$ denotes the tensor product. The results on τ and τ^2 are as follows.

Theorem 1.1 ([6, Theorem 6.6] and [8, Theorem 4.2]). *The following three conditions are equivalent:*

(i) *τ is extendible to RP^m for every $m \geq n$.*
(ii) *τ is stably extendible to RP^m for every $m \geq n$.*
(iii) *$n = 1$, 3 or 7.*

Theorem 1.2 ([4, Theorem 4]). *The following three conditions are equivalent:*

(i) *τ^2 is extendible to RP^m for every $m \geq n$.*
(ii) *τ^2 is stably extendible to RP^m for every $m \geq n$.*
(iii) *$1 \leq n \leq 16$.*

Received by the editors June 1, 2004 and, in revised form, August 30, 2004.
2000 *Mathematics Subject Classification.* Primary 55R50; Secondary 55N15.
Key words and phrases. Vector bundle, tangent bundle, real projective space, extendibility, stable extendibility, KO-theory, K-theory.

The first purpose of this paper is to obtain the complete answer for the k-fold power τ^k. Let $\phi(n)$ be the number of integers s such that $0 < s \le n$ and $s \equiv 0$, 1, 2 or 4 mod 8. Then we have

Theorem A. *For the k-fold power $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ (k-fold) of the tangent bundle $\tau(RP^n)$, the following three conditions are equivalent:*

(i) *τ^k is extendible to RP^m for every $m \ge n$.*
(ii) *τ^k is stably extendible to RP^m for every $m \ge n$.*
(iii) *There is an integer a satisfying*

$$(n+2)^k - n^k \le a2^{\phi(n)+1} \le (n+2)^k + n^k.$$

If $k = 1$, the condition (iii) is equivalent to the condition: $n = 1$, 3 or 7, and if $k = 2$, it is equivalent to the condition: $1 \le n \le 16$. (Note that $2^{\phi(n)} > n + 1$ for $n \ne 1$, 3, 7, and that $2^{\phi(n)} > n^2 + 2n + 2$ for $n \ge 17$.) Hence Theorem A is a generalization of Theorems 1.1 and 1.2. The results on $c\tau$ and $c\tau^2$ are as follows.

Theorem 1.3 ([5, Theorem 1])**.** *The following three conditions are equivalent:*

(i) *$c\tau$ is extendible to RP^m for every $m \ge n$.*
(ii) *$c\tau$ is stably extendible to RP^m for every $m \ge n$.*
(iii) *$1 \le n \le 5$ or $n = 7$.*

Theorem 1.4 ([4, Theorem 5])**.** *The following three conditions are equivalent:*

(i) *$c\tau^2$ is extendible to RP^m for every $m \ge n$.*
(ii) *$c\tau^2$ is stably extendible to RP^m for every $m \ge n$.*
(iii) *$1 \le n \le 17$.*

The second purpose of this paper is to obtain the complete answer for the complexification $c\tau^k$ of τ^k. For a real number x, let $[x]$ be the largest integer n with $n \le x$. Then we have

Theorem B. *For the complexification $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$ of the k-fold power τ^k, the following three conditions are equivalent:*

(i) *$c\tau^k$ is extendible to RP^m for every $m \ge n$.*
(ii) *$c\tau^k$ is stably extendible to RP^m for every $m \ge n$.*
(iii) *There is an integer b satisfying*

$$(n+2)^k - n^k \le b2^{[n/2]+1} \le (n+2)^k + n^k.$$

If $k = 1$, condition (iii) is equivalent to the condition $1 \le n \le 5$ or $n = 7$, and if $k = 2$, it is equivalent to the condition $1 \le n \le 17$. (Note that $2^{[n/2]} > n + 1$ for $n = 6$ or $n \ge 8$, and that $2^{[n/2]} > n^2 + 2n + 2$ for $n \ge 18$.) Hence Theorem B is a generalization of Theorems 1.3 and 1.4.

