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TOPOLOGICAL ENTROPY AND AF SUBALGEBRAS OF GRAPH C^*-ALGEBRAS

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ABSTRACT. Let $\mathcal{A}_E$ be the canonical AF subalgebra of a graph C^*-algebra $C^*(E)$ associated with a locally finite directed graph E. For Brown and Voiculescu's topological entropy $ht(\Phi_E)$ of the canonical completely positive map Φ_E on $C^*(E)$, $ht(\Phi_E) = ht(\Phi_E|_{\mathcal{A}_E}) = h_l(E) = h_b(E)$ is known to hold for a finite graph E, where $h_l(E)$ is the loop entropy of Gurevic and $h_b(E)$ is the block entropy of Salama. For an irreducible infinite graph E, the inequality $h_l(E) \leq ht(\Phi_E|_{\mathcal{A}_E})$ has recently been known. It is shown in this paper that

$$ht(\Phi_E|_{\mathcal{A}_E}) \leq \max\{h_b(E), h_b({}^tE)\},$$

where ${}^tE$ is the graph E with the direction of the edges reversed. Some irreducible infinite graphs E_p ($p > 1$) with $ht(\Phi_E|_{\mathcal{A}_{E_p}}) = \log p$ are also examined.

1. INTRODUCTION

Voiculescu [22] introduced a notion of topological entropy $ht(\alpha)$ for an automorphism α of a nuclear unital C^*-algebra A to measure the growth of α^n as $n \to \infty$ using the fact that a nuclear C^*-algebra has the completely positive approximation property. The definition extends very well to automorphisms of exact C^*-algebras (as done by Brown in [4]) due to the deep result by Kirchberg [13] that exact C^*-algebras are nuclearly embeddable. But without effort one can define $ht(\Phi)$ even for a completely positive (cp) map on an exact C^*-algebra as described in [2]. Since a C^*-subalgebra of an exact C^*-algebra is always exact, if $\Phi : A \to A$ is a cp map on an exact C^*-algebra A and B is a Φ-invariant C^*-subalgebra of A, then $ht(\Phi|_B)$ can be defined and the monotonicity $ht(\Phi|_B) \leq ht(\Phi)$ holds.

The topological entropy has been computed in several cases; for example, the equality $ht(\alpha * \beta) = \max\{ht(\alpha), ht(\beta)\}$ for the reduced free product automorphism $\alpha * \beta$ was proved in [1], when the free product is with amalgamation over a finite-dimensional C^*-algebra. Also Dykema [9] showed that $ht(\alpha) = 0$ for certain classes of automorphisms α of reduced amalgamated free products of C^*-algebras, which turns out to extend Størmer's result [21] that the Connes-Størmer entropy of the free shift automorphism of the II_1-factor $L(F_\infty)$ is zero.

In this paper we are concerned with the topological entropy of the shift type cp maps on C^*-algebras arising from directed graphs. A typical one is the canonical

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cp map $\Phi_A : \mathcal{O}_A \to \mathcal{O}_A$ of the Cuntz-Krieger algebra $\mathcal{O}_A$ given by

$$\Phi_A(x) = \sum_{i=1}^{n} s_i x s_i^*,$$

where $s_1, \ldots, s_n$ are the partial isometries that generate $\mathcal{O}_A$. The reason we call Φ_A shift type is that $\mathcal{O}_A$ contains a Φ_A-invariant commutative C^*-subalgebra $\mathcal{D}_A$ which is isomorphic to $C(X_A)$ in such a way that the restriction $\Phi_A|_{\mathcal{D}_A}$ corresponds to the shift map σ_{X_A} on the (compact) shift space X_A associated with the incidence matrix A. The topological entropy of Φ_A is then computed (see [5, 2, 11, 19]) as $ht(\Phi_A) = \log r(A)$ ($r(A)$ is the spectral radius of A). But $\log r(A) = h_{top}(X_A)$ is a well-known fact, so that one can deduce by [8] that $ht(\Phi_A) = ht(\Phi_A|_{\mathcal{D}_A})$. On the other hand, $\mathcal{O}_A$ also contains another important Φ_A-invariant C^*-subalgebra $\mathcal{A}_A$ which is an AF algebra with $\mathcal{D}_A \subset \mathcal{A}_A$. Thus by monotonicity of entropy, we have $ht(\Phi_A) = ht(\Phi_A|_{\mathcal{A}_A}) = ht(\Phi_A|_{\mathcal{D}_A})$.

