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REPRESENTING CONDITIONAL EXPECTATIONS AS ELEMENTARY OPERATORS

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Abstract. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. We call a linear operator from $\mathcal{A}$ to $\mathcal{B}$ an elementary conditional expectation if it is simultaneously an elementary operator and a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. We give necessary and sufficient conditions for the existence of a faithful elementary conditional expectation of a prime unital C^*-algebra onto a subalgebra containing the identity element. We give a description of all faithful elementary conditional expectations. We then use these results to give necessary and sufficient conditions for certain conditional expectations to be index-finite (in the sense of Watatani) and we derive an inequality for the index.

1. Elementary operators

Let $\mathcal{A}$ be a C^*-algebra. $\Phi : \mathcal{A} \to \mathcal{A}$ is called an elementary operator if either $\Phi = 0$ or there exists $\{x_i\}_{i=1}^n; \{y_i\}_{i=1}^n$, two finite sets each containing the same number of linearly independent elements in $\mathcal{A}$, such that $\Phi(a) = \sum_{i=1}^n x_i a y_i$. A C^*-algebra $\mathcal{A}$ is called prime if for any non-zero $x, y \in \mathcal{A}$, there exists $z \in \mathcal{A}$ such that $xzy \neq 0$. It was first noted by Mathieu [4] that elementary operators on prime C^*-algebras have many interesting properties. Many of the results in section 4 of [4] will be used later in this paper. We will also need the following results.

Lemma 1.1. *Let $\mathcal{A}$ be a prime C^*-algebra, let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be two linearly independent sets in $\mathcal{A}$ and let $c_{ij} \in \mathbb{C}$ for $1 \leq i, j \leq n$. Then $\sum_{1\leq i,j\leq n} c_{ij} x_i m y_j = 0$ for all $m \in \mathcal{A}$ implies that $c_{ij} = 0$ for all i, j.*

Proof. Let C be the n by n matrix whose (i, j) entry is c_{ij}. Let $C = UDV$ be its singular value decomposition, where U and V are unitary matrices and D is a diagonal matrix with non-negative entries. Now let $a_k = \sum_{i=1}^n u_{ik} x_i$ and $b_k = \sum_{j=1}^n v_{kj} y_j$. Note that $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are both linearly independent sets and $\sum_{k=1}^n d_{kk} a_k m b_k = 0$ for all $m \in \mathcal{A}$. Hence Theorem 4.1 of [4] implies that $C = 0$.

Any completely positive elementary operator Φ has a representation of the form $\Phi(a) = \sum_{i=1}^n x_i^* a x_i$ for some linearly independent set of elements $\{x_i\}_{i=1}^n$ ([4],

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Theorem 4.10). Since any conditional expectation is completely positive, we will only consider elementary operators of this form from here on.

Corollary 1.2. *Let $\mathcal{A}$ be a prime C^*-algebra. Let V be a finite-dimensional subspace of $\mathcal{A}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $\{x_i\}_{i=1}^n$ be an orthonormal basis of V and let $\{y_k\}_{k=1}^n$ be any n-tuple of elements in V. Then $\sum_{i=1}^n x_i^* a x_i = \sum_{k=1}^n y_k^* a y_k$ for all $a \in A$ if and only if $\{y_k\}_{k=1}^n$ is also an orthonormal basis of V.*

Proof. Suppose $\sum_{i=1}^n x_i^* a x_i = \sum_{k=1}^n y_k^* a y_k$. We can write $y_k = \sum_{i=1}^n c_{ik} x_i$ where $c_{ik} = \langle y_k, x_i \rangle$. Let C denote the n by n matrix with the $(i,j)^{\text{th}}$ entry equal to c_{ij} for all i, j. Then $\sum_{i=1}^n x_i^* a x_i = \sum_{k=1}^n y_k^* a y_k = \sum_{i,j=1}^n [CC^*]_{ij} x_j^* a x_i$ ($[CC^*]_{ij}$ is the $(i,j)^{\text{th}}$ entry of the matrix CC^*) and hence $\sum_{i,j=1}^n [CC^* - I]_{ij} x_j^* a x_i = 0$. C is therefore unitary by the previous lemma and $\{y_k\}_{k=1}^n$ is an orthonormal basis. The converse may be proved by reversing our steps.

2. Conditional expectations

We begin by recalling the concept of a conditional expectation onto a subalgebra first introduced in [5].

