A NOTE ON THE STABILITY OF LOCAL ZETA FUNCTIONS

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ABSTRACT. We show the existence of an interval of stability under small perturbations of local zeta functions corresponding to non-trivial local solutions of an elliptic equation with Lipschitz coefficients.

RÉSUMÉ. Nous démontrons l'existence d'un intervalle de stabilité pour la fonction zêta associée à une équation uniformément elliptique du second ordre à coefficients lipschitziens.

1. Introduction and statement of the results

Let u be a function defined and continuous in the ball B_{2R} of radius $2R$ centered at the origin of $\mathbb{R}^N$. For $0 < 2r < R$ and $\lambda \in \mathbb{C}$ consider the integral

$$(1.1) \qquad J_u(\lambda) \; = \; \int_{B_r} |u(x)|^{-\lambda}\, dV,$$

where dV is the Lebesgue measure on $\mathbb{R}^N$. For a fixed u the above integral converges absolutely when the real part $\Re(\lambda)$ of λ is negative and defines a holomorphic function of λ in this region. A natural question is to find spaces of functions for which $J_u(\lambda)$ has a holomorphic extension to the right of the imaginary axis for any u in the considered space. A second question is to ask if the extension is uniform with respect to "small" perturbations of u. In other words we are asking for classes of functions for which

$$(1.2) \qquad t_o(u) \; = \; \sup\, \{t = \Re(\lambda) \,|\, J_u(\lambda) < \infty\}$$

is strictly positive and furthermore for any $t < t_o(u)$ there exists $\epsilon > 0$, such that, for every function v in the considered class of functions and satisfying

$$\sup_{x \in B_{2R}} |u(x) - v(x)| \; < \; \epsilon,$$

we have $t_o(v) \geq t_o(u)$.

If we consider the class of functions that are analytic near the origin, it is a classical result that $J_u(\lambda)$ can be extended to the whole complex plane as a meromorphic function with poles on the positive real axis; see [BG], [A], [Be], [Ka], [M] and [Bj]. Having this we are led to study the so-called multiplicity, i.e., the value of the smallest pole. It is clear that the stability to which we referred in the last paragraph is a question about the multiplicity. It is important to have in mind Varchenko's example [V] according to which if $N \geq 3$ we do not have stability in

the class of real analytic functions. For sharp results in the setting or real analytic functions on $\mathbb{R}^N$ or holomorphic functions on $\mathbb{C}^N$ we refer the reader to [DK], [N], [PSS], [PS1] and [PS2].

Now consider the space of all local solutions of a fixed uniformly elliptic equation with Lipschitz continuous coefficients a_{ij},

$$(1.3) \qquad \mathcal{L}u \;=\; \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = 0.$$

The main result of this note is the following theorem.

Theorem 1.1. *Let L be a uniformly elliptic operator with Lipschitz coefficients as in (1.3) and u a non-trivial solution of $\mathcal{L}u = 0$ on B_{2R}. There exist positive numbers R_o and t_u, such that, for any $t < t_u$ there exists an $\epsilon > 0$ with the following property: for every solution v of $\mathcal{L}v = 0$ on B_{2R} satisfying*

$$\sup_{x \in B_{2R_o}} |u(x) - v(x)| \;<\; \epsilon,$$

we have $I_t(v) \overset{def}{=} \int_B |v(x)|^{-2t}\, dV < \infty$ for any ball B, $B \subset 2B \subset R_o$, i.e., the local integrability is an open condition for t's in the interval $t < t_u$.

As a consequence we can define the multiplicity for any solution u of the considered elliptic equation by setting

$$(1.4) \qquad t_o \;=\; \sup\,\{t \,|\, I_t(u) < \infty\}.$$

The above theorem shows that $t_o > 0$ and there is an interval around $t = 0$ on which we have stability under small perturbations. Our proof yields a precise lower bound on t_u; see (2.16). The very interesting questions of determining the value of t_o or showing stability on the whole interval $t < t_o$ are not addressed in this note.

