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SPECTRAL PICTURES OF AB AND BA

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ABSTRACT. The spectral pictures of products AB and BA of Banach space operators are compared; in particular when one of them is 'of index zero'.

We shall describe an operator A in the algebra $B(X)$ of bounded linear operators on a Banach space X as being *of index zero* whenever there is Banach space isomorphism

$$A^{-1}(0) \;\cong\; X/\mathrm{cl}\,A(X). \tag{0.1}$$

For example a Fredholm operator has this property if and only if its Fredholm index is zero. In particular, finite-dimensional operators and normal operators acting on a Hilbert space are of index zero. A 'quasi-affinity', in the sense of an operator which is one-to-one and of dense range, is another kind of example. D. Djordjevic [Dj] has essentially noticed that

1. Theorem. *If $A \in B(X)$ is of index zero and $B \in B(X)$, there is the implication*

$$AB \textit{ invertible}, \Longleftrightarrow BA \textit{ invertible} \tag{1.1}$$

and

$$AB \textit{ Fredholm} \iff BA \textit{ Fredholm}\,, \tag{1.2}$$

in which case

$$\mathrm{index}(AB) = \mathrm{index}(B) = \mathrm{index}(BA)\,. \tag{1.3}$$

Proof. If BA is invertible, then A is left invertible, in particular one-to-one with closed range; hence by index zero it has dense range, which now makes it invertible. For invertible A the implication (1.1) is clear. If conversely AB is invertible, then A is right invertible, in particular onto; hence by index zero it is one-to-one and again invertible. This gives (1.1) both ways; towards (1.2) argue that if BA is Fredholm, then A is upper semi-Fredholm, having closed range and finite-dimensional null space. If also A is "index zero" in the sense of (0.1), then its closed range must have finite codimension making it Fredholm, in which case (1.2) is clear. Conversely if AB is Fredholm, then A is lower semi-Fredholm, in the sense of having a closed range of finite codimension; hence it is also a finite-dimensional null space and again

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Fredholm. This gives (1.2) both ways; finally since $\text{index}(A) = 0$ also, the usual index-of-product formula gives (1.3). $\square$

If $A \in B(X)$, write $\sigma(A)$, $\sigma^{\text{left}}(A)$, $\sigma^{\text{right}}(A)$, $\sigma_{\text{ess}}(A)$, $\sigma_{\text{ess}}^{\text{left}}(A)$, and $\sigma_{\text{ess}}^{\text{right}}(A)$ for the spectrum, the left spectrum, the right spectrum, the essential spectrum, the left essential spectrum, and the right essential spectrum, respectively, of A. If $A \in B(X)$, a *hole* in $\sigma_{\text{ess}}(A)$ is a bounded component of $\mathbb{C} \setminus \sigma_{\text{ess}}(A)$ and a *pseudohole* in $\sigma_{\text{ess}}(A)$ is a component of $\sigma_{\text{ess}}(A) \setminus \sigma_{\text{ess}}^{\text{left}}(A)$ or $\sigma_{\text{ess}}(A) \setminus \sigma_{\text{ess}}^{\text{right}}(A)$. The *spectral picture* of A, denoted $\mathcal{SP}(A)$, is the structure consisting of the set $\sigma_{\text{ess}}(A)$, the collection of holes and pseudoholes in $\sigma_{\text{ess}}(A)$, and the indices associated with those holes and pseudoholes. Write $K(X)$ for the ideals of compact operators on X.

We now have:

2. Theorem. *If $A \in B(X)$ is of index zero and $B \in B(X)$, then*

$$\sigma(BA) = \sigma(AB); \tag{2.1}$$

$$\sigma^{\text{left}}(AB) \subseteq \sigma^{\text{left}}(BA) \quad \text{and} \quad \sigma^{\text{right}}(BA) \subseteq \sigma^{\text{right}}(AB); \tag{2.2}$$

$$\sigma_{\text{ess}}(BA) = \sigma_{\text{ess}}(AB); \tag{2.3}$$

$$\sigma_{\text{ess}}^{\text{left}}(AB) \subseteq \sigma_{\text{ess}}^{\text{left}}(BA) \quad \text{and} \quad \sigma_{\text{ess}}^{\text{right}}(BA) \subseteq \sigma_{\text{ess}}^{\text{right}}(AB)\ . \tag{2.4}$$

