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AF-ALGEBRAS AND THE TAIL-EQUIVALENCE RELATION ON BRATTELI DIAGRAMS

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ABSTRACT. We show that the C^*-algebra associated to the tail-equivalence relation on a Bratteli diagram, according to a procedure recently introduced by the first-named author and A. Lopes, is isomorphic to the AF-algebra of the diagram. More generally we consider an approximately proper equivalence relation $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ on a compact space X for which the quotient maps $\pi_n : X \to X/\mathcal{R}_n$ are local homeomorphisms. We show that the algebra associated to $\mathcal{R}$ under the above-mentioned procedure is isomorphic to the groupoid C^*-algebra $C^*(\mathcal{R})$.

1. INTRODUCTION

In [2] A. Lopes and the first-named author introduced the notion of approximately proper equivalence relations over a possibly non-commutative C^*-algebra and constructed an associated C^*-algebra, given a certain family of conditional expectations. In this paper we wish to compare that construction to some well-known constructions, namely the groupoid C^*-algebra for an approximately proper equivalence relation in the commutative context and, in particular, AF-algebras.

Let X be a compact topological space. An equivalence relation $\mathcal{R} \subseteq X \times X$ is said to be *proper* when the quotient space $X/\mathcal{R}$ is Hausdorff. $\mathcal{R}$ is said to be *approximately proper* when $\mathcal{R}$ is the union of an increasing family $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ of proper equivalence relations.

The main goal of this paper is to discuss a class of examples of approximately proper equivalence relations and to compute the associated C^*-algebras $C^*(\mathcal{R}, \mathcal{E})$ according to the prescription introduced in [2], where $\mathcal{E}$ is a suitable family of conditional expectations. Among the examples we consider the most concrete one is the tail-equivalence relation on a Bratteli diagram. The resulting algebra, not surprisingly, turns out to be the AF-algebra corresponding to the given diagram.

The key feature of this example is that the quotient maps

$$\pi_n : X \to X/\mathcal{R}_n$$

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are local homeomorphisms. This implies that each $\mathcal{R}_n$ is an open subset of $\mathcal{R}_{n+1}$ (Proposition 6.1) and hence, viewing $\mathcal{R}$ as a locally compact groupoid with the inductive limit topology, we have that $\mathcal{R}$ is r-discrete. We therefore consider general approximately proper equivalence relations for which the π_n are local homeomorphisms and then show that the groupoid C^*-algebra $C^*(\mathcal{R})$ is isomorphic (Theorem 6.2) to $C^*(\mathcal{R}, \mathcal{E})$.

If $\mathcal{R}$ is the tail-equivalence relation on a Bratteli diagram it is well known that the associated groupoid C^*-algebra coincides with the AF-algebra of the diagram. So the isomorphism between the latter and $C^*(\mathcal{R}, \mathcal{E})$ may be obtained as an application of Theorem 6.2. Nevertheless we feel that a direct proof of the isomorphism between $C^*(\mathcal{R}, \mathcal{E})$ and the AF-algebra shows some interesting features of the construction involved in a very concrete way.

2. Preliminaries

Let $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ be a directed graph, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. Recall (see [1]) that $\mathcal{D}$ is a Bratteli diagram if:

(a) one is given a decomposition
$$\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$$
(we adopt the convention according to which the set $\mathbb{N}$ of natural numbers starts with zero) of $\mathcal{V}$ as the union of pairwise disjoint, finite, non-empty sets $\mathcal{V}_n$,

(b) for every edge $\epsilon \in \mathcal{E}$, if the source $s(\epsilon)$ of ϵ lies in $\mathcal{V}_n$, then its range $r(\epsilon)$ lies in $\mathcal{V}_{n+1}$,

(c) for every n, the set of edges from $\mathcal{V}_n$ to $\mathcal{V}_{n+1}$ is finite.

We will moreover assume, for simplicity that:

(d) $\mathcal{V}_0$ is a singleton,

(e) every vertex is the source of some edge,

(f) every vertex is the range of some edge, except for the single vertex in $\mathcal{V}_0$.

Assumptions (e) and (f) above correspond to the fact that the embeddings between the finite-dimensional sub-algebras of the associated AF-algebra [1] are unital and injective.

For each $n \in \mathbb{N}$, let us denote by $\mathcal{E}_n$ the subset of $\mathcal{E}$ given by
$$\mathcal{E}_n = \{\epsilon \in \mathcal{E} \colon s(\epsilon) \in \mathcal{V}_n\}.$$
By (b) above one obviously has that $r(\epsilon) \in \mathcal{V}_{n+1}$ for every $\epsilon \in \mathcal{E}_n$. Therefore $\mathcal{E}_n$ is precisely the finite set referred to in (c).

By a *path* in $\mathcal{D}$ we will mean, as usual, a finite or infinite sequence $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ of edges such that the range of each edge α_n coincides with the source of the following edge in the sequence. For a finite path $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n)$ we will say that the *length* of α is the number of edges involved (as opposed to the number of vertices).

If a path α is such that $s(\alpha_0) \in \mathcal{V}_0$, then it clearly follows that $\alpha_n \in \mathcal{E}_n$ for every n. If moreover α is infinite, then it may be considered as an element of the Cartesian product
$$\prod_{n\in\mathbb{N}} \mathcal{E}_n.$$

Equipping each $\mathcal{E}_n$ with the discrete topology it becomes a compact space by (c) and hence the Cartesian product above is compact for the product topology. It is not hard to see that the subset

$$\Omega = \left\{ \alpha \in \prod_{n \in \mathbb{N}} \mathcal{E}_n \colon \alpha \text{ is a path} \right\}$$

is a closed subset and hence also compact.

The main object of interest in this work is the equivalence relation on Ω, sometimes called the *tail-equivalence relation*, defined by

$$\alpha \sim \beta$$

if and only if there exists $n \in \mathbb{N}$ such that $\alpha_k = \beta_k$, for all $k \geq n$.