Definition 2.1. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{B} \subseteq \mathcal{A}$ be a C^*-subalgebra. Then we call $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ if it satisfies the following three properties:

(1) $E(b) = b$, $\forall b \in \mathcal{B}$,
(2) $E(b_1 a b_2) = b_1 E(a) b_2 \quad \forall a \in \mathcal{A}, \forall b_1, b_2 \in \mathcal{B}$,
(3) $a \geq 0 \implies E(a) \geq 0$ for all $a \in \mathcal{A}$.

We say that E is a faithful conditional expectation if, in addition to the three conditions above, E satsifies the following:

(4) If $E(a^* a) \neq 0$ for all non-zero $a \in \mathcal{A}$.

We say that E is an elementary conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$, if E is an elementary operator on $\mathcal{A}$ which is also a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$.

Conditional expectations may not be unique. If $\mathcal{A}$ has a faithful tracial state τ, there is a unique conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ satisfying $\tau(E(a)) = \tau(a)$ for all $a \in \mathcal{A}$ which is called the τ-preserving conditional expectation. If we consider $\mathcal{A}$ to be a Hilbert space with the inner product $\langle x, y \rangle = \tau(y^* x)$, the τ-preserving conditional expectation is the orthogonal projection on the subspace $\mathcal{B}$.

3. Finite-dimensional C^*-algebras

Let $\mathcal{M}$ be a finite-dimensional C^*-algebra (the identity element will be denoted $1_{\mathcal{M}}$ or just 1 if it is clear which C^*-algebra it belongs to). It is well known that every finite-dimensional C^*-algebra is isomorphic to a direct sum of matrix algebras; hence $\mathcal{M} \cong \bigoplus_{k=1}^q M_{n_k}(\mathbb{C})$ for some finite set of natural numbers $\{n_k\}_{k=1}^q$. We can form a basis of $\mathcal{A}$ consisting of matrix units $\{\{e_{ij}^{(k)}\}_{i,j=1}^{n_k}\}_{k=1}^q$ satisfying $e_{ij}^{(k)} e_{rs}^{(t)} = \delta_{jr} \delta_{kt} e_{is}^{(t)}$ and $e_{ij}^{(k)*} = e_{ji}^{(k)}$. We note that there are q minimal central projections of $\mathcal{M}$, and they are of the form $c_k = \sum_{i=1}^{n_k} e_{ii}^{(k)}$. The span of the minimal central projections is $Z(\mathcal{M})$, the center of $\mathcal{M}$. We note that the minimal central projections are independent of the choice of matrix units. We will define the left-regular trace τ on $\mathcal{M}$ as follows: $\tau(c_k) = n_k^2$ (hence $\tau(1) = dim(\mathcal{M})$ and $\tau(e_{ij}^{(k)}) = \delta_{ij} n_k$ for any set of matrix units). The terminology here comes from the fact that this τ can be viewed as the trace (sum of the diagonal matrix entries) of the left regular matrix representation of $\mathcal{M}$ (see pg. 424 of [1] for more details), although we will not use this fact in what follows. We end this section with the following useful elementary result.

Lemma 3.1. *Let $\mathcal{M}$ be a finite-dimensional C^*-algebra and let I be a proper right ideal of $\mathcal{M}$. Then there exists a non-zero $m \in \mathcal{M}$ such that $mI = 0$.*

4. Faithful elementary conditional expectations

We recall that if $\mathcal{B}$ is a C^*-subalgebra of a C^*-algebra $\mathcal{A}$, $B' \cap \mathcal{A} = \{a \in \mathcal{A} | ab = ba, \forall b \in \mathcal{B}\}$ and is called the relative commutant of $\mathcal{B}$ in $\mathcal{A}$.

Lemma 4.1. *Let $\mathcal{A}$ be a unital prime C^*-algebra and let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{A}} \in \mathcal{B}$. Suppose there exists E, a faithful elementary conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ and let $\{x_i\}_{i=1}^n$ be a linearly independent set of elements in $\mathcal{A}$ such that $E(a) = \sum_{i=1}^n x_i^* a x_i$. Then $\mathcal{B}' \cap \mathcal{A} = span(\{x_i\}_{i=1}^n)$. (In particular $\mathcal{B}' \cap \mathcal{A}$ must be finite dimensional).*