Further if $0 \in \mathbb{C}$ is not in any pseudohole of either AB or BA, then

$$\mathcal{SP}(AB) = \mathcal{SP}(BA)\ . \tag{2.5}$$

Proof. It is familiar ([Ba], [GGK], [Ha], [LYR]) that

$$\omega(AB) \setminus \{0\} = \omega(BA) \setminus \{0\} \tag{2.6}$$

for the spectrum $\omega = \sigma$, as well as for the left, right, essential, left essential and right essential spectrum: thus (2.1) and (2.3) follow from (1.1) and (1.2). For the same reason (2.2) and (2.4) depend only on the fate of $0 \in \mathbb{C}$. Thus if BA has a left inverse, then A is left invertible, which together with having "index zero" makes it invertible, which now gives AB a left inverse:

$$CBA = I \Longrightarrow CB = A^{-1} \Longrightarrow ACB = I$$

means that B is now left invertible, and hence also AB. The argument for right invertibility is exactly the same. For (2.4) suppose BA is upper semi-Fredholm. Then by Atkinson's theorem BA is left invertible modulo $K(X)$ and so $I - U(BA) \in K(X)$ for some $U \in B(X)$. Note that A is upper semi-Fredholm and hence by assumption, it is Fredholm of index zero. Remembering ([Ha, Theorem 6.5.2]) that a Fredholm operator of index zero can be written as the sum of an invertible and a finite rank operator, write $A = V + K$, where V is invertible and K is of finite rank. Then

$$I - UB(V + K) \in K(X) \Longrightarrow I - UBV \in K(X) \Longrightarrow I - VUB \in K(X),$$

which implies B is upper semi-Fredholm and hence, so is AB, giving the first inclusion of (2.4), and the argument for the second is the same. Finally, to see (2.5), it is effective to remember ([GGK, p. 38]) that with no restriction on A,

$$\begin{pmatrix} AB - I & 0 \\ 0 & I \end{pmatrix} = F \begin{pmatrix} BA - I & 0 \\ 0 & I \end{pmatrix} E, \tag{2.7}$$

where

$$E := \begin{pmatrix} B & I \\ AB - I & A \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} A & I - AB \\ -I & B \end{pmatrix}$$

are both invertible. Thus from (2.7),

$$\text{index}(AB - I) = \text{index}(F) + \text{index}(BA - I) + \text{index}(E) = \text{index}(BA - I). \tag{2.8}$$

This implies that whenever $\lambda \neq 0$ is in a hole or pseudohole common to AB and BA, then the value of the index for that pseudohole is the same for both. Thus if $0 \in \mathbb{C}$ is not in any pseudohole of either AB or BA, then we can conclude that $\mathcal{SP}(AB) = \mathcal{SP}(BA)$. This proves (2.5). □

We would remark that 0 can be in a pseudohole of AB but not in a pseudohole of BA, or vice versa, but that if 0 is in the polynomially convex hull of a pseudohole of AB, then it is also in the polynomially convex hull of a pseudohole of BA, and vice versa. On the other hand, none of the inclusions in (2.2) and (2.4) can be replaced by equality:

3. Example. If $X = \ell_2$ and

$$A(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (0, x_2, 0, x_4, 0, x_6, \ldots)\ , \tag{3.1}$$

$$B(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots)\ , \tag{3.2}$$

$$B'(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (x_2, x_4, x_6, x_8, x_{10}, x_{12}, \ldots), \tag{3.3}$$

then A is of index zero, AB is left invertible but BA is not upper semi-Fredholm, while $B'A$ is right invertible but AB' is not lower semi-Fredholm. Also BA is of index zero while AB is not.