This paper is arranged as follows. In Section 2 we establish the formulas on the power $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ of the tangent bundle $\tau(RP^n)$ of the real projective n-space RP^n. In Section 3 we apply the results in Section 2 to the problem of extendibility and stable extendibility of the k-fold power τ^k and prove Theorem A by using Theorem 4.1 in [8]. In Section 4 we establish the formulas on the complexification $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$ of τ^k. In Section 5 we apply the results in Section 4 to the problem of extendibility and stable extendibility of $c\tau^k$ and prove Theorem B by using Theorem 2.1 in [8].

2. The k-fold power of the tangent bundle of $\mathbb{RP}^n$

In this section we establish the formulas on the k-fold power of the tangent bundle $\tau = \tau(RP^n)$. Let ξ_n denote the canonical line bundle over RP^n.

Lemma 2.1. *Let $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ (k-fold) denote the k-fold power of the tangent bundle $\tau = \tau(RP^n)$. Then, for any positive integer r, the following hold in the Grothendick group $KO(RP^n)$:*

(1) $\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n - 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\}$,
(2) $\tau^{2r} = -2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r}\}$.

Proof. It is well known that $\tau = (n+1)\xi_n - 1$ in $KO(RP^n)$. Hence formula (1) clearly holds for $r = 1$.

Assume that formula (1) holds for $r \geq 1$. Then

$$\begin{aligned}\tau^{2r} &= \tau \otimes \tau^{2r-1} \\ &= \{(n+1)\xi_n - 1\}[2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n - 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\}] \\ &= -2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r}\},\end{aligned}$$

since $\xi_n \otimes \xi_n = 1$. So formula (2) holds for $r \geq 1$.

Assume that formula (2) holds for $r \geq 1$. Then

$$\begin{aligned}\tau^{2r+1} &= \tau \otimes \tau^{2r} \\ &= \{(n+1)\xi_n - 1\}[2^{-1}\{(n+2)^{2r} + n^{2r}\} - 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n] \\ &= 2^{-1}\{(n+2)^{2r+1} + n^{2r+1}\}\xi_n - 2^{-1}\{(n+2)^{2r+1} - n^{2r+1}\},\end{aligned}$$

since $\xi_n \otimes \xi_n = 1$. So formula (1) holds for $r + 1$.

Hence formulas (1) and (2) hold for any positive integer r by induction on r. □

The following result is used in our proofs.

Theorem 2.2 (cf. [2, Theorem 1.5, p. 100])**.** *Two t-dimensional F-vector bundles over an n-dimensional CW-complex which are stably equivalent are equivalent if $\langle((n+2)/f) - 1\rangle \leq t$, where $\langle x\rangle$ denotes the smallest integer n with $x \leq n$ and $f = 1$ or 2 according as $F = R$ or C.*

We establish the formula on τ^k, as follows.

Theorem 2.3. *Let $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ (k-fold) denote the k-fold power of the tangent bundle $\tau = \tau(RP^n)$. Then, for any positive integer r, the following hold:*

(1) $\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n$,
(2) $\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}$,

where, in the equalities (1) and (2), a positive integer k denotes the k-dimensional trivial bundle over RP^n and $\oplus$ denotes the Whitney sum.

Proof. (1) By Lemma 2.1(1), we have

$$\tau^{2r-1} + 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n$$

in $KO(RP^n)$. Since

$$\begin{aligned}&\dim[\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\}] \\ &\quad = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\} > n = \dim RP^n,\end{aligned}$$

the equality

$$\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n$$

holds as R-vector bundles by Theorem 2.2.

(2) By Lemma 2.1(2), we have

$$\tau^{2r} + 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}$$

in $KO(RP^n)$. Since $\dim[\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n] = 2^{-1}\{(n+2)^{2r} + n^{2r}\} > n = \dim RP^n$, the equality

$$\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}$$

holds as R-vector bundles by Theorem 2.2. □

Moreover, the next theorem follows from Lemma 2.1.

Theorem 2.4. *For any positive integer r and any integer a, the following hold in $KO(RP^n)$:*

(1) $\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\}\xi_n$
$+ 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\}$,

(2) $\tau^{2r} = 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r} + n^{2r}\}\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r} - a2^{\phi(n)+1}\}$.