Proof. First note that $E(ab) = E(a)b$ for all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$ and hence $\sum_{i=1}^n x_i^* a(x_i b - b x_i) = 0$. Therefore, by Theorem 4.1 of [4], $x_i \in \mathcal{B}' \cap \mathcal{A}$ for all i. Now let u be a unitary element in $\mathcal{B}' \cap \mathcal{A}$. Then $uE(a) = E(a)u$ for all $x \in \mathcal{A}$. Hence $\sum_{i=1}^n u x_i^* a x_i = \sum_{i=1}^n x_i^* a x_i u$ and by Corollary 4.7 of [4], $span(\{x_i u\}_{i=1}^n) = span(\{x_i\}_{i=1}^n)$. Since every element in a unital C^*-algebra can be expressed as a linear combination of four unitaries, $span(\{x_i\}_{i=1}^n)$ is a right ideal in $\mathcal{B}' \cap \mathcal{A}$. Similarly, $span(\{x_i^*\}_{i=1}^n)$ is a left ideal and hence $span(\{x_i^* x_j\}_{i,j=1}^n)$ is a two-sided ideal which contains the identity and hence is equal to $\mathcal{B}' \cap \mathcal{A}$. In particular, $\mathcal{B}' \cap \mathcal{A}$ is a finite-dimensional C^*-algebra. Hence, if $span(\{x_i\}_{i=1}^n)$ is a proper right ideal of $\mathcal{B}' \cap \mathcal{A}$, then there exists a non-zero $m \in \mathcal{B}' \cap \mathcal{A}$ such that $m x_i = 0$ for all i. But then $E_{\mathcal{B}}(m^* m) = 0$ which contradicts faithfulness. So $\mathcal{B}' \cap \mathcal{A} = span(\{x_i\}_{i=1}^n)$.

Theorem 4.2. *Let $\mathcal{A}$ be a unital prime C^*-algebra and let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{A}} \in \mathcal{B}$. Then there exists E, a faithful elementary conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ if and only if $\mathcal{B}$ is the relative commutant of a finite-dimensional C^*-subalgebra of $\mathcal{A}$ containing $1_{\mathcal{A}}$.*

Proof. Let $E(a) = \sum_{i=1}^n x_i^* a x_i$ be a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. Let $\mathcal{M} = \mathcal{B}' \cap \mathcal{A}$; $\mathcal{M}$ is finite dimensional by the previous lemma and $\mathcal{B} \subseteq \mathcal{M}' \cap \mathcal{A}$. Then $\sum_{i=1}^n x_i^* x_i = 1$ and $E(a) = a$ for all $a \in \mathcal{M}' \cap \mathcal{A}$. Hence $\mathcal{B} = \mathcal{M}' \cap \mathcal{A}$.

Now suppose $\mathcal{B} = \mathcal{M}' \cap \mathcal{A}$, where $\mathcal{M} \cong \bigoplus_{k=1}^q M_{n_k}(\mathbb{C})$. Now consider the elementary operator $E_0(a) = \sum_{k=1}^q \frac{1}{n_k} \sum_{i,j=1}^{n_k} e_{ji}^{(k)} a e_{ij}^{(k)}$, where $\{\{e_{ij}^{(k)}\}_{i,j=1}^{n_k}\}_{k=1}^q$ are a set of matrix units for $\mathcal{M}$. Note that $e_{rs}^{(t)} E_0(a) = \frac{1}{n_t} \sum_{i,j=1}^{n_t} e_{ri}^{(t)} a e_{is}^{(t)} = E_0(a) e_{rs}^{(t)}$ so $E_0(a) \in \mathcal{B}$. We also have $E_0(1) = \sum_{k=1}^q \sum_{j=1}^{n_k} e_{jj}^{(k)} = 1$. Now it can easily be verified that E_0 satisfies all of the conditions of a faithful conditional expectation. If $\mathcal{A}$ has any tracial states, it is easy to see that E_0 is the unique trace-preserving conditional expectation for any trace on $\mathcal{A}$.

We note that the "if" direction of the above result remains true even if $\mathcal{A}$ is not prime. We can show that E_0 is independent of the choice of matrix units by noting that any set of normalized matrix units $\{\{n_k^{-\frac{1}{2}} e_{ij}^{(k)}\}_{i,j=1}^{n_k}\}_{k=1}^q$ is an orthonormal basis of $\mathcal{M}$ with respect to the inner product $\langle x, y \rangle = \tau(y^* x)$, where τ is the left regular trace of $\mathcal{M}$ and then by using Corollary 1.2. We will call $E_0(a) = \sum_{k=1}^{q} \frac{1}{n_k} \sum_{i,j=1}^{n_k} e_{ji}^{(k)} a e_{ij}^{(k)}$ the minimal conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ (in the next section, we will explain the reason for calling E_0 minimal). We can now characterize all faithful elementary conditional expectations in terms of E_0.