Proof. Observe that

$$BB' = A \neq I = B'B\ ;\ AB = B \neq BA\ ;\ B'A = B' \neq AB'\ ,$$

and look at the null space of BA and the closure of the range of AB'. □

In Example 3, a straightforward calculation shows that $\mathcal{SP}(AB)$ and $\mathcal{SP}(BA)$ has only one pseudohole H_0 whose polynomially convex hull contains 0: with $\mathbb{D}$ the open unit disk

$$H_0(AB) = \mathbb{D} \ \text{ with index}\, H_0(AB) = -\infty;$$
$$H_0(BA) = \mathbb{D} \setminus \{0\} \ \text{ with index}\, H_0(BA) = -\infty.$$

On the other hand, from (2.7) we can see that for each $\lambda \neq 0$,

$$(AB - \lambda I)^{-1}(0) \cong (BA - \lambda I)^{-1}(0) \quad \text{and} \quad X/\text{cl}\,(AB - \lambda I)(X) \cong X/\text{cl}\,(BA - \lambda I)(X),$$

which implies that with no restriction on A and B,

$$AB - \lambda I \text{ “of index zero”} \iff BA - \lambda I \text{ “of index zero”}, \quad \lambda \neq 0. \tag{3.4}$$

However if $\lambda = 0$, Example 3 shows that (3.4) may fail though each of A, B, AB and BA has closed range and A is of index zero. As a trivial sort of dual to Theorem 1, it is clear that

$$A \text{ invertible} \implies (BA \text{ index zero} \iff AB \text{ index zero}); \tag{3.5}$$

by Example 3 this does not extend to A of index zero.

We however have:

4. Proposition. *If X is separable Hilbert space and if each of AB, BA and B has closed range, then if A is also Fredholm there is equivalence*

$$BA \quad \text{“of index zero”} \iff AB \quad \text{“of index zero”}. \tag{4.1}$$

Proof. If either AB or BA is Fredholm, then this is contained in (2.3), and if either BA is upper semi-Fredholm or AB is lower semi-Fredholm, then this is contained in (2.4). Thus we may assume that the null space of BA is infinite dimensional and that the range of AB is of finite codimension: on separable space X this implies

$$(BA)^{-1}(0) \cong X \cong X/(AB)X\,; \tag{4.2}$$

we therefore have to show

$$(AB)^{-1}(0) \cong X \iff X/(BA)X \cong X\,. \tag{4.3}$$

We claim

$$(BA)^{-1}(0) \cong X \Longrightarrow B^{-1}(0) \cong X \iff (AB)^{-1}(0) \cong X \tag{4.4}$$

and

$$X/(AB)X \cong X \Longrightarrow X/B(X) \cong X \iff X/(BA)X \cong X\,; \tag{4.5}$$

this is because of the isomorphisms [Ha, (6.5.4.6), (6.5.4.7)]

(4.6)
$$(AB)^{-1}(0)/B^{-1}(0) \cong A^{-1}(0) \cap B(X); \quad AX/(AB)X \cong X/(BX + A^{-1}(0))\,,$$

(4.7)
$$(BA)^{-1}(0)/A^{-1}(0) \cong B^{-1}(0) \cap A(X)\,; \quad BX/(BA)X \cong X/(AX + B^{-1}(0))\,.$$

If $A^{-1}(0)$ is finite dimensional, then the first part of (4.6) gives the second part of (4.4), while the first comes from the first part of (4.7). If $A(X)$ is of finite codimension, then the second part of (4.7) gives the second part of (4.5), while the first comes from the second part of (4.6); alternatively take adjoints in (4.4). □

The spectral picture $\mathcal{SP}(T)$ determines whether an operator is “quasitriangular” [AFV], and whether it is “compalent” to another operator [BDF].

Recall ([Pe, Definition 4.8]) that $T \in B(H)$ for a Hilbert space H is called *quasitriangular* if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections of finite rank in $B(H)$ that converges strongly to 1 and satisfies $||P_nTP_n - TP_n|| \to 0$. The set of quasitriangular operators can be characterized as the set of all sums of the form $T_0 + K$, where T_0 is triangular and $K \in K(H)$ (cf. [Pe, Corollary 4.19]). We have:

5. Corollary. *If $A \in B(H)$ is of index zero, then AB is quasitriangular if and only if BA is quasitriangular.*

Proof. By Apostol, Foias, and Voiculescu [AFV] the operator T is quasitriangular if and only if $\mathcal{SP}(T)$ contains no hole or pseudohole with negative index. □