Proof. Subtracting $a2^{\phi(n)}(\xi_n - 1) = 0$ (cf. [1, Theorem 7.4]) from equality (1) in Lemma 2.1, we have equality (1) above, and adding $a2^{\phi(n)}(\xi_n - 1) = 0$ to equality (2) in Lemma 2.1, we have the equality (2) above. □

3. Extendibility and stable extendibility of the k-fold power $\tau^k = \tau(\mathbb{RP}^n) \otimes \cdots \otimes \tau(\mathbb{RP}^n)$

Theorem 3.1. *Assume that there is an integer a satisfying*

$$(n+2)^k - n^k \leq a2^{\phi(n)+1} \leq (n+2)^k + n^k.$$

Then τ^k is extendible to RP^m for every $m \geq n$.

Proof. If $k = 1$, the inequalities imply $a = 1$ and $n = 1$, 3 or 7, and if $n = 1$, 3 or 7, $\tau(RP^n)$ is trivial. Hence the results clearly hold for $n = 1$ or $k = 1$. So we may restrict our attention to the case $n > 1$ and $k > 1$.

In case k is odd, let $k = 2r - 1$, where r is an integer > 1. Then, by the assumption, we have

$$2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\} \geq 0$$

and

$$2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\} \geq 0.$$

Hence Theorem 2.4(1) implies that the equality

$$\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\}\xi_n \oplus 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\}$$

holds by Theorem 2.2, since $\dim \tau^{2r-1} = n^{2r-1} > n = \dim RP^n$ for $n > 1$ and $r > 1$. So τ^{2r-1} is extendible to RP^m for every $m \geq n$, since ξ_n and the trivial bundle over RP^n are extendible to RP^m for every $m \geq n$.

In case k is even, let $k = 2r$, where r is a positive integer. Then, by the assumption, we have

$$2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r} + n^{2r}\} \geq 0$$

and

$$2^{-1}\{(n+2)^{2r}+n^{2r}-a2^{\phi(n)+1}\}\geq 0.$$

Hence Theorem 2.4(2) implies that the equality

$$\tau^{2r}=2^{-1}\{a2^{\phi(n)+1}-(n+2)^{2r}+n^{2r}\}\xi_n\oplus 2^{-1}\{(n+2)^{2r}+n^{2r}-a2^{\phi(n)+1}\}$$

holds by Theorem 2.2, since $\dim\tau^{2r}=n^{2r}>n=\dim RP^n$ for $n>1$ and $r>0$. So τ^{2r} is extendible to RP^m for every $m\geq n$. □

The following result is Theorem 4.1 in [8] which is the stably extendible version of Theorem 6.2 in [6].

Theorem 3.2. *Let ζ be a t-dimensional R-vector bundle over RP^n. Assume that there is a positive integer l such that ζ is stably equivalent to $(t+l)\xi_n$, and $t+l<2^{\phi(n)}$. Then $n<t+l$ and ζ is not stably extendible to RP^m for every $m\geq t+l$.*

Theorem 3.3. *Assume that there is an integer a satisfying*

$$(n+2)^k+n^k-2^{\phi(n)+1}<a2^{\phi(n)+1}<(n+2)^k-n^k.$$

Then τ^k is not stably extendible to RP^m for every $m\geq 2^{-1}\{(n+2)^k+n^k-a2^{\phi(n)+1}\}$ if k is odd, and for every $m\geq 2^{-1}\{(a+1)2^{\phi(n)+1}-(n+2)^k+n^k\}$ if k is even.

Proof. If k is odd, let $k=2r-1$. Then putting

$$\zeta=\tau^{2r-1},\quad t=n^{2r-1}\quad\text{and}\quad l=2^{-1}\{(n+2)^{2r-1}-n^{2r-1}-a2^{\phi(n)+1}\}$$

in Theorem 3.2, we obtain the result by Theorem 2.4(1) and Theorem 3.2, since $t+l<2^{\phi(n)}$ and $l>0$ by the assumption.