We shall give the proof of Theorem 1.1 at the end of Section 2. In Section 3 we make a comment on the stability in the class of holomorphic functions.

2. A_p WEIGHTS AND THE STABILITY FOR LOCAL ZETA FUNCTIONS

The key point in the proof of Theorem 1.1 is that every solution of an elliptic equation is an A_p weight for a suitable $p > 0$ (cf. [GL1], [GL2] and see also [GG]) and furthermore the A_p norm is controlled by the so-called frequency function. It is well known that a reverse Hölder inequality implies the A_p property for some p; [CF] and [St2]. In our case a reverse Hölder inequality will hold for the function u^2 and all $s > 1$ (see Theorem 2.4) and our task is to write down the consequences of this fact. For ease of exposition and in order to keep track of the dependence on the various parameters we shall state all the results that lead to the A_p property of solutions to elliptic equations in divergence form with Lipschitz coefficients. It is worth comparing our result with the case of polynomial functions, namely for any polynomial P on $\mathbb{R}^N$ the function $\log|P|$ belongs to $\mathrm{BMO}(\mathbb{R}^N)$ (cf. [St1]), hence the finiteness of the integral $I_t(P)$ for some $t > 0$. Moreover (see [RSt]) $|P|$ satisfies a reverse Hölder inequality and $|P|^a$ is an A_p weight if $-1 < ad < p - 1$, $p > 0$, where d is the degree of the polynomial P. However the A_p norm depends on the degree d while in our case we have control of the A_p norm by the frequency of the function u.

Turning to our problem, let $\mathcal{L}$ be a second-order elliptic operator of the type

$$(2.1) \qquad \mathcal{L}u = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right).$$

We shall assume $A = \left(a_{ij}(x)\right), i, j = 1, ..., N$, is a Lipschitz, symmetric and uniformly elliptic matrix. Thus $a_{ij} = a_{ji}$ and there exist $\lambda, \Lambda > 0$ such that for all $x, y, \eta \in \mathbb{R}^N$ and $i, j = 1, ..., N$ we have

$$(2.2) \qquad \lambda|\eta|^2 \ \leq \ <A(x)\eta, \eta> \ \leq \ \lambda^{-1}|\eta|^2 \qquad |a_{ij}(x) - a_{ij}(y)| < \Lambda|x - y|.$$

Let $B_R(x)$ denote the ball with center x and radius R, and let ρ be the distance to its center. Suppose u is a weak solution of $\mathcal{L}u = 0$ in some ball $B_R(x)$. For $r < R$ we define, correspondingly, the height $H_u(r)$ and the Dirichlet (energy) integral $D_u(r)$ of u on the ball $B_r(x)$ by the following formulas:

$$H_{x,u}(r) \ = \ \int_{\partial B_r(x)} u^2\langle A\nabla\rho, \nabla\rho\rangle d\sigma,$$

$$D_{x,u}(r) \ = \ \int_{B_r(x)} \langle A\nabla u, \nabla u\rangle dV.$$

We shall use the notation $H(r)$ and $D(r)$ when there is no ambiguity about the function and point we are using. The following lemma provides the key properties of the height and energy associated to a fixed solution u. We refer the reader to [GL1], [K] or [GV] for the proof.

Lemma 2.1. *Given a linear uniformly elliptic operator $\mathcal{L}$ as above, there exists a constant $M = M(\lambda, \Lambda, N)$ such that for every solution $\mathcal{L}u = 0$ in $B_R(x)$ the height and energy functions of u satisfy the following inequalities for a.e. $r \in (0, R)$:*

$$\left|H^{'}(r) - \frac{N-1}{r}H(r) - 2D(r)\right| \leq M\,H(r)$$

and

$$D^{'}(r) \geq 2\int_{\partial B_r(x)} \frac{\langle A\nabla u, \nabla\rho\rangle^2}{\langle A\nabla\rho, \ \nabla\rho\rangle}\,dV + \frac{N-2}{r}D(r) - MD(r).$$