Quite commonly one finds that the quotient topological space $\Omega / \sim$ is very badly behaved, often being chaotic (the only open sets being the empty set and the whole space). This is the case, e.g. for the Bratteli diagram of the CAR algebra, namely the diagram in which each $\mathcal{V}_n$ consists of a single vertex and each $\mathcal{E}_n$ consists of exactly two edges (necessarily both joining the vertex in $\mathcal{V}_n$ to the vertex in $\mathcal{V}_{n+1}$).

Returning to the general case consider, for each $n \in \mathbb{N}$, the equivalence relation on Ω defined by

$$\alpha \sim_n \beta$$

if and only if $\alpha_k = \beta_k$, for all $k \geq n$.

It is apparent that the equivalence class of each $\alpha \in \Omega$ under "$\sim_n$" is determined by the infinite sub-path $(\alpha_n, \alpha_{n+1}, \dots)$ and that the quotient space $\Omega / \sim_n$ is homeomorphic to the space of all infinite paths starting at some vertex in $\mathcal{V}_n$. So this quotient is a well-behaved Hausdorff space. In other words "$\sim_n$" is a proper equivalence relation.

Denote by

$$\mathcal{R} = \{(\alpha, \beta) \in \Omega \times \Omega \colon \alpha \sim \beta\} \text{ and } \mathcal{R}_n = \{(\alpha, \beta) \in \Omega \times \Omega \colon \alpha \sim_n \beta\}.$$

Observe that, according to the strictly technical definition of equivalence relations, $\mathcal{R}$ <u>is</u> the equivalence relation "$\sim$" and $\mathcal{R}_n$ <u>is</u> "$\sim_n$". From now on we will therefore refer to "$\sim$" and "$\sim_n$" as $\mathcal{R}$ and $\mathcal{R}_n$, respectively.

Notice that $\mathcal{R}$ is the increasing union of the $\mathcal{R}_n$ and hence $\mathcal{R}$ is an approximately proper equivalence relation.

Since the $\mathcal{R}_n$ are proper it is easy to see that the sub-C^*-algebra

$$C(\Omega; \mathcal{R}_n)$$

of $C(\Omega)$ formed by the functions which are constant on each $\mathcal{R}_n$-equivalence class is $*$-isomorphic to $C(\Omega/\mathcal{R}_n)$. Our next immediate goal will be to describe a certain conditional expectation from $C(\Omega)$ onto $C(\Omega; \mathcal{R}_n)$.

For each $n \in \mathbb{N}$ and each $\alpha \in \Omega$ we will let

$$\mathcal{R}_n(\alpha) = \{\beta \in \Omega \colon \beta \sim_n \alpha\},$$

that is, $\mathcal{R}_n(\alpha)$ is the equivalence class of α relative to $\mathcal{R}_n$. Observe that a path β lies in $\mathcal{R}_n(\alpha)$ if and only if it is of the form

$$\beta = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \alpha_n, \alpha_{n+1}, \dots),$$

where $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1})$ is any path joining (the single vertex in) $\mathcal{V}_0$ to the source of α_n.

For each vertex $v \in \mathcal{V}$ let $\#v$ denote the number of paths joining $\mathcal{V}_0$ to v. Clearly $\#v$ is finite and non-zero for every v. It is also clear from our discussion above that the number of elements in $\mathcal{R}_n(\alpha)$ coincides with $\#s(\alpha_n)$.

2.1. **Proposition.** *Given $n \in \mathbb{N}$ and $f \in C(\Omega)$, the complex-valued function $E_n^0(f)$ defined on Ω by*

$$E_n^0(f)|_\alpha = \sum_{\beta \in \mathcal{R}_n(\alpha)} f(\beta), \quad \forall \alpha \in \Omega, \tag{†}$$

is continuous.

Proof. Suppose first that $f(\alpha)$ depends only on the first m coordinates of α, that is,

$$f(\alpha) = g(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m), \quad \forall \alpha \in \Omega,$$

where g is some complex function defined on $\prod_{k=0}^m \mathcal{E}_k$. Supposing without loss of generality that $m > n$, one easily sees that $E_n^0(f)|_\alpha$ likewise depends only on the first m coordinates of α. So $E_n^0(f)$ is continuous.

Returning to the general case observe that the sup-norm $\|E_n^0(f)\|_\infty$ is bounded by $K\|f\|_\infty$, where K is the maximum number of summands in (†) (that is, $K = \max\{\#v \colon v \in \mathcal{V}_n\}$). Now apply the Stone-Weierstrass Theorem to write f as the uniform limit of a sequence $\{f_k\}_k$ formed by functions f_k, each of which depends only on finitely many coordinates of its argument. We conclude that $E_n^0(f)$ is the uniform limit of the sequence $\{E_n^0(f_k)\}_k$ and hence that $E_n^0(f)$ is continuous. □

2.2. **Proposition.** *Given $n \in \mathbb{N}$ and $f \in C(\Omega)$ consider the complex-valued function $E_n(f)$ defined on Ω by*

$$E_n(f)|_\alpha = \frac{1}{\#s(\alpha_n)} \sum_{\beta \in \mathcal{R}_n(\alpha)} f(\beta). \tag{‡}$$

Then E_n is a conditional expectation from $C(\Omega)$ onto $C(\Omega; \mathcal{R}_n)$. Moreover for every $m \geq n$ one has that $E_n E_m = E_m E_n = E_m$. In particular, E_n commutes with E_m.

Proof. Note that $\#s(\alpha_n)$ is a continuous function of α since it depends only on one coordinate of α, namely α_n. It then follows from (2.1) that $E_n(f) = \#s(\alpha_n)^{-1} E_n^0(f) \in C(\Omega)$.

Observe that the number of summands in (‡) is exactly $\#s(\alpha_n)$. Therefore $E_n(f)|_\alpha$ is precisely the arithmetic mean of the values of f on the equivalence class of α relative to $\mathcal{R}_n$. So it easily follows that E_n is a conditional expectation onto $C(\Omega; \mathcal{R}_n)$.