Recall that $T \in B(H)$ for a Hilbert space H is called *essentially normal*, if $T^*T - TT^* \in K(H)$ and that operators T_1 and T_2 in $B(H)$ are said to be *compalent* if there exists a unitary operator $W \in B(H)$ and a compact operator $K \in K(H)$ such that $WT_1W^* + K = T_2$. Then by the beautiful Brown-Douglas-Fillmore theorem ([BDF]) we have:

6. Corollary. *Let $A \in B(H)$ be of index zero. If AB and BA are essentially normal, then AB and BA are compalent.*

Proof. If AB and BA are essentially normal, then neither of them have any pseudoholes, so that (2.5) holds. Now the result follows from the Brown-Douglas-Fillmore theorem—if T_1 and T_2 are essentially normal, then T_1 and T_2 are compalent if and only if $\mathcal{SP}(T_1) = \mathcal{SP}(T_2)$. $\square$

We write $\operatorname{Lat}(A)$ for the invariant subspace lattice of $A \in B(X)$, and recall that a "quasiaffinity" is one-to-one with dense range; obviously if A is not a quasiaffinity, then either its null space or the closure of its range will be in $\operatorname{Lat}(A)$. We observe

7. Proposition. *If A, B in $B(X)$ are such that BA is a quasiaffinity, then*

$$\operatorname{Lat}(BA) \textit{ nontrivial } \Longrightarrow \operatorname{Lat}(AB) \textit{ nontrivial.} \tag{7.1}$$

Proof. By assumption A is one-to-one and B has dense range. We claim that if $N \in \operatorname{Lat}(BA)$ is nontrivial, then

$$M = \operatorname{cl}(AN) \Longrightarrow M \in \operatorname{Lat}(AB) \text{ with } \{0\} \neq M \neq X. \tag{7.2}$$

The invariance of M is clear; $M \neq \{0\}$ is because A is one-to-one and N is nonzero; $M \neq X$ is because B is dense and $N \neq X$. $\square$

Not everything in $\operatorname{Lat}(AB)$ need be derived in this way from $\operatorname{Lat}(BA)$: for example if A is the forward and B the backward shift, look at $(AB)^{-1}(0)$.

An operator $T \in B(H)$ for a Hilbert space H has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with the same null space as T. Associated with T, there is a useful related operator $T_\epsilon := |T|^\epsilon U |T|^{1-\epsilon}$ $(0 \leq \epsilon \leq 1)$ called the *generalized Aluthge transform* of T of order ϵ ([Al]). If $\epsilon = \frac{1}{2}$ this really is the Aluthge transform while if $\epsilon = 0$ we get back T itself; if $\epsilon = 1$, then this is what Carl Pearcy has called the "Duggal transplant" of T.

We recapture [JKP, Corollary 1.12]:

8. Corollary. *Let $T \in B(H)$. If $\mathcal{SP}(T)$ has no pseudoholes, then $\mathcal{SP}(T) = \mathcal{SP}(T_\epsilon)$ for each $0 \leq \epsilon \leq 1$.*

Proof. Let $T = U\,|T|$ be the polar decomposition of T. Note that $|T|^\epsilon$ is of index zero. Now applying Theorem 2 with $A := |T|^\epsilon$ and $B := U|T|^{1-\epsilon}$ gives the result. $\square$

9. Corollary. *If $T \in B(H)$ is a quasiaffinity and $0 \leq \epsilon \leq 1$, then $\operatorname{Lat}(T)$ is nontrivial if and only if $\operatorname{Lat}(T_\epsilon)$ is nontrivial.*

Proof. Let $T = U\,|T|$ be the polar decomposition of T. Note that if T is a quasiaffinity, then T_ϵ is a quasiaffinity and U is a unitary operator. Write $A := |T|^\epsilon$ and $B := U|T|^{1-\epsilon}$. Now applying Proposition 7 with $T = BA$ and $T_\epsilon = AB$ gives implication one way, and for the other way reverse them. $\square$

10. Remark. If $f(\lambda)$ is a holomorphic function on a neighbourhood of $\sigma(AB)$ with $f(0) = 0$, then for $A, B \in B(X)$ we can see that

$$f(AB) = AC \quad \text{and} \quad f(BA) = CA \quad \text{for some } C \in B(X)$$

(cf. [Ba, Corollary 8]). Thus the results of this paper can be extended to $f(AB)$ and $f(BA)$ with such a function f.

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