Note that the inequality for the height function shows that $H'(r) > 0$ for $0 < r < \frac{N-1}{M}$ when $H(r)$ is defined. It will be convenient then to define

$$(2.3) \qquad R_o = \min\{\frac{R}{8}, \frac{N-1}{M}\},$$

and hence $H'(r) > 0$ for $0 < r \leq R_o$. Therefore, if u is a non-trivial solution (i.e. $u \not\equiv 0$), then $H(r) \neq 0$ on $(0, R_o]$. We remark that if $\mathcal{L}$ is the laplacian, we have $M = 0$ and the height is an increasing function of r wherever it is defined.

From now on we shall consider only non-trivial solutions. For every ball $B_r(x)$ contained in the domain on which we are given a non-trivial solution, u we define the frequency function by

$$F_{x,u}(r) \ \stackrel{def}{=} \ \frac{rD_{x,u}(r)}{H_{x,u}(r)}\,e^{2Mr}.$$

Lemma 2.1 and the above observations imply the following monotonicity theorem.

Theorem 2.2 ([GL1]). *Suppose u is a non-trivial solution of $\mathcal{L}u = 0$ in $B_R(x)$. Under these conditions the frequency $F_{x,u}(r)$ is a continuous monotone non-decreasing function of $r \in (0, R_o]$, with $R_o = \min\{\frac{R}{8}, \frac{N-1}{M}\}$.*

With the help of the monotonicity one proves that the measure $u^2\, dV$ is a doubling measure.

Theorem 2.3 ([GL1]). *Suppose u is a weak non-trivial solution of (2.1) in $B_R(x)$. With*

$$(2.4) \qquad \tilde{C}_o \;=\; \lambda^2 2^N e^M 2^{2F_{x,u}(R_o)},$$

we have

$$\int_{B_{2r}(x)} u^2\, dV \;\leq\; \tilde{C}_o \int_{B_r(x)} u^2\, dV$$

for all $0 < 2r \leq R_o$.

The doubling property implies the following reverse Hölder inequalities.

Theorem 2.4 ([GL1]). *Suppose u is a weak non-trivial solution of (2.1) in $B_R(x)$ and let $w = u^2$. There exists a positive constant $\mu = \mu(\lambda, \Lambda, N)$ such that with*

$$\tilde{C}_1 \;=\; \frac{\mu \tilde{C}_o}{2^N},$$

we have for all $s > 1$ and $0 < 2r \leq R_o$,

$$(2.5) \qquad \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} w^s\, dV \right)^{1/s} \leq \; \tilde{C}_1\, \frac{1}{|B_r(x)|} \int_{B_r(x)} w\, dV.$$

The constant μ is the constant that appears in the classical Moser estimate bounding the maximum of u^2 on a ball by the average of u^2 on a twice as big ball. In the case of laplacian $\mu = 2^N$ can be seen by the mean value property of harmonic functions.

Remark 2.5. A similar result holds for $w = |\nabla u|^2$ for some $s > 1$.

Let us denote by B_{R_o} the ball centered at the origin with radius R_o, i.e., $B_{R_o} = B_{R_o}(0)$. Given a non-trivial solution u on some ball B_{2R}, let F_u be the constant defined by

$$(2.6) \qquad F_u \;=\; \max\{F_{x,u}(R_o) \mid x \in B_{R_o}\},$$

where R_o was defined in (2.3). By the monotonicity property F_u bounds the frequencies on all balls B such that $2B \subset B_{R_o}$, where for a ball B we denote by $2B$ the ball with the same center as B and twice its radius. It is clear that if u is a non-trivial solution we have that the doubling property of Theorem 2.3 and the reverse Hölder inequality of Theorem 2.4 hold for all balls $B \subset 2B \subset B_{R_o}$ with the same constants C_o and C_1, replacing correspondingly $\tilde{C}_o$ and $\tilde{C}_1$, and defined respectively by

$$(2.7) \qquad C_o \;=\; \lambda^2 2^N e^M 2^{2F_u}$$

and

$$(2.8) \qquad C_1 \;=\; \lambda^2 \mu e^M 2^{2F_u}.$$

Let us also observe that since we have the reverse Hölder inequality for all $s > 1$ and C_1 is independent of s, we can let $s \to \infty$ and obtain the maximum of w on

the ball. Alternatively, we can directly use Theorem 2.3 and Moser's estimate. In either case, the conclusion is the following theorem.