Given that $n \leq m$ it is clear that $C(\Omega; \mathcal{R}_n) \supseteq C(\Omega; \mathcal{R}_m)$ and hence E_n coincides with the identity on $C(\Omega; \mathcal{R}_m)$. It follows that $E_n E_m = E_m$.

It remains to prove that $E_m E_n = E_m$. In order to do so let $\mathcal{V}_n = \{v_1, \ldots, v_p\}$, and for every $i = 1, \ldots, p$ let X_i denote the set of all paths from $\mathcal{V}_0$ to v_i. Given $\alpha \in \Omega$, let Y_i be the set of all paths from v_i to $s(\alpha_m)$. Therefore a path β is in $\mathcal{R}_m(\alpha)$ if and only if β is of the form

$$\beta = xy\overline{\alpha} \text{ (juxtaposition of paths)},$$

where, for some $i = 1, \ldots, p$, one has that $x \in X_i$, $y \in Y_i$, and $\overline{\alpha} = (\alpha_m, \alpha_{m+1}, \ldots)$.

Fix $i = 1, \ldots, p$, and $y \in Y_i$. We claim that for any $f \in C(\Omega)$, we have that

$$\sum_{x \in X_i} f(xy\overline{\alpha}) = \sum_{x \in X_i} E_n(f)|_{xy\overline{\alpha}}. \tag{$\star$}$$

In order to see why this is so choose $x_0 \in X_i$. The $\mathcal{R}_n$-equivalence class of $x_0 y\overline{\alpha}$ is therefore formed by all paths of the form $xy\overline{\alpha}$, for $x \in X_i$. Therefore we have by definition that

$$E_n(f)|_{x_0 y\overline{\alpha}} = \frac{1}{\#v_i} \sum_{x \in X_i} f(xy\overline{\alpha}).$$

Replacing f by $E_n(f)$ above, and observing that $E_n^2 = E_n$, we have

$$E_n(f)|_{x_0 y\overline{\alpha}} = \frac{1}{\#v_i} \sum_{x \in X_i} E_n(f)|_{xy\overline{\alpha}},$$

and $(\star)$ follows by applying the definition of $E_n(f)$ on the left-hand side above.

Returning to the proof that $E_m = E_m E_n$, observe that

$$\begin{aligned} E_m(f)|_\alpha &= \frac{1}{\#s(\alpha_m)} \sum_{\beta \in \mathcal{R}_m(\alpha)} f(\beta) = \frac{1}{\#s(\alpha_m)} \sum_{i=1}^{p} \sum_{y \in Y_i} \sum_{x \in X_i} f(xy\overline{\alpha}) \\ &\overset{(\star)}{=} \frac{1}{\#s(\alpha_m)} \sum_{i=1}^{p} \sum_{y \in Y_i} \sum_{x \in X_i} E_n(f)|_{xy\overline{\alpha}} = E_m(E_n(f))|_\alpha. \end{aligned}$$ □

It is our goal in this work to describe the C^*-algebra associated to $\mathcal{R}$ and the collection of conditional expectations $\mathcal{E} = \{E_n\}_n$ under the procedure described in [2] and to prove it to be isomorphic to the AF-algebra arising from the Bratteli diagram $\mathcal{D}$.

Recall from [2] that the Toeplitz algebra of the pair $(\mathcal{R}, \mathcal{E})$, denoted $\mathcal{T}(\mathcal{R}, \mathcal{E})$, is the universal C^*-algebra generated by a copy of $C(\Omega)$ and a sequence $\{\hat{e}_n\}_{n \in \mathbb{N}}$ of projections subject to the relations:

(i) $\hat{e}_0 = 1$,
(ii) $\hat{e}_{n+1}\hat{e}_n = \hat{e}_{n+1}$,
(iii) $\hat{e}_n f \hat{e}_n = E_n(f)\hat{e}_n$,

for all $f \in C(\Omega)$ and $n \in \mathbb{N}$.

As in [2], for each $n \in \mathbb{N}$ we will denote by $\widehat{\mathcal{K}}_n$ the closed linear span of the set

$$\{f\hat{e}_n g \colon f, g \in C(\Omega)\}$$

within $\mathcal{T}(\mathcal{R}, \mathcal{E})$. By [2, 2.4] we have that $\hat{\mathcal{K}}_n$ is a $*$-subalgebra of $\mathcal{T}(\mathcal{R}, \mathcal{E})$.

Still, according to [2], a *redundancy* is, by definition, a finite sequence

$$(k_0, \ldots, k_n) \in \prod_{i=0}^{n} \widehat{K}_i,$$

such that $\sum_{i=0}^{n} k_i x = 0$, for all $x \in \widehat{K}_n$.

The ideal of $\mathcal{T}(\mathcal{R}, \mathcal{E})$ generated by the sums $\sum_{i=0}^{n} k_i$, for all redundancies $(k_0, \ldots, k_n)$, is called the *redundancy ideal*. The C^*-algebra for the pair $(\mathcal{R}, \mathcal{E})$, denoted by $C^*(\mathcal{R}, \mathcal{E})$, is then defined [2, 2.7] as the quotient of $\mathcal{T}(\mathcal{R}, \mathcal{E})$ by the redundancy ideal. We will denote by e_n the image of $\hat{e}_n$ in $C^*(\mathcal{R}, \mathcal{E})$.

The main goal of this work is therefore to prove:

2.3. **Theorem.** *Let $\mathcal{D}$ be a Bratteli diagram and let $\mathcal{R}$ be the tail-equivalence relation on the infinite path space of $\mathcal{D}$. If $\mathcal{E} = \{E_n\}_n$ is the sequence of conditional expectations defined above, then $C^*(\mathcal{R}, \mathcal{E})$ is isomorphic to the AF-algebra associated to the Bratteli diagram $\mathcal{D}$.*

3. Systems of matrix units in $C^*(\mathcal{R}, \mathcal{E})$

In this section we will introduce systems of matrix units within $C^*(\mathcal{R}, \mathcal{E})$ which will correspond to the matrix units of the AF-algebra associated to the Bratteli diagram $\mathcal{D}$ and will eventually lead us to the proof of (2.3).