Theorem 4.3. *Let $\mathcal{A}$ be a unital prime C^*-algebra, let $\mathcal{M}$ be a finite-dimensional C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{A}} \in \mathcal{M}$, $\mathcal{B} = \mathcal{M}' \cap \mathcal{A}$ and let E_0 be the minimal conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. Let $E(x) = \sum_{i=1}^{n} x_i^* a x_i$ where the $\{x_i\}_{i=1}^{n}$ is a basis of $\mathcal{M}$. Then E is a faithful conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ if and only if $E(x) = E_0(rxr)$ for some positive invertible $r \in \mathcal{M}$ such that $E_0(r^2) = 1$.*

Proof. We note that the "if" direction can easily be seen to be true; we will only prove the "only if" direction. Suppose E is a faithful conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ and let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathcal{M}$ defined by $\langle x_i, x_j \rangle = \delta_{ij}$. We note that $x_i m = \sum_{j=1}^{n} \langle x_i m, x_j \rangle x_j$ and $m x_j^* = (x_j m^*)^* = \sum_{i=1}^{n} \langle x_i, x_j m^* \rangle x_i^*$. Since E maps $\mathcal{A}$ onto $\mathcal{B}$, $\sum_{i,j=1}^{n} \langle x_i, x_j m^* \rangle x_i^* a x_j = \sum_{j=1}^{n} m x_j^* a x_j = \sum_{i=1}^{n} x_i^* a x_i m = \sum_{i,j=1}^{n} \langle x_i m, x_j \rangle x_i^* a x_j$ for all $m \in \mathcal{M}$. Hence the inner product has the property that $\langle xm, y \rangle = \langle x, ym^* \rangle$ for all $x, y, m \in \mathcal{M}$ and there exists an positive invertible $r \in \mathcal{M}$ such that $\langle x, y \rangle = \tau(y^* r^{-2} x)$, where τ is the left regular trace on $\mathcal{M}$. Therefore the set $\{\{\frac{1}{n_k} r e_{ij}^{(k)}\}_{i,j=1}^{n_k}\}_{k=1}^{q}$ is an orthonormal basis with respect to our inner product and $E(x) = E_0(rxr)$ by Corollary 1.2 and $E_0(r^2) = E(1) = 1$.

5. Index finiteness

Definition 5.1. Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ and let E be a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. A finite set of pairs of elements in $\mathcal{A}$, $\{(u_i, v_i)\}_{i=1}^{n}$ is called a quasi-basis for E if $\sum_{i=1}^{n} u_i E(v_i a) = a = \sum_{i=1}^{n} E(a u_i) v_i$ for any $a \in \mathcal{A}$. A conditional expectation E is said to be of index-finite type if there exists a quasi-basis for E.

We note that the quasi-basis may not be unique. An index-finite E always admits a quasi-basis of the form $\{(u_i, u_i^*)\}_{i=1}^{n}$ ([6], Lemma 2.1.6). In the special case where $\mathcal{B}' \cap \mathcal{A}$ is finite dimensional, if one conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is of index-finite type, then every faithful conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is of index-finite type ([6], Proposition 2.10.2). In the following theorem, $\mathbb{C}$ is the C^*-subalgebra of $\mathcal{A}$ consisting of all scalar multiples of the identity element.

Theorem 5.2. *Let $\mathcal{A}$ be a unital prime C^*-algebra, let $\mathcal{M}$ be a finite-dimensional C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{A}} \in \mathcal{M}$, and let $\mathcal{B} = \mathcal{M}' \cap \mathcal{A}$. Further, let τ denote the left regular trace of $\mathcal{M}$. Then the following are equivalent:*

(1) *Any faithful conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is of index-finite type.*

(2) *There exists F, an elementary operator on $\mathcal{A}$, such that $F|_{\mathcal{M}}$ is the τ-preserving faithful conditional expectation of $\mathcal{M}$ onto $\mathbb{C}$.*

(3) *There exists G, an elementary operator on $\mathcal{A}$, such that $G|_{Z(\mathcal{B})}$ is the τ-preserving faithful conditional expectation of $Z(\mathcal{B})$ onto $\mathbb{C}$.*

(4) *No proper two-sided ideal of $\mathcal{A}$ contains a minimal central projection of $\mathcal{B}$.*

If any of these four equivalent conditions hold, then any conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is an elementary conditional expectation.