Theorem 2.6. *Suppose u is a weak non-trivial solution of* (2.1) *in B_{2R}, for some $R > 0$. Let $M = M(\lambda, \Lambda, N)$ be the constant from Lemma 2.1, $R_o = \min\{\frac{R}{8}, \frac{N-1}{M}\}$ and $w = u^2$. For every ball $B \subset 2B \subset B_{R_o}$ the following inequality holds:*

$$(2.9) \qquad \sup_B w \ \le \ C_1 \frac{1}{|B|} \int_B w \, dV,$$

where C_1 is defined in (2.8).

It is well known that a reverse Hölder inequality for some $s > 1$ implies that $w = u^2$ is an A_p weight for a suitable p depending on s. The Muckenhoupt weight class $A_p(B_{R_o})$ consists of all non-negative measurable functions w for which

$$\|w\|_{A_p} \ \overset{def}{=} \ \sup \left[\frac{1}{|B|} \int_B w(x) \, dV \right] \left[\frac{1}{|B|} \int_B w(x)^{1-p'} \, dV \right]^{p-1} < \infty,$$

where the sup is over all balls B with $2B \subset B_{R_o}$.

In our case we have (2.9), so we are going to look for an estimate on the possible p's. Let us define

$$(2.10) \qquad p_o \ = \ \inf\{p > 1 \, | \, w \in A_p\}.$$

In the following theorem we show that

$$p_o \ \le \ \frac{1 + \frac{1}{m} \log_2 \sigma_N}{\frac{1}{m} \log_2 \sigma_N},$$

where σ_N denotes the area of the unit sphere in $\mathbb{R}^N$. The number m, related to the doubling constant for dyadic cubes, is defined by

$$(2.11) \qquad 2^m \ = \ C_o^2,$$

where C_o is defined in (2.7). As is customary we denote $w(Q) = \int_Q w \, dV$ and for $s \ge 1$ we define s' to be the conjugate exponent $\frac{1}{s} + \frac{1}{s'} = 1$.

Theorem 2.7. *If u is a non-trivial solution of $Lu = 0$ in B_{2R}, then $u^2 \in A_p(B_{R_o})$ for every $p = \bar{s}'$ satisfying*

$$\bar{s} - 1 \ < \ \frac{1}{m} \log_2 \sigma_N,$$

where m is the doubling exponent from (2.11) *and σ_N is the area of the unit sphere in $\mathbb{R}^N$.*

The fact that w is an A_p weight for some p was proven in Theorem 1.1 in [GL1]. Our task here is to prove the explicit bound on p, which gives a precise version of the result in [GL1]. In the proof we will use the following lemma which is a simple corollary from Theorem 2.6.

Lemma 2.8. *Suppose $w \in L^1_{loc}(\Omega)$, Ω a domain, is a non-negative function such that* (2.9) *holds true for every ball $B \subset 2B \subset \Omega$.*
   a) *For every $0 \ < \ \gamma \ < \ 1$ and δ given by*

$$\delta \ = \ 1 \ - \ \left(\frac{1-\gamma}{C_1} \right) \qquad (\ \Rightarrow \ \ 0 \ < \ \delta \ < \ 1 \ ),$$

we have the following property: whenever $E \subset B$ and $w(E) \leq \gamma w(B)$ it must be

$$|E| \leq \delta|B|.$$

b) *For every γ, $0 < \gamma < 1$, and δ given by*

$$\delta = \frac{1}{\sigma_N} - \frac{1}{\sigma_N}\left(\frac{1-\gamma}{C_1}\right) \qquad (\Rightarrow \quad 0 < \delta < 1),$$

we have the following property: whenever $E \subset Q$ and $w(E) \leq \gamma w(Q)$, it must be

$$|E| \leq \delta|Q|,$$

where Q is any cube such that $4Q \subset \Omega$.