For each finite path $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ starting at $\mathcal{V}_0$ and each $\alpha \in \Omega$ we will say that

$$\gamma \leq \alpha$$

when $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \gamma$, that is, whenever α "starts with γ". Moreover, we will let I_γ be the characteristic function of the set

$$\Omega_\gamma = \{\alpha \in \Omega \colon \gamma \leq \alpha\}.$$

We may then write

$$I_\gamma(\alpha) = [\gamma \leq \alpha], \quad \forall \alpha \in \Omega,$$

where the brackets stand for the obvious boolean-valued function.

For each vertex $v \in \mathcal{V}_n$ let I^v be the characteristic function of the set

$$\Omega^v = \{\alpha \in \Omega \colon s(\alpha_n) = v\}.$$

Both Ω_γ and Ω^v are clopen sets and hence the corresponding characteristic functions I_γ and I^v are continuous.

Before we proceed it will be convenient to extend the notion of the "range of an edge" in order to apply it to finite paths also. So, given a finite path $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ we will let

$$r(\gamma) := r(\gamma_{n-1}),$$

so that the range of a finite path is understood to be the range of its last edge, as one would naturally expect.

If $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ is a finite path starting at $\mathcal{V}_0$ and $v \in \mathcal{V}_n$, it is easy to see that

$$I_\gamma I^v = \begin{cases} I_\gamma, & \text{if } r(\gamma) = v, \\ 0, & \text{if } r(\gamma) \neq v. \end{cases}$$

We will write this as

$$I_\gamma I^v = [r(\gamma) = v] I_\gamma. \tag{3.1}$$

3.2. **Lemma.** *Let $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ be a finite path starting at $\mathcal{V}_0$. Then*

$$E_n(I_\gamma) = \frac{I^{r(\gamma)}}{\# r(\gamma)}.$$

Proof. This is just a computation based on the definition of E_n and is left for the reader. □

We are now ready to introduce the matrix units which constitute the core of this section.

3.3. **Definition.** Given paths $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ and $\delta = (\delta_0, \delta_1, \delta_2, \ldots, \delta_{n-1})$ of the same length n, let $e^n_{\gamma,\delta}$ be the element of $C^*(\mathcal{R}, \mathcal{E})$ given by

$$e^n_{\gamma,\delta} = \# r(\gamma) I_\gamma e_n I_\delta,$$

where e_n is the image of $\hat{e}_n$ in $C^*(\mathcal{R}, \mathcal{E})$, as already mentioned.

3.4. **Lemma.** *Let γ, δ, and n be as above. Then $e^n_{\gamma,\delta} = 0$ whenever $r(\gamma) \neq r(\delta)$.*

Proof. It suffices to show that $(I_\gamma e_n I_\delta)(I_\gamma e_n I_\delta)^* = 0$. In order to prove this note that

$$(I_\gamma e_n I_\delta)(I_\gamma e_n I_\delta)^* = I_\gamma e_n I_\delta e_n I_\gamma$$
$$= I_\gamma E_n(I_\delta) e_n I_\gamma \overset{(3.2)}{=} \frac{1}{\#r(\delta)} I_\gamma I^{r(\delta)} e_n I_\gamma \overset{(3.1)}{=} \frac{1}{\#r(\delta)}[r(\gamma) = r(\delta)] I_\gamma e_n I_\gamma. \qquad \square$$

One should therefore concentrate on the $e^n_{\gamma,\delta}$ for which $r(\gamma) = r(\delta)$. For these we have:

3.5. **Lemma.** *Let $\gamma, \delta, \zeta, \eta$ be finite paths of length n with $r(\gamma) = r(\delta)$ and $r(\zeta) = r(\eta)$. Then*

$$e^n_{\gamma,\delta} e^n_{\zeta,\eta} = [\delta = \zeta] e^n_{\gamma,\eta}.$$

Proof. We have

$$\begin{aligned} e^n_{\gamma,\delta} e^n_{\zeta,\eta} &= \#r(\gamma)\#r(\zeta) I_\gamma e_n I_\delta I_\zeta e_n I_\eta \\ &= [\delta = \zeta]\#r(\gamma)^2 I_\gamma e_n I_\delta e_n I_\eta \\ &= [\delta = \zeta]\#r(\gamma)^2 I_\gamma E_n(I_\delta) e_n I_\eta \\ &= [\delta = \zeta]\#r(\gamma)^2 I_\gamma \frac{I^{r(\delta)}}{\#r(\delta)} e_n I_\eta \\ &= [\delta = \zeta]\#r(\gamma) I_\gamma I^{r(\delta)} e_n I_\eta \\ &= [\delta = \zeta]\#r(\gamma) I_\gamma e_n I_\eta \\ &= [\delta = \zeta] e^n_{\gamma,\eta}, \end{aligned}$$

concluding the proof. $\square$

4. Finite index

There is not much more one can say about the present situation before proving that the expectations E_n introduced above are of index-finite type according to [7]. This must therefore be our next goal.

4.1. **Proposition.** *Let $n \in \mathbb{N}$ be fixed and let Ω_n denote the set of all finite paths of length n starting at $\mathcal{V}_0$. For each $\gamma \in \Omega_n$ let*

$$u_\gamma = \sqrt{\#r(\gamma)} I_\gamma.$$

Then the set $\{u_\gamma\}_{\gamma\in\Omega_n}$ is a quasi-basis for E_n. In particular E_n is of index-finite type.