Proof. (1) $\implies$ (2) Let $\{x_j\}_{j=1}^{m}$ be an orthonormal basis of $\mathcal{M}$ with respect to the inner product $\langle x, y \rangle = \tau(y^* x)$ with $x_1 = (\tau(1))^{-\frac{1}{2}} 1$. Now $E_0(a) = \sum_{j=1}^{m} x_j^* a x_j$ is the minimal conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. Let $\{(u_i, v_i)\}_{i=1}^n$ be a quasi-basis for E_0 and let $F(a) = \frac{1}{\tau(1)} \sum_{i=1}^n u_i a v_i$. Then

$$a = \sum_{i=1}^n u_i E(v_i a) = \sum_{j=1}^m \sum_{i=1}^n \tau(1) F(x_j) a x_j.$$

So $F(x_i) = 0$ for $i = 2, 3, ..., n$ and $F(1) = 1$. Hence $F|_{\mathcal{M}}$ is the τ-preserving faithful conditional expectation of $\mathcal{M}$ onto $\mathbb{C}$.

(2) $\implies$ (1) Let $F(x) = \sum_{i=1}^n u_i a v_i$ where $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ are both linearly independent sets. We can show that $\{(u_i, v_i)\}_{i=1}^n$ is a quasi-basis for E_0 by reversing the steps of our previous argument.

(2) $\implies$ (3) Simply take $G = F$.

(3) $\implies$ (2) Let E_0 be the minimal conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. Note that $E_0|_{\mathcal{M}}$ is a τ-preserving conditional expectation of $\mathcal{M}$ onto $Z(\mathcal{B})$. Hence let $F(a) = G(E_0(a))$ for all $a \in \mathcal{A}$.

(3) $\implies$ (4) G maps any minimal central projection of $\mathcal{B}$ into a multiple of the identity. Since any elementary operator will map a two-sided ideal into itself, no proper two-sided ideal of $\mathcal{A}$ contains a minimal central projection of $\mathcal{B}$.

(4) $\implies$ (3) If any ideal of $\mathcal{A}$ contains a non-zero element of $Z(\mathcal{B})$, then it also contains a minimal central projection of $\mathcal{B}$. So any non-zero linear combination of the minimal central projections cannot be in any proper two-sided ideal of $\mathcal{A}$. The existence of G now follows from Proposition 1.1 of [3].

The last sentence in the statement of the theorem is an immediate consequence of Proposition 2.10.9 of [6].

We note that the fourth of the equivalent conditions is satisfied in certain important cases such as when $\mathcal{A}$ is simple or when $Z(\mathcal{B}) = \mathbb{C}$.

Watatani has defined the index of a conditional expectation E having a quasi-basis $\{(u_i, v_i)\}_{i=1}^n$ as follows: $ind(E) = \sum_{i=1}^n u_i v_i$ and has shown that $ind(E)$ is independent of the choice of quasi-basis and is in $Z(\mathcal{A})$ [6] (in our case, $\mathcal{A}$ is prime; so $Z(\mathcal{A}) = \mathbb{C}$).

Theorem 5.3. *Let $\mathcal{A}$ be a unital prime C^*-algebra, let $\mathcal{M}$ be a finite-dimensional C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{A}} \in \mathcal{M}$, and let $\mathcal{B} = \mathcal{M}' \cap \mathcal{A}$. Let E be a faithful conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. If E is index-finite, then $dim(\mathcal{M}) \leq ind(E)$ with equality if and only if E is the minimal conditional expectation.*

Proof. Let E_0 be the minimal conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ and let $\{(u_i, v_i)\}_{i=1}^n$ be a quasi-basis for E_0. By Theorem 4.3, $E(a) = E_0(rar)$ for some positive invertible $r \in \mathcal{M}$ such that $E_0(r^2) = 1$. Then $\{(u_i r^{-1}, r^{-1} v_i)\}_{i=1}^n$ is a quasi-basis for E. Using an argument in the first paragraph of the proof of Theorem 5.2, $ind(E) = \sum_{i=1}^n u_i r^{-2} v_i = \tau(r^{-2})$, where τ is the left regular trace on $\mathcal{M}$. Now $\tau(1) dim(\mathcal{M}) = (\tau(1))^2 \leq \tau(r^2)\tau(r^{-2}) = \tau(E_0(r^2))\tau(r^{-2}) = \tau(1)\tau(r^{-2}) = \tau(1) ind(E)$. The inequality step follows from the Cauchy-Schwarz inequality, and hence equality holds only when $r = 1$ and $E = E_0$.

E_0 minimizes the index and is therefore a minimal conditional expectation in the usual sense (as in section 2.12 of [6]), thus justifying our use of the terminology. If $\mathcal{A}$ has a tracial state, the conditional expectation which minimizes the index is also the one which preserves all the tracial states of $\mathcal{A}$. We also note that our inequality seems to be a Watatani index analogue of an inequality for the Jones index described in Jones' original paper (in the Remark after Corollary 2.2.3 in [2]).

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