If k is even, let $k=2r$. Then putting

$$\zeta=\tau^{2r},\quad t=n^{2r}\quad\text{and}\quad l=2^{-1}\{(a+1)2^{\phi(n)+1}-(n+2)^{2r}-n^{2r}\}$$

in Theorem 3.2, we obtain the result by Theorem 2.4(2) and Theorem 3.2, since $t+l<2^{\phi(n)}$ and $l>0$ by the assumption. □

Proof of Theorem A. (i) clearly implies (ii). (iii) implies (i) by Theorem 3.1. To show that (ii) implies (iii), we prove the contraposition. Assume that every integer a satisfies

$$a2^{\phi(n)+1}<(n+2)^k-n^k\text{ or }(n+2)^k+n^k<a2^{\phi(n)+1}.$$

Assume that there are integers a with $a2^{\phi(n)+1}<(n+2)^k-n^k$. Then we define A as the maximum integer such that $A2^{\phi(n)+1}<(n+2)^k-n^k$. If A satisfies $A2^{\phi(n)+1}\leq(n+2)^k+n^k-2^{\phi(n)+1}$, we have $(n+2)^k-n^k\leq(A+1)2^{\phi(n)+1}\leq(n+2)^k+n^k$, and these are inconsistent with our assumption. Hence A satisfies $(n+2)^k+n^k-2^{\phi(n)+1}<A2^{\phi(n)+1}<(n+2)^k-n^k$. So, by Theorem 3.3, τ^k is not stably extendible to RP^m for every $m\geq 2^{-1}\{(n+2)^k+n^k-A2^{\phi(n)+1}\}$ if k is odd, and for every $m\geq 2^{-1}\{(A+1)2^{\phi(n)+1}-(n+2)^k+n^k\}$ if k is even.

Assume that there are integers a with $(n+2)^k+n^k<a2^{\phi(n)+1}$. Then we define B as the minimum integer such that $(n+2)^k+n^k<B2^{\phi(n)+1}$. If B satisfies $B2^{\phi(n)+1}\geq(n+2)^k-n^k+2^{\phi(n)+1}$, we have $(n+2)^k-n^k\leq(B-1)2^{\phi(n)+1}\leq(n+2)^k+n^k$, and these are inconsistent with our assumption. Hence B satisfies $(n+2)^k+n^k-2^{\phi(n)+1}<(B-1)2^{\phi(n)+1}<(n+2)^k-n^k$. So, by Theorem 3.3, τ^k is not stably extendible to RP^m for every $m\geq 2^{-1}\{(n+2)^k+n^k-(B-1)2^{\phi(n)+1}\}$ if k is odd, and for every $m\geq 2^{-1}\{B2^{\phi(n)+1}-(n+2)^k+n^k\}$ if k is even. □

4. The complexification of the k-fold power of $\tau(\mathbb{RP}^n)$

Complexifying the equalities (1) and (2) in Lemma 2.1, we immediately have

Lemma 4.1. *Let $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$ denote the complexification of the k-fold power τ^k of the tangent bundle $\tau = \tau(RP^n)$. Then, for any positive integer r, the following hold in the Grothendick group $K(RP^n)$:*

(1) $c\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}c\xi_n - 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\}$,
(2) $c\tau^{2r} = -2^{-1}\{(n+2)^{2r} - n^{2r}\}c\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r}\}$.

Complexifying the equalities (1) and (2) in Theorem 2.3, we immediately have

Theorem 4.2. *For the complexification $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$ of the k-fold power τ^k of the tangent bundle $\tau = \tau(RP^n)$, the following hold:*

(1) $c\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}c\xi_n$,
(2) $c\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}c\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}$.

Furthermore, the next theorem follows from Lemma 4.1.

Theorem 4.3. *For any positive integer r and any integer b, the following hold in $K(RP^n)$:*

(1) $c\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - b2^{[n/2]+1}\}c\xi_n$
$+ 2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r-1} + n^{2r-1}\}$,
(2) $c\tau^{2r} = 2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r} + n^{2r}\}c\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r} - b2^{[n/2]+1}\}$.

Proof. Subtracting $b2^{[n/2]}(\xi_n - 1) = 0$ (cf. [1, Theorem 7.3]) from equality (1) in Lemma 4.1, we have equality (1) above, and adding $b2^{[n/2]}(\xi_n - 1) = 0$ to equality (2) in Lemma 4.1, we have equality (2) above. □

5. Extendibility and stable extendibility of the complexification $c\tau^k = c(\tau(\mathbb{RP}^n) \otimes \cdots \otimes \tau(\mathbb{RP}^n))$

The proofs of the following Theorems 5.1 and 5.3 are parallel to those of Theorems 3.1 and 3.3, respectively.