We note that $4Q$ stays for the cube with the same center as Q and four times its size. With the help of the above lemma we can prove our main theorem.

Proof of Theorem 2.7. By the usual invariances it is enough to prove the claim for w being a measure on a unit dyadic cube Q_o satisfying

$$w(Q_o) = |Q_o| = 1.$$

Let m be such that $2^m = C_o^2$. By Theorem 2.3 it follows that for any dyadic cube Q and $\tilde{Q}$ a dyadic parent of Q,

(2.12) $$w(\tilde{Q}) \leq 2^m w(Q).$$

Define

$$f(x) = \frac{1}{w(x)}\chi_{Q_o}(x)$$

and the corresponding dyadic maximal function

$$Mf(x) = \sup_{x \in Q}\frac{1}{|Q|}\int_Q fw,$$

where the sup is taken over all dyadic cubes containing x. Now we have

$$\int_{Q_o} w^{1-\bar{s}}\,dV = \int_{Q_o} f^{\bar{s}-1}\,dV \leq \int_{Q_o}(Mf)^{\bar{s}-1}\,dV$$

$$\leq \int_{Q_o \cap \{Mf \leq 1\}}(Mf)^{\bar{s}-1}\,dV + \sum_{k=0}^{\infty}\int_{E^k \backslash E^{k+1}}(Mf)^{\bar{s}-1}\,dV,$$

where we define

$$E^k = \{x \in Q_o|\,Mf(x) > 2^{nk}\}$$

with n a fixed constant, to be chosen. The first integral in the above sum is bounded by one, so we have to estimate the integrals in the sum.

For this we use the Calderon-Zygmund decomposition, i.e., given $\alpha > 0$ we decompose the set $\{Mf > \alpha\}$ as a disjoint union of dyadic cubes $\cup Q_j$, such that,

(2.13) $$\alpha < \frac{1}{|Q_j|}\int_{Q_j} fw\,dV \leq 2^m\alpha.$$

Furthermore, if $\alpha_1 > \alpha_2$, each dyadic cube in the decomposition at level α_1 is contained in a dyadic cube in the decomposition at level α_2.

Going back to the estimate of each of the terms in the sum, let us decompose each of the sets E^k using, correspondingly, $\alpha = 2^{nk}$ as described above. Therefore we have

$$\int_{E^k \setminus E^{k+1}} (Mf)^{\bar{s}-1} \, dV \leq 2^{n(k+1)(\bar{s}-1)} \int_{E^k \setminus E^{k+1}} dV \ \leq \ 2^{n(k+1)(s-1)} |E^k|.$$

Note that

$$w(E^k \cap Q) \ \leq \ 2^{m-n} \, w(Q),$$

for every cube Q in the decomposition of E^{k-1}. Indeed, by the properties of the decomposition we have

$$w(E^k \cap Q) \ = \ \sum_{Q_j} w(Q_j),$$

where the sum is over all cubes Q_j making E^k and such that $Q_j \subset Q$. Using (2.13) we obtain

$$w(E^k \cap Q) \ = \ \sum_{Q_j} w(Q_j) \ \leq \ \frac{1}{2^{nk}} \int_{Q_j} fw \, dV \ \leq \ \frac{1}{2^{nk}} \int_Q fw \, dV$$

$$\leq \ \frac{1}{2^{nk}} 2^m 2^{n(k-1)} \, w(Q) \qquad (Q \text{ is of height } E^{k-1}!)$$

$$\leq \ 2^{m-n} \, w(Q).$$

At this point we note that for any $n > m$ the above inequality combined with Lemma 2.8 gives

$$|E^k| \leq \delta |Q|,$$

where

$$(2.14) \qquad\qquad \delta \ = \ \frac{1}{\sigma_N} \Big[1 \ - \ \Big(\frac{1 - 2^{m-n}}{C_1} \Big) \Big], \qquad n > m.$$

