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ZERO SETS AND INTERPOLATING SETS IN FOCK SPACES

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ABSTRACT. An example is constructed to show that interpolating sets for Fock spaces are not necessarily zero sets.

1. INTRODUCTION

For $1 \le p < \infty$ and $\alpha > 0$, the *Fock space* F^p_α consists of all entire functions f for which

$$\|f\|^p_{p,\alpha} = \int_{\mathbb{C}} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) < \infty.$$

The space F^∞_α consists of all entire functions f for which

$$\|f\|_{\infty,\alpha} = \sup_{z\in\mathbb{C}} |f(z)|\, e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

A sequence of distinct complex numbers $\{z_j\}$ is called an *interpolating set for* F^p_α if for every sequence $\{a_j\}$ with the property

$$\sum_{j=1}^{\infty} \left| a_j e^{-\frac{\alpha}{2}|z_j|^2} \right|^p < \infty,$$

there is a function $f \in F^p_\alpha$ solving the interpolation problem

$$f(z_j) = a_j, \qquad j = 1, 2, \ldots .$$

A sequence $\Gamma = \{z_j\}$ is said to be *uniformly discrete* if there is a constant $\delta > 0$ such that

$$|z_j - z_k| > \delta, \qquad j \ne k.$$

The supremum of all such δ is called the *separation constant* of Γ. For $\zeta \in \mathbb{C}$ and $r > 0$, let $B(\zeta, r)$ denote the disk $|z - \zeta| < r$. Let $n(\Gamma, \zeta, r)$ be the number of points of Γ that lie in $B(\zeta, r)$. The *upper uniform density* of Γ is

$$D^+(\Gamma) = \limsup_{r\to\infty} \sup_{\zeta\in\mathbb{C}} \frac{n(\Gamma, \zeta, r)}{r^2}.$$

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Seip and Wallstén [4, 5] characterized interpolating sets in terms of their upper uniform densities, as follows.

Theorem (Seip–Wallstén)**.** *Let $1 \le p < \infty$ and $\alpha > 0$. A complex sequence Γ is an interpolating set for F^p_α if and only if it is uniformly discrete and $D^+(\Gamma) < \alpha$.*

The theorem shows that for a fixed α, the interpolating sets for F^p_α are independent of p. Qualitatively, an interpolating set must be sparse in $\mathbb{C}$.

A set $\Gamma = \{z_j\}$ is called a *zero set* of a space X of analytic functions if there is a function $f \in X$ whose zeros are precisely Γ, with the usual convention that repeated elements of Γ are zeros of prescribed order. The *Bergman space* A^p consists of those functions f analytic in the unit disk $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty.$$

Horowitz [2] showed that every subset of an A^p zero set is also an A^p zero set (see also Chapter 4 of [3]). However, Zhu [6] showed that a subset of an F^p_α zero set need not be an F^p_α zero set.

A sequence of distinct points $\{z_j\}$ in the disk $\mathbb{D}$ is called an *interpolating set for* A^p if for every sequence $\{a_j\}$ with

$$\sum_{j=1}^{\infty} |a_j|^p \left(1 - |z_j|^2\right)^2 < \infty,$$

there is a function $f \in A^p$ solving the interpolation problem

$$f(z_j) = a_j, \qquad j = 1, 2, \ldots.$$

It is easy to show that interpolating sets for A^p are also A^p zero sets. Given an interpolating set $\{z_j\}$ for A^p, let $f \in A^p$ be a solution to

$$f(z_1) = 1, \qquad f(z_j) = 0 \quad \text{for } j \ge 2.$$

Then the non-zero function $(z - z_1)f(z)$ belongs to A^p and vanishes on all z_j. Now Horowitz's result quoted above allows us to conclude that there is some function in A^p vanishing *precisely* on the set $\{z_j\}$.