Proof. Let $\gamma \in \Omega_n$ and let $f \in C(\Omega)$. Then, for every $\alpha \in \Omega$, we have

$$\begin{aligned} I_\gamma E_n(I_\gamma f)|_\alpha &= [\gamma \le \alpha] \frac{1}{\#s(\alpha_n)} \sum_{\beta \in \mathcal{R}_n(\alpha)} [\gamma \le \beta] f(\beta) \\ &= [\gamma \le \alpha] \frac{1}{\#r(\gamma)} f(\alpha). \end{aligned}$$

It follows that

$$\sum_{\gamma\in\Omega_n} u_\gamma E_n(u_\gamma f)|_\alpha = \sum_{\gamma\in\Omega_n} \#r(\gamma) I_\gamma E_n(I_\gamma f)|_\alpha = \sum_{\gamma\in\Omega_n} [\gamma \le \alpha] f(\alpha) = f(\alpha),$$

so that $\sum_{\gamma\in\Omega_n} u_\gamma E_n(u_\gamma f) = f$, as desired. $\square$

One of the first consequences of this is given in:

4.2. **Proposition.** *For every $\gamma \in \Omega_n$ one has that $I_\gamma = e^n_{\gamma,\gamma}$.*

Proof. Let $\{u_\delta\}_{\delta\in\Omega_n}$ be the quasi-basis for E_n given by (4.1). By [2, 6.2i] we have that

$$1 = \sum_{\delta\in\Omega_n} u_\delta e_n u_\delta.$$

It follows that

$$I_\gamma = \sum_{\delta\in\Omega_n} I_\gamma u_\delta e_n u_\delta = u_\gamma e_n u_\gamma = \#r(\gamma) I_\gamma e_n I_\gamma = e^n_{\gamma,\gamma}. \qquad \square$$

By [2, 3.7] we have that the canonical embedding of A into $C^*(\mathcal{R},\mathcal{E})$ is injective. Therefore, since I_γ is obviously non-zero, it follows that

$$e^n_{\gamma,\delta} e^n_{\delta,\gamma} = e^n_{\gamma,\gamma} = I_\gamma \neq 0,$$

whenever $\gamma,\delta \in \Omega_n$ are such that $r(\gamma) = r(\delta)$. In particular, $e^n_{\gamma,\delta} \neq 0$.

Fixing $v \in \mathcal{V}_n$ the sub-C^*-algebra of $C^*(\mathcal{R},\mathcal{E})$ generated by the set $\{e^n_{\gamma,\delta} \colon r(\gamma) = r(\delta) = v\}$ is therefore easily seen to be isomorphic to $M_{\#v}(\mathbb{C})$ by (3.5). If $\mathcal{V}_n = \{v_1,\ldots,v_p\}$, then the sub-$C^*$-algebra generated by $\{e^n_{\gamma,\delta} \colon \gamma,\delta\in\Omega_n\}$ is isomorphic to

$$m_{\#v_1}(\mathbb{C}) \oplus M_{\#v_2}(\mathbb{C}) \oplus \cdots \oplus M_{\#v_p}(\mathbb{C}), \tag{4.3}$$

by (3.4).

In order to see how the $e^n_{\gamma,\delta}$ relate to each other for different n's we need a quasi-basis for the restriction of E_{n+1} to $C(\Omega;\mathcal{R}_n)$. By [2, 6.1] such a quasi-basis is simply given by $\{E_n(u_\gamma)\}_{\gamma\in\Omega_{n+1}}$. In order to obtain a concrete expression for the $E_n(u_\gamma)$ we need to introduce yet another characteristic function of interest. Given an edge $\epsilon \in \mathcal{E}_n$ we will denote by ${}^\epsilon I$ the characteristic function of the clopen set

$$\{\alpha\in\Omega \colon \alpha_n = \epsilon\}.$$

Thus

$${}^\epsilon I(\alpha) = [\alpha_n = \epsilon], \quad \forall \alpha\in\Omega.$$

Observing that ${}^\epsilon I(\alpha)$ depends only on α_n,and hence is constant on the $\mathcal{R}_n$-equivalence class of α, it is clear that ${}^\epsilon I \in C(\Omega;\mathcal{R}_n)$.

Given $\gamma \in \Omega_{n+1}$, say $\gamma = (\gamma_0,\gamma_1,\ldots,\gamma_{n-1},\gamma_n)$, observe that

$$I_\gamma = I_{\gamma'}{}^{\gamma_n} I,$$

where $\gamma' = (\gamma_0,\gamma_1,\ldots,\gamma_{n-1})$. So

$$E_n(I_\gamma) = E_n(I_{\gamma'}{}^{\gamma_n} I) = E_n(I_{\gamma'})^{\gamma_n} I \overset{(3.2)}{=} \frac{I^{r(\gamma')}}{\#r(\gamma')}{}^{\gamma_n} I = \frac{{}^{\gamma_n} I}{\#r(\gamma')}.$$

It follows that

$$E_n(u_\gamma) = \sqrt{\#r(\gamma)} E_n(I_\gamma) = \frac{\sqrt{\#r(\gamma)}}{\#r(\gamma')}{}^{\gamma_n} I.$$

By [2, 6.2.i] we conclude that

$$e_n = \sum_{\gamma\in\Omega_{n+1}} E_n(u_\gamma) e_{n+1} E_n(u_\gamma) = \sum_{\gamma\in\Omega_{n+1}} \frac{\#r(\gamma)}{\#r(\gamma')^2}{}^{\gamma_n} I e_{n+1}{}^{\gamma_n} I.$$

If $\zeta, \eta \in \Omega_n$ are such that $r(\zeta) = r(\eta)$, we then have that

$$e^n_{\zeta,\eta} = \#r(\zeta) I_\zeta e_n I_\eta = \sum_{\gamma \in \Omega_{n+1}} \frac{\#r(\zeta)\#r(\gamma)}{\#r(\gamma')^2} I_\zeta{}^{\gamma_n} I e_{n+1} I_\eta{}^{\gamma_n} I$$
$$= \sum_{\gamma \in \Omega_{n+1}} \frac{\#r(\zeta)\#r(\gamma)}{\#r(\gamma')^2} [r(\zeta) = s(\gamma_n)] I_{\zeta\gamma_n} e_{n+1} I_{\eta\gamma_n} = \cdots .$$