Since the cubes of E^{k-1} are contained in the cubes of E^k, we obtain by summing over the cubes of E^{k-1}

$$|E^k| \ \leq \ \delta |E^{k-1}| \qquad \Rightarrow \qquad |E^k| \ \leq \ \delta^k.$$

Putting the estimates together we prove

$$(2.15) \qquad \int_{Q_o} w^{1-\bar{s}} \ \leq \ 1 \ + \ \frac{1}{\delta} \sum_{k=0}^{\infty} \big(2^{n(\bar{s}-1)} \delta \big)^{k+1} \ = \ 1 \ + \ \frac{2^{n(\bar{s}-1)}}{1 - \delta 2^{n(\bar{s}-1)}} \ < \ \infty,$$

provided $2^{n(\bar{s}-1)} \delta \ < \ 1$, i.e., $\bar{s} - 1 \ < \ \frac{1}{n} \log_2 \delta^{-1}$. Let us consider the function $\log_2 \delta^{-1/n}$ on the interval (m, ∞), where $\delta = \delta(n)$ is the function in (2.14). Then clearly the integral (2.15) is finite if

$$\bar{s} \ - \ 1 \ < \ \sup_{n > m} \log_2 \delta^{-1/n} \ = \ \log_2 \sup_{n \geq m} \delta^{-1/n}.$$

Since $\delta(m) = \sigma_N^{-1}$ it follows that the integral in (2.15) is finite for any $\bar{s}$ such that

$$\bar{s} - 1 \ < \ \log_2 \delta^{-1/n} |_{n=m} \ = \ \frac{1}{m} \log_2 \sigma_N.$$

This concludes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let u be a fixed non-trivial solution of $\mathcal{L}u = 0$ in B_{2R}. Let R_o be the constant defined in Theorem 2.2. Define the stability multiplicity of u by

$$(2.16) \qquad t_u = \frac{1}{m} \log_2 \sigma_N.$$

From Theorem 2.7 for $t < t_u$ we have $I_t(u) < \infty$ on any ball B with $2B \subset B_{R_o}$, hence the first claim of our theorem.

To prove the stability result, it will be enough to show

$$(2.17) \qquad |F_u - F_v| = |\sup_{x \in B_{R_o}} F_{x,u}(R_o) - \sup_{x \in B_{R_o}} F_{x,v}(R_o)| \le C \sup_{B_{2R_o}} |u - v|.$$

Indeed, assuming the above inequality and taking into account that $m = 2 \log_2 C_o$ depends continuously on the quantity F_u (see (2.7) and (2.6)) it follows that there exists an $\epsilon > 0$, such that, if $\sup_{B_{2R_o}} |u - v| < \epsilon$ and $t < t_u$, then $t_u - t > |t_u - t_v|$, and hence $t < t_v$.

At this point we are left with proving (2.17) to which task we turn. Let $x \in B_{R_o}$ and let v be any solution of $\mathcal{L}v = 0$ satisfying

$$\sup_{B_{2R_o}} |u - v| < \frac{1}{2} \sup_{B_{2R_o}} |u|.$$

We have

$$|F_{x,u}(R_o) - F_{x,v}(R_o)| \le \frac{R_o\, e^{2MR_o}}{H_{x,u}(R_o) H_{x,v}(R_o)} \Big[D_{x,u}(R_o) |H_{x,u}(R_o) - H_{x,u}(R_o)|$$
$$+ H_{x,u}(r) |D_{x,u}(R_o) - D_{x,u}(R_o)| \Big].$$

For v as above and any $y \in B_{R_o}(x)$ the triangle inequality implies

$$|u^2(y) - v^2(y)| \le \frac{5}{2} \sup_{B_{2R_o}} |u|.$$

Hence the height of v is bounded by the height of u as follows:

$$|H_{x,u}(R_o) - H_{y,v}(R_o)| \le \frac{3}{2} \lambda N R_o^{N-1} \sigma_N \sup_{B_{2R_o}} |u| \sup_{B_{R_o}} |u - v|.$$