The Cuntz-Krieger algebras $\mathcal{O}_A$ are now well understood as graph C^*-algebras $C^*(E) = C^*(s_e, p_v)$ associated with finite directed graphs E, and the cp map Φ_A of $\mathcal{O}_A$ is interpreted as the map $\Phi_E : C^*(E) \to C^*(E)$ given by $\Phi_E(x) = \sum_{e\in E^1} s_e x s_e^*$. Hence if E is a finite graph (possibly with sinks) which contains an infinite path, it follows that $ht(\Phi_E) = ht(\Phi_E|_{\mathcal{A}_E}) = ht(\Phi_E|_{\mathcal{D}_E}) = \log r(A_E)$, where $\mathcal{A}_E$ is the AF subalgebra of $C^*(E)$ corresponding to $\mathcal{A}_A$ in $\mathcal{O}_A$ and A_E is the edge matrix of E (see [11]).

If E is infinite but locally finite, then the map Φ_E is known to be a contractive cp map, and furthermore if E is irreducible and $\mathcal{A}_E$ is the canonical AF subalgebra of $C^*(E)$, the inequality $h_l(E) \le ht(\Phi_E|_{\mathcal{A}_E})$ is known to hold [11]. The purpose of the present paper is then to give an upper bound for $ht(\Phi_E|_{\mathcal{A}_E})$, and we actually prove the following (see Theorem 3.10):

$$ht(\Phi_E|_{\mathcal{A}_E}) \le \max\{h_b(E), h_b({}^tE)\}.$$

In particular, for an irreducible infinite graph E_p constructed in [20] so that $h_l(E_p) = h_b(E_p) = \log p$ ($p > 1$), we have $ht(\Phi_{E_p}|_{\mathcal{A}_{E_p}}) = \log p$.

We believe that the result would be helpful to compute the entropy $ht(\Phi_E)$ of Φ_E on the whole graph C^*-algebra $C^*(E)$.

2. Preliminaries

2.1. **Graph C^*-algebras.** A (directed) graph is a quadruple $E = (E^0, E^1, r, s)$ of the vertex set E^0, the edge set E^1, and the range, source maps $r, s : E^1 \to E^0$. A family $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ of mutually orthogonal projections p_v and partial isometries s_e is called a *Cuntz-Krieger E-family* if the following relations hold:

$$\begin{aligned} &s_e^* s_e = p_{r(e)}, \quad s_e s_e^* \le p_{s(e)}, \\ &p_v = \sum_{s(e)=v} s_e s_e^*, \quad \text{if } 0 < |s^{-1}(v)| < \infty. \end{aligned}$$

The *graph C^*-algebra* $C^*(E)$ is then defined to be a C^*-algebra generated by a universal Cuntz-Krieger E-family (see [16, 17, 3]). If E is *row-finite*, that is, each vertex emits only finitely many vertices, the relations can be written as (with the edge matrix A_E of E)

$$s_e^* s_e = \sum_{f\in E^1} A_E(e,f) s_f s_f^*;$$

hence the family is also called a Cuntz-Krieger A_E-*family*.

Given a $\{0,1\}$ matrix B such that each row has only finitely many non-zero entries (row-finite), let E be the graph with the vertex matrix B. Then by definition $C^*(E)$ is generated by a Cuntz-Krieger A_E-family. But it is also generated by a Cuntz-Krieger B-family [17, Proposition 4.1]. Hence many results on C^*-algebras of $\{0,1\}$ matrices can be applied to graph C^*-algebras even though not all $\{0,1\}$ matrices can occur as edge matrices of some graphs.