Denoting by $\mathcal{E}_\zeta$ the set of all edges whose source is $r(\zeta)$, so that the juxtaposition $\zeta\epsilon$ lies in Ω_{n+1} if and only if $\epsilon \in \mathcal{E}_\zeta$, the above equals

$$\cdots = \sum_{\epsilon \in \mathcal{E}_\zeta} \sum_{\substack{\gamma \in \Omega_{n+1} \\ \gamma_n = \epsilon}} \frac{\#r(\zeta)\#r(\gamma)}{\#r(\gamma')^2} I_{\zeta\epsilon} e_{n+1} I_{\eta\epsilon} = \sum_{\epsilon \in \mathcal{E}_\zeta} \sum_{\substack{\gamma \in \Omega_{n+1} \\ \gamma_n = \epsilon}} \frac{\#r(\epsilon)}{\#r(\zeta)} I_{\zeta\epsilon} e_{n=1} I_{\eta\epsilon}$$
$$= \sum_{\epsilon \in \mathcal{E}_\zeta} \#r(\epsilon) I_{\zeta\epsilon} e_{n+1} I_{\eta\epsilon} = \sum_{\epsilon \in \mathcal{E}_\zeta} e^{n+1}_{\zeta\epsilon,\eta\epsilon}.$$

Summarizing we have

$$e^n_{\zeta,\eta} = \sum_{\epsilon \in \mathcal{E}_\zeta} e^{n+1}_{\zeta\epsilon,\eta\epsilon},$$

from which one easily deduces the proof of the following:

4.4. **Theorem.** *For each $n \in \mathbb{N}$, let A_n be the closed *-sub-algebra of $C^*(\mathcal{R}, \mathcal{E})$ generated by the set $\{e^n_{\gamma,\delta} \colon \gamma, \delta \in \Omega_n\}$ (see (4.3)). Then $A_n \subseteq A_{n+1}$ and the inclusion of these algebras is determined by the n-th stage in the Bratteli diagram $\mathcal{D}$ as in [1]. Therefore the closure of the union of the A_n is isomorphic to the AF-algebra associated to $\mathcal{D}$.*

In order to prove that $C^*(\mathcal{R}, \mathcal{E})$ in fact coincides with $\overline{\bigcup_{n \in \mathbb{N}} A_n}$, it suffices to show that $C^*(\mathcal{R}, \mathcal{E})$ is generated by all the $e^n_{\gamma,\delta}$, which we now set out to do.

4.5. **Proposition.** *The sub-C^*-algebra of $C^*(\mathcal{R}, \mathcal{E})$ generated by the set*

$$\{e^n_{\gamma,\delta} \colon n \in \mathbb{N}, \gamma, \delta \in \Omega_n\}$$

coincides with $C^*(\mathcal{R}, \mathcal{E})$.

Proof. Let A be the sub-C^*-algebra of $C^*(\mathcal{R}, \mathcal{E})$ generated by the set in the statement. It is enough to show that A contains $C(\Omega)$ and all the e_n. By (4.2) we have that every $I_\gamma \in A$, and since the set $\{I_\gamma \colon n \in \mathbb{N}, \gamma \in \Omega_n\}$ generates $C(\Omega)$ by the Stone-Weierstrass Theorem, we have that $C(\Omega) \subseteq A$. Given $n \in \mathbb{N}$ note that $1 = \sum_{\gamma \in \Omega_n} I_\gamma$. So

$$e_n = \left(\sum_{\gamma \in \Omega_n} I_\gamma \right) e_n \left(\sum_{\delta \in \Omega_n} I_\delta \right) = \sum_{\gamma,\delta \in \Omega_n} I_\gamma e_n I_\delta = \sum_{\gamma,\delta \in \Omega_n} \#r(\gamma)^{-1} e^n_{\gamma,\delta}. \qquad \square$$

The proof of (2.3) therefore follows from our last two results.

5. Proper equivalence relations

In preparation for our study of more general approximately proper equivalence relations let us take a moment to study proper ones.

Given a compact topological space X and a proper equivalence relation $\mathcal{R}$ on X, let $\Omega = X/\mathcal{R}$, which is a Hausdorff space (precisely because $\mathcal{R}$ is assumed to be a proper equivalence relation), and let $\pi \colon X \to \Omega$ be the quotient map.

We will assume, from now on, that π is a local homeomorphism. In particular, the equivalence class of each $x \in X$ must be a discrete, and hence finite, set.

We may view $\mathcal{R}$ as a locally compact r-discrete groupoid as follows. The topology of $\mathcal{R}$ is the relative topology of $X \times X$ (product topology), the multiplication operation is defined by

$$(x_1, x_2)(y_1, y_2) = (x_1, y_2)$$

whenever $(x_1, x_2),\ (y_1, y_2) \in \mathcal{R}$ are such that $x_2 = y_1$, and the inversion operation is given by

$$(x, y)^{-1} = (y, x),$$

for $(x, y) \in \mathcal{R}$.

By [3, Theorem 2.8] (see also [4, Theorem 2.2]) one has a very concrete faithful representation of $C^*(\mathcal{R})$, which we would now like to describe.

Let M_0 be the pre-Hilbert module over $C(\Omega)$ as follows. As a vector space M_0 coincides with $C(X)$, and we define

$$\langle \xi, \eta \rangle(w) = \sum_{\pi(x)=w} \overline{\xi(x)}\eta(x), \quad \forall \xi, \eta \in M_0, \quad \forall w \in \Omega,$$

and

$$\xi f|_x = \xi(x) f(\pi(x)), \quad \forall \xi \in M_0, \quad \forall f \in C(\Omega), \quad \forall x \in X.$$

Denote by M the completion of M_0 and by $\mathcal{L}(M)$ the C^*-algebra of adjointable operators on M. The representation of $C^*(\mathcal{R})$ referred to above is by operators on M and is determined by the formula

$$k\xi|_x = \sum_{\pi(y)=\pi(x)} k(x, y)\xi(y),$$

where k is a any continuous function on $\mathcal{R}$, $\xi \in M_0$, and $x \in X$. In particular, note that each $f \in C(X)$ defines a continuous function k_f on $\mathcal{R}$ by

$$k_f(x, y) = \begin{cases} f(x), & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases}$$

and the action of k_f is simply by pointwise multiplication. In the following we will identify k_f with f and hence think of $C(X)$ as a subalgebra of $C^*(\mathcal{R})$.