Furthermore, let us observe that $h_o = \inf_{x \in B_{R_o}} H_{x,u}(R_o)$ is a positive constant, $h_o > 0$, since u is a non-trivial solution. On the other hand, Caccioppoli's inequality yields for any $x \in B_{R_o}$ the inequality

$$|D_{x,u}(R_o) - D_{y,v}(R_o)| \le C \sup_{B_{2R_o}} |u - v|,$$

with a constant C depending on Λ, λ, R_o, N and $\sup_{B_{2R_o}} |u|$. Putting the above estimates together we come to

$$|F_{x,u}(R_o) - F_{x,v}(R_o)| \le C \sup_{B_{2R_o}} |u - v|,$$

with a constant $C = C\big(\lambda,\ \Lambda, N,\ \sup_{B_{2R_o}} |u|,\ \inf_{x \in B_{R_o}} \int_{\partial B_{R_o}(x)} u^2\, d\sigma\big)$, which concludes the proof of the theorem. $\square$

3. Local zeta functions for holomorphic functions

Let us now consider integrals of the type

$$(3.1) \qquad\qquad I_t(f) \;=\; \int_{B_r} |f(z)|^{-2t}\, dV,$$

where $B_R \subset \mathbb{C}^N$ is the ball of radius R centered at the origin and f is holomorphic in B_{2R}. Using the convexity of $u \mapsto u^{-t}$, $u \in \mathbb{R}$, and the fact that the real and imaginary parts of f are harmonic, we immediately obtain the following theorem.

Theorem 3.1. *Let f be a not identically vanishing holomorphic function on B_{2R_o} and let $0 < 2r < R_o$. There exists a $t(f) > 0$ such that for any $t < t(f)$ there exists an $\epsilon > 0$ such that for every g holomorphic in B_{2R_o} and satisfying*

$$\sup_{z \in B_{2r}} |f(z) - g(z)| \;<\; \epsilon,$$

we have $I_t(g) < \infty$.

As pointed out by the referee, the above theorem can also be proved by invoking the Weierstrass Preparation theorem with parameters, as formulated for example in [PSS].

It is an interesting question if Theorem 1.1 in the case of L being the laplacian is in fact identical to the above theorem. We recall the following fact (see Proposition 6.2.2 of [Av]) which provides a link between the two classes of functions.

Theorem 3.2. *Let h be function harmonic in the ball $B_R \subset \mathbb{R}^n$ and for $0 < r < R$ let $\tilde{h}$ be its holomorphic extension in the Lie ball $U_r \subset \mathbb{C}^N$. If*

$$m(r) \;=\; \max\{|h(x)| : |x| = r\}$$

and

$$M(r') \;=\; \max\{|\tilde{h}(z)| : |z| = r'\}, \qquad r' < \frac{r}{\sqrt{2}} < \frac{R}{\sqrt{2}},$$

then

$$M(r') \;\leq\; 3 \cdot 2^{\frac{N}{2}-1}\Big(1 - 2\frac{r'^2}{r^2}\Big)^{-\frac{N}{2}} m(r).$$

Recall that the Lie ball is defined by

$$U_r \;=\; \Big\{ z \in \mathbb{C}^N : \Big(|z|^2 \;+\; (|z|^4 - |\sum_{j=1}^{N} z_j^2|^2)^{1/2} \Big)^{1/2} < r \Big\}.$$

The above theorem can be extended to p-harmonic functions (see Theorem 6.1.1 [A]) or solutions of linear elliptic equations with real analytic coefficients ([DS] and [E]).

Theorem 3.2 alone is not enough to show equivalence of the holomorphic and harmonic cases since restriction of a locally integrable function to lower-dimensional hyper-planes can result in a locally non-integrable function. It seems worth investigating if restricting holomorphic functions to a fixed lower-dimensional surface gives rise to different stability results depending on the position of the surface.

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