The corresponding argument fails in the Fock space on two accounts. The factor $(z - z_1)$ is unbounded in $\mathbb{C}$, so for $f \in F^p_\alpha$ the function $(z - z_1)f(z)$ need not belong to F^p_α. Also, in view of Zhu's result, a non-zero function in F^p_α may vanish on a set Γ without Γ being an F^p_α zero set. In fact, the corresponding conclusion fails as well. In the next section of the paper we will show that, contrary to the situation in Bergman spaces, an interpolating set for F^p_α need not be an F^p_α zero set.

2. An interpolating set that is not a zero set

Before giving the construction, we recall some background from the theory of entire functions. Proofs can be found, for instance, in Boas [1].

Let f be an entire function, and let $M(r, f) = \max_{|z|=r} |f(z)|$. The *order* of f is

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

If $0 < \rho < \infty$, the *type* of f is

$$\tau = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho}.$$

It is clear from the above definition that if a function f has order ρ, then for every $\varepsilon > 0$,

$$M(r,f) = O(e^{r^{\rho+\varepsilon}}), \qquad r \to \infty.$$

Thus any function with order $\rho < 2$ must be in F^p_α. Conversely, we can use the fact that point-evaluation functionals are bounded to determine the order and type of a function in the Fock space fairly precisely; see Theorem 2 of Zhu [6].

Lemma 1. *Every function $f \in F^p_\alpha$ has order $\rho \le 2$. If $\rho = 2$, then f has type $\tau \le \frac{\alpha}{2}$.*

For any entire function f, let its zeros z_j be enumerated so that $|z_1| \le |z_2| \le \dots$. The *convergence exponent of* $\{z_j\}$ is the infimum ρ_1 of the positive numbers β for which

$$\sum_j \frac{1}{|z_j|^\beta} < \infty.$$

Let $n(r)$ be the number of zeros of f inside the disk $B(0,r)$. If f has order ρ with $0 < \rho < \infty$, let

$$S(r) = \sum_{0<|z_j|<r} \frac{1}{(z_j)^\rho}.$$

The following lemma is a well-known consequence of Jensen's theorem.

Lemma 2. *Let f be an entire function of order ρ, and let ρ_1 be the convergence exponent of its zeros. Then $\rho_1 \le \rho$.*

One can show, via the Hadamard factorization theorem, that an entire function with non-integer order ρ has finite type if and only if $n(r) = O(r^\rho)$. If ρ is an integer, there is an additional restriction. The following theorem is due to Lindelöf (see Boas [1], p. 27).

Theorem (Lindelöf)**.** *If the order of an entire function f is a positive integer ρ, then f is of finite type if and only if both $n(r) = O(r^\rho)$ and the sums $S(r)$ are bounded.*

Lindelöf's theorem will be the key to our construction.

Theorem. *An interpolating set in the Fock space F^p_α need not be an F^p_α zero set.*

Proof. Let $\delta > 0$. For each integer $k \ge 1$, let Γ_k be the set of $(k+1)$ points evenly spaced in the first quadrant on the circle $|z| = k\delta$, including the points $k\delta$ and $ik\delta$. Let $\Gamma = \bigcup_{k=1}^\infty \Gamma_k$, and write $\Gamma = \{z_j\}$.

First we observe that the distance between two neighboring points in Γ_k is $2k\delta \sin\frac{\pi}{4k} > \delta$, so we may choose $\delta > \frac{2}{\sqrt{\alpha}}$ to ensure that Γ is uniformly discrete with a separation constant larger than $\frac{2}{\sqrt{\alpha}}$. We claim that the resulting set Γ is an interpolating set for F^p_α. Indeed, if the set $\{z_j\}$ is uniformly discrete with separation constant δ, then the disks $B\left(z_j, \frac{\delta}{2}\right)$ are pairwise disjoint. Given any $\zeta \in \mathbb{C}$ and