By the references mentioned above this representation establishes an isomorphism from $C^*(\mathcal{R})$ to the C^*-algebra $\mathcal{K}(M)$ of generalized compact operators on M.

Let ρ be a normalized potential. Therefore ρ is a strictly positive continuous function on X such that for every $x \in X$ one has that

$$\sum_{\pi(y)=\pi(x)} \rho(y) = 1.$$

As in section 2 define an expectation E from $C(X)$ to $C(X, \mathcal{R})$ by

$$E(f)|_x = \sum_{\pi(y)=\pi(x)} \rho(y) f(y).$$

Observe that the correspondence between normalized potentials ρ and faithful conditional expectations E from $C(X)$ to $C(X; \mathcal{R})$ given by the formula above is a bijection.

Also define $\check{e} \in C(\mathcal{R}) \subseteq C^*(\mathcal{R})$ by

$$\check{e}(x, y) = \rho(x)^{1/2}\rho(y)^{1/2}.$$

5.1. **Lemma.** (i) *$\check{e}$ is a projection in $C^*(\mathcal{R})$.*
(ii) *$\check{e}f\check{e} = E(f)\check{e}$, for all $f \in C(X)$.*
(iii) *$C^*(\mathcal{R})$ coincides with the closed linear span of the set $\{f\check{e}g\colon f, g \in C(X)\}$.*

Proof. Given $\xi, \eta \in M$ we denote by $\theta_{\xi,\eta}$ the generalized compact operator on M given by

$$\theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle, \quad \forall \zeta \in M.$$

With this notation observe that $\check{e}$ corresponds precisely to the operator $\theta_{\xi,\xi}$, for $\xi = \rho^{1/2}$.

By identifying $C^*(\mathcal{R})$ with its image under the above representation one checks (i) and (ii) without difficulty. For (iii), we have

$$f\check{e}g = f\theta_{\xi,\xi}g = \theta_{f\xi,\overline{g}\xi}$$

for all $f, g \in C(X)$. This shows that the closed linear span of the set referred to in (iii) contains all finite rank operators, hence is equal to $\mathcal{K}(M) = C^*(\mathcal{R})$. $\square$

If we consider instead the inner product

$$\langle\xi, \eta\rangle_\rho(\omega) = \sum_{\pi(x)=\omega} \overline{\xi(x)}\eta(x)\rho(x),$$

we obtain the Hilbert module M^ρ, which is isomorphic to M via the map

$$j_\rho\colon \xi \in M^\rho \mapsto \xi\rho^{1/2} \in M.$$

6. Approximately proper equivalence relations

Given a compact topological space X and an approximately proper equivalence relation $\mathcal{R}$ on X, say $\mathcal{R}$ is the increasing union of a family $\{\mathcal{R}_n\}_n$ of proper equivalence relations on X, we may also view $\mathcal{R}$ as a locally compact groupoid with operations as above and topology defined in such a way that a subset $U \subseteq \mathcal{R}$ is open if and only if $U \cap \mathcal{R}_n$ is open in $\mathcal{R}_n$ for every n. This defines a topology on $\mathcal{R}$ called the *inductive limit topology.*

Let $X_n = X/\mathcal{R}_n$. As before we will assume that the quotient mappings

$$\pi_n\colon X \to X_n$$

are local homeomorphisms. These are the working assumptions of [6, Section 2], where the following result can also be found.

6.1. **Proposition.** *Under the hypothesis that each π_n is a local homeomorphism one has that $\mathcal{R}_n$ is an open subset of $\mathcal{R}_{n+1}$.*

Proof. Let $(x, y) \in \mathcal{R}_n$. Choose open sets $U \ni x$ and $V \ni y$ such that both π_n and π_{n+1} are injective on both U and V, and moreover such that $\pi_n(U) = \pi_n(V)$. Observe that the latter condition means that for each $x' \in U$ there exists $y' \in V$ such that $(x', y') \in \mathcal{R}_n$.

We claim that $(U \times V) \cap \mathcal{R}_{n+1} \subset \mathcal{R}_n$. In fact, if $(x', y') \in (U \times V) \cap \mathcal{R}_{n+1}$, then, using the observation at the end of the last paragraph, there exists $y'' \in V$ such that $(x', y'') \in \mathcal{R}_n$. But then $(x', y'') \in \mathcal{R}_{n+1}$ as well, so

$$\pi_{n+1}(y'') = \pi_{n+1}(x') = \pi_{n+1}(y').$$

Since π_{n+1} is injective on V we have $y'' = y'$ and hence $(x', y') = (x', y'') \in \mathcal{R}_n$. This shows our claim and hence also that (x, y) is an interior point of $\mathcal{R}_n$ in the relative topology of $\mathcal{R}_{n+1}$. Therefore, $\mathcal{R}_n$ is open in $\mathcal{R}_{n+1}$. □

It clearly follows that $\mathcal{R}_n$ is open in $\mathcal{R}_m$ for every $m > n$ and hence we see that $\mathcal{R}_n$ is open[1] in $\mathcal{R}$ by definition of the inductive limit topology.

Observe that the fact that the $\mathcal{R}_n$ are open in $\mathcal{R}$ also ensures that the inclusion maps

$$\mathcal{R}_n \hookrightarrow \mathcal{R}$$

are homeomorphism onto their images and hence we may view the $\mathcal{R}_n$ (with the product topology) as topological subspaces of $\mathcal{R}$.

In particular $\mathcal{R}_0$, the unit space of $\mathcal{R}$, is open in $\mathcal{R}$ and hence $\mathcal{R}$ is an r-discrete groupoid [5, I.2.6]. More precisely, $\mathcal{R}$ is an étale groupoid (this means that the range and source maps are local homeomorphisms) and therefore it is possible to define its groupoid C^*-algebra $C^*(\mathcal{R})$. By definition of the topology of R, we have

$$C^*(\mathcal{R}) = \overline{\bigcup_{n\in\mathbb{N}} C^*(\mathcal{R}_n)}.$$

The following is the main result of this section.