Theorem 5.1. *Assume that there is an integer b satisfying*

$$(n+2)^k - n^k \leq b2^{[n/2]+1} \leq (n+2)^k + n^k.$$

Then $c\tau^k$ is extendible to RP^m for every $m \geq n$.

Proof. In case k is odd, let $k = 2r - 1$, where r is a positive integer. Then, by the assumption, we have

$$2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - b2^{[n/2]+1}\} \geq 0$$

and

$$2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r-1} + n^{2r-1}\} \geq 0.$$

Hence Theorem 4.3(1) implies that the equality

$$c\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - b2^{[n/2]+1}\}c\xi_n \oplus 2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r-1} + n^{2r-1}\}$$

holds by Theorem 2.2, since $\dim c\tau^{2r-1} = n^{2r-1} \geq \langle (n+2)/2 - 1 \rangle = \langle n/2 \rangle$. So $c\tau^{2r-1}$ is extendible to RP^m for every $m \geq n$, since $c\xi_n$ and the trivial bundle over RP^n are extendible to RP^m for every $m \geq n$.

In case k is even, let $k = 2r$, where r is a positive integer. Then, by the assumption, we have

$$2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r} + n^{2r}\} \geq 0$$

and

$$2^{-1}\{(n+2)^{2r} + n^{2r} - b2^{[n/2]+1}\} \geq 0.$$

Hence Theorem 4.3(2) implies that the equality

$$c\tau^{2r} = 2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r} + n^{2r}\}c\xi_n \oplus 2^{-1}\{(n+2)^{2r} + n^{2r} - b2^{[n/2]+1}\}$$

holds by Theorem 2.2, since $\dim c\tau^{2r} = n^{2r} \geq \langle (n+2)/2 - 1\rangle = \langle n/2\rangle$. So $c\tau^{2r}$ is extendible to RP^m for every $m \geq n$. □

The following result is Theorem 2.1 in [8] which is the stably extendible version of Theorem 4.2 for $d = 1$ in [7].

Theorem 5.2. *Let ζ be a t-dimensional C-vector bundle over RP^n. Assume that there is a positive integer l such that ζ is stably equivalent to $(t+l)c\xi_n$, and $t+l < 2^{[n/2]}$. Then $n < 2t+2l$ and ζ is not stably extendible to RP^m for every $m \geq 2t+2l$.*

Theorem 5.3. *Assume that there is an integer b satisfying*

$$(n+2)^k + n^k - 2^{[n/2]+1} < b2^{[n/2]+1} < (n+2)^k - n^k.$$

Then $c\tau^k$ is not stably extendible to RP^m for every $m \geq (n+2)^k + n^k - b2^{[n/2]+1}$ if k is odd, and for every $m \geq (b+1)2^{[n/2]+1} - (n+2)^k + n^k$ if k is even.

Proof. If k is odd, let $k = 2r-1$. Then putting

$$\zeta = c\tau^{2r-1}, \quad t = n^{2r-1} \quad \text{and} \quad l = 2^{-1}\{(n+2)^{2r-1} - n^{2r-1} - b2^{[n/2]+1}\}$$

in Theorem 5.2, we obtain the result by Theorem 4.3(1) and Theorem 5.2, since $t + l < 2^{[n/2]}$ and $l > 0$ by the assumption.

If k is even, let $k = 2r$. Then putting

$$\zeta = c\tau^{2r}, \quad t = n^{2r} \quad \text{and} \quad l = 2^{-1}\{(b+1)2^{[n/2]+1} - (n+2)^{2r} - n^{2r}\}$$

in Theorem 5.2, we obtain the result by Theorem 4.3(2) and Theorem 5.2, since $t + l < 2^{[n/2]}$ and $l > 0$ by the assumption. □