$r > 0$, if $z_j \in B(\zeta, r)$, the triangle inequality implies that $B\left(z_j, \frac{\delta}{2}\right) \subset B\left(\zeta, r + \frac{\delta}{2}\right)$. Therefore,

$$\bigcup_{z_j \in B(\zeta,r)} B\left(z_j, \tfrac{\delta}{2}\right) \subset B\left(\zeta, r + \tfrac{\delta}{2}\right).$$

A comparison of areas now shows that

$$\pi \left(\tfrac{\delta}{2}\right)^2 n(\Gamma, \zeta, r) \leq \pi \left(r + \tfrac{\delta}{2}\right)^2.$$

Thus $n(\Gamma, \zeta, r)$ has an upper bound independent of the center ζ, so it follows that

$$D^+(\Gamma) = \limsup_{r\to\infty} \sup_{\zeta \in \mathbb{C}} \frac{n(\Gamma, \zeta, r)}{r^2} \leq \limsup_{r\to\infty} \frac{4\left(r + \frac{\delta}{2}\right)^2}{\delta^2 r^2} = \frac{4}{\delta^2}.$$

Thus the choice $\delta > \frac{2}{\sqrt{\alpha}}$ implies $D^+(\Gamma) < \alpha$, so by the Seip–Wallstén theorem, Γ is an interpolating set for F^p_α.

Now suppose for purpose of contradiction that Γ is the zero set of some function $f \in F^p_\alpha$. Observe that Γ has the property

$$\sum_{j=1}^{\infty} \frac{1}{|z_j|^\beta} = \sum_{k=1}^{\infty} \frac{k+1}{(k\delta)^\beta} < \infty,$$

for each $\beta > 2$, whereas the series diverges for $\beta = 2$. Thus the convergence exponent $\rho_1 = 2$, and it follows from Lemma 2 that f has order $\rho \geq 2$. But $\rho \leq 2$ by Lemma 1, so we conclude that $\rho = 2$. Since ρ is a positive integer, Lindelöf's theorem applies.

For $m\delta < r \leq (m+1)\delta$, we calculate

$$\begin{aligned} S(r) &= \sum_{|z_j|<r} \frac{1}{z_j^2} = \sum_{k=1}^{m} \frac{1}{(k\delta)^2} \sum_{\ell=0}^{k} e^{-i\pi\ell/k} \\ &= \sum_{k=1}^{m} \frac{1}{(k\delta)^2} \frac{1 + e^{-i\pi/k}}{1 - e^{-i\pi/k}} = \sum_{k=1}^{m} \frac{1}{(k\delta)^2} \frac{\cos\frac{\pi}{2k}}{i \sin\frac{\pi}{2k}} \\ &\sim -\frac{2i}{\pi\delta^2} \sum_{k=1}^{m} \frac{1}{k} \sim -\frac{2i}{\pi\delta^2} \log m \sim -\frac{2i}{\pi\delta^2} \log r \end{aligned}$$

as $r \to \infty$, which shows that the sums $S(r)$ are unbounded. By Lindelöf's theorem, this implies that f has infinite type. However, this contradicts Lemma 1, which says that every function in F^p_α with $\rho = 2$ has finite type $\tau \leq \frac{\alpha}{2}$. Therefore, Γ cannot be the zero set of a function in F^p_α. □

Remark. The construction above relies on the fact that one can find interpolating sets for F^p_α whose exponent of convergence ρ_1 is exactly 2. It is easy to see that every set $\Gamma = \{z_j\}$ with $\rho_1 < 2$ is an F^p_α zero set. In fact, the canonical product P formed by $\{z_j\}$ has order $\rho = \rho_1 < 2$ (see Boas [1], p. 19). Thus $P \in F^p_\alpha$, and Γ is an F^p_α zero set. The same argument shows that if Γ is the zero set of some $f \in F^p_\alpha$ with order $\rho < 2$, then $\rho_1 \leq \rho < 2$, so every subset of Γ is also an F^p_α zero set.

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