6.2. **Theorem.** *Let X be a compact Hausdorff space and let $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ be an approximately proper equivalence relation on X, where the $\mathcal{R}_n$ form an increasing sequence of proper equivalence relations such that the quotient maps $\pi_n\colon X \to X/\mathcal{R}_n$ are local homeomorphisms. Let $\mathcal{E} = (E_n)$ be a sequence of faithful conditional expectations defined on $C(X)$ with $E_n(C(X)) = C(X; \mathcal{R}_n)$ and $E_{n+1} \circ E_n = E_n$ for all n. Then $C^*(\mathcal{R})$ and $C^*(\mathcal{R}, \mathcal{E})$ are naturally isomorphic.*

Proof. Lemma 5.1 provides a sequence of projections $\{\check{e}_n\}_{n\in\mathbb{N}} \subseteq C^*(\mathcal{R})$ such that

(i) $\check{e}_0 = 1$,
(ii) $\check{e}_n\check{e}_{n-1} = \check{e}_n$,
(iii) $\check{e}_n f \check{e}_n = E_n(f)\check{e}_n$, for all $f \in C(X)$, and
(iv) $C^*(\mathcal{R})$ is generated as a C^*-algebra by the $\check{e}_n$'s and $C(X)$.

Therefore, we have a surjective $*$-homomorphism $\widehat{\Psi}\colon \mathcal{T}(\mathcal{R}, \mathcal{E}) \to C^*(\mathcal{R})$, which is the identity when restricted to $C(X)$ and such that $\widehat{\Psi}(\hat{e}_n) = \check{e}_n$.

Recall that, for each $n \in \mathbb{N}$, the closed linear span of the set $\{f\hat{e}_n g\colon f, g \in C(X)\}$ within $\mathcal{T}(\mathcal{R}, \mathcal{E})$ is denoted $\widehat{\mathcal{K}}_n$. By Lemma 5.1 the image of $\widehat{\mathcal{K}}_n$ in $C^*(\mathcal{R})$ is then precisely $C^*(\mathcal{R}_n)$.

Since the latter form an increasing sequence of C^*-algebras, one sees that $\widehat{\Psi}$ vanishes on the redundancy ideal. So let

$$\Psi\colon C^*(\mathcal{R}, \mathcal{E}) \to C^*(\mathcal{R})$$

be the quotient map.

[1]By contrast note that $\mathcal{R}_n$ is not necessarily open in $\mathcal{R}$ if the latter is equipped with the product topology. This is partly the reason why the product topology is not appropriate and is replaced by the inductive limit topology here.

Let $\mathcal{K}_n$ be the image of $\widehat{K}_n$ in $C^*(\mathcal{R},\mathcal{E})$ under the quotient map, and let $e_n \in \mathcal{K}_n$ be the corresponding image of $\hat{e}_n$. By [2, 6.2] and the fact that the E_n are of index-finite type, we have that the $\mathcal{K}_n$ are also increasing. In order to show that Ψ is injective, one therefore needs only show it is isometric on each $\mathcal{K}_n$. In order to accomplish this fix $n \in \mathbb{N}$ and let M_n and M_n^ρ be as in section 5 with respect to X and $\mathcal{R}_n$. Let Φ denote the composition

$$\Phi\colon \mathcal{K}_n \xrightarrow{\Psi} C^*(\mathcal{R}_n) \to \mathcal{L}(M_n) \xrightarrow{Ad_{j_\rho}} \mathcal{L}(M_n^\rho),$$

where the middle arrow is the representation of section 5 and $Ad_{j\rho}$ is conjugation by j_ρ. By direct computation one checks that $\Phi(f) = f$, for every $f \in C(X)$, and $\Phi(e_n) = p_n$, where p_n is the projection in $\mathcal{L}(M_n^\rho)$ defined by

$$p_n(\xi) = E_n(\xi), \quad \forall \xi \in C(X).$$

It therefore follows that the range of Φ is precisely the *reduced C^*-basic construction* defined in [7, 2.1.2]. By [7, 2.2.9] it is isomorphic to the *unreduced C^*-basic construction* which therefore has the universal property of [7, 2.2.7]. It follows that there is a *-homomorphism from the range of Φ to $\mathcal{K}_n$ which is the identity when restricted to $C(X)$ and which sends p_n to e_n. It must therefore be the inverse of Φ and hence Φ must be injective. ☐

6.3. *Remark.* The above theorem shows that, up to isomorphism, $C^*(\mathcal{R},\mathcal{E})$ does not depend on $\mathcal{E}$.

6.4. *Remark.* It is shown in [6, 3.4] that, when $\mathcal{R} = \bigcup \mathcal{R}_n$ is an approximately proper étale equivalence relation as above, there is a bijection between sequences of faithful conditional expectations $\mathcal{E} = (E_n)$ defined on $C(X)$ with $E_n(C(X)) = C(X;\mathcal{R}_n)$ and $E_{n+1} \circ E_n = E_n$ for all n and cocycles D in $Z^1(\mathcal{R},\mathbb{R}_+^*)$. Let us describe this bijection, starting with D in $Z^1(\mathcal{R},\mathbb{R}_+^*)$. For each n, the restriction D_n of D to $\mathcal{R}_n$ can be written under the form $D_n(x,y) = \rho_n(x)/\rho_n(y)$. The so-called potential ρ_n in $C(X,\mathbb{R}_+^*)$ is unique under the normalization condition $\sum_{xR_ny} \rho_n(x) = 1$ for all y. The conditional expectation E_n is given by $E_n(f)(y) = \sum_{xR_ny} f(x)\rho_n(x)$. The sequence of expectations introduced in section 2 corresponds to the cocycle $D = 1$. In their study of KMS states for some gauge actions, A. Lopes and the first author [2, Section 7] use the framework of a pair $(\mathcal{R},\mathcal{E})$ and a sequence of potentials, while the second author [6, Section 3] uses groupoids and cocycles. The above theorem also shows that, in the case of an approximately proper étale equivalence relation, both approaches are equivalent.

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