Proof of Theorem B. (i) implies (ii) clearly. (iii) implies (i) by Theorem 5.1. To show that (ii) implies (iii), we prove the contraposition. Assume that every integer b satisfies

$$b2^{[n/2]+1} < (n+2)^k - n^k \text{ or } (n+2)^k + n^k < b2^{[n/2]+1}.$$

Assume that there are integers b with $b2^{[n/2]+1} < (n+2)^k - n^k$. Then we define C as the maximum integer such that $C2^{[n/2]+1} < (n+2)^k - n^k$. If C satisfies $C2^{[n/2]+1} \leq (n+2)^k + n^k - 2^{[n/2]+1}$, we have $(n+2)^k - n^k \leq (C+1)2^{[n/2]+1} \leq (n+2)^k + n^k$, and these are inconsistent with our assumption. Hence C satisfies $(n+2)^k + n^k - 2^{[n/2]+1} < C2^{[n/2]+1} < (n+2)^k - n^k$. So, by Theorem 5.3, $c\tau^k$ is not stably extendible to RP^m for every $m \geq (n+2)^k + n^k - C2^{[n/2]+1}$ if k is odd, and for every $m \geq (C+1)2^{[n/2]+1} - (n+2)^k + n^k$ if k is even.

Assume that there are integers b with $(n+2)^k + n^k < b2^{[n/2]+1}$. Then we define D as the minimum integer such that $(n+2)^k + n^k < D2^{[n/2]+1}$. If D satisfies $D2^{[n/2]+1} \geq (n+2)^k - n^k + 2^{[n/2]+1}$, we have $(n+2)^k - n^k \leq (D-1)2^{[n/2]+1} \leq (n+2)^k + n^k$, and these are inconsistent with our assumption. Hence D satisfies

$(n+2)^k + n^k - 2^{[n/2]+1} < (D-1)2^{[n/2]+1} < (n+2)^k - n^k$. So, by Theorem 5.3, $c\tau^k$ is not stably extendible to RP^m for every $m \geq (n+2)^k + n^k - (D-1)2^{[n/2]+1}$ if k is odd, and for every $m \geq D2^{[n/2]+1} - (n+2)^k + n^k$ if k is even. □

References

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632. MR0139178 (25:2614)
2. D. Husemoller, *Fibre bundles*, Second edition, Graduate Texts in Mathmatics **20**, Springer-Verlag, New York-Hiderberg-Berlin, 1975. MR0370578 (51:6805)
3. M. Imaoka and K. Kuwana, *Stably extendible vector bundles over the quaternionic projective spaces*, Hiroshima Math. J. **29** (1999), 273–279. MR1704248 (2000h:55027)
4. T. Kobayashi and K. Komatsu, *Extendiblity and stable extendibility of vector bundles over real projective spaces*, Hiroshima Math. J. **31** (2001), 99–106. MR1820697 (2001j:55018)
5. T. Kobayashi and T. Yoshida, *Extendible and stably extendible vector bundles over real projective spaces*, J. Math. Soc. Japan **55** (2003), 1053–1059. MR2003759 (2004i:55022)
6. T. Kobayashi, H. Maki and T. Yoshida, *Remarks on extendible vector bundles over lens spaces and real projective spaces*, Hiroshima Math. J. **5** (1975), 487–497. MR0385857 (52:6716)
7. T. Kobayashi, H. Maki and T. Yoshida, *Extendiblity with degree d of the complex vector bundles over lens spaces and projective spaces*, Mem. Fac. Sci Kochi Univ. (Math.) **1** (1980), 23–33. MR0561386 (81c:55027)
8. T. Kobayashi, H. Maki and T. Yoshida, *Stably extendible vector bundles over the real projective spaces and the lens spaces*, Hiroshima Math. J. **29** (1999), 631–638. MR1728614 (2000k:55023)
9. R. L. E. Schwarzenberger, *Extendible vector bundles over real projective space*, Quart. J. Math. Oxford (2) **17** (1966), 19–21. MR0195105 (33:3310)

Department of Mathematics, Faculty of Science, Kochi University, Kochi 780–8520, Japan

E-mail address: kteiichi@lime.ocn.ne.jp

Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739–8526, Japan

E-mail address: d042710@math.sci.hiroshima-u.ac.jp

Department of Mathematics, Faculty of Integrated Arts and Sciences, Hiroshima University, Higashi-Hiroshima 739–8521, Japan

E-mail address: t-yosida@mis.hiroshima-u.ac.jp