We call a graph E *locally finite* if each vertex receives and emits only finitely many edges. Throughout this paper we consider only locally finite graphs and adopt the notation in [16]. If a finite path $\alpha \in E^*$ of length $|\alpha| > 0$ is a return path, that is, $s(\alpha) = r(\alpha)$, then α is called a *loop* at $v = s(\alpha)$. A graph E is said to be *irreducible* if for any two vertices v, w there is a finite path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. It is known that if E is irreducible and every loop has an exit, then $C^*(E)$ is simple ([16]).

2.2. **Topological entropy of cp maps.** Let A be a C^*-algebra, $\pi : A \to B(H)$ a faithful $*$-representation, and $Pf(A)$ the set of all finite subsets of A. For $\omega \in Pf(A)$ and $\delta > 0$, put

$$\begin{aligned} CPA(\pi, A) = \{(\phi, \psi, B) \mid \phi : A \to B, \psi : B \to B(H) &\text{ are contractive cp maps} \\ &\text{and } B \text{ is a } C^*\text{-algebra with } \dim B < \infty\}, \\ rcp(\pi, \omega, \delta) = \inf\{rank(B) \mid (\phi, \psi, B) \in CPA(\pi, A),\ &\|\psi\circ\phi(x) - \pi(x)\| < \delta, \\ &\text{for all } x \in \omega\}, \end{aligned}$$

where $rank(B)$ denotes the dimension of a maximal abelian subalgebra of B.

Since the cp δ-rank $rcp(\pi, \omega, \delta)$ is independent of the choice of π ([2, 4]) and graph C^*-algebras $C^*(E)$ are nuclear ([15]) we may write $rcp(\omega, \delta)$ for $rcp(\pi, \omega, \delta)$ assuming that $C^*(E) \subset B(H)$ for a Hilbert space H.

Definition 2.1 ([2, 4, 22])**.** Let $A \subset B(H)$ be a C^*-algebra and let $\Phi : A \to A$ be a cp map. Put

$$\begin{aligned} ht(\Phi, \omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log\big(rcp(\bigcup_{i=0}^{n-1} \Phi^i(\omega), \delta)\big), \\ ht(\Phi, \omega) &= \sup_{\delta>0} ht(\Phi, \omega, \delta). \end{aligned}$$

Then $ht(\Phi) := \sup_{\omega\in Pf(A)} ht(\Phi, \omega)$ is called the *topological entropy* of Φ.

Remark 2.2. We refer the reader to [2] and [4] for the following useful properties and their proofs. Let A be an exact C^*-algebra and let $\Phi : A \to A$ be a cp map.

(a) If $\theta : A \to B$ is a C^*-isomorphism, then $ht(\Phi) = ht(\theta\Phi\theta^{-1})$.
(b) Let $\tilde{A}$ be the unital C^*-algebra obtained by adjoining a unit and let $\tilde{\Phi} : \tilde{A} \to \tilde{A}$ be the extension of Φ. Then $ht(\tilde{\Phi}) = ht(\Phi)$.
(c) If $A_0 \subset A$ is a Φ-invariant C^*-subalgebra of A, $ht(\Phi|_{A_0}) \le ht(\Phi)$.

We will use the following Arveson's extension theorem several times.

Arveson Extension Theorem (see [4]). *Let A be a unital C^*-algebra, $S \subset A$ a unital subspace with $S = S^*$, and $\phi : S \to B$ a contractive cp map where $B = B(H)$ or $\dim(B) < \infty$. Then ϕ extends to a cp map $\bar{\phi} : A \to B$. If S is a C^*-subalgebra of A, then we obtain a unital cp extension of ϕ even when S does not contain the unit of A.*

If E is a locally finite graph, the map $\Phi_E : C^*(E) \to C^*(E)$, defined by

$$\Phi_E(x) = \sum_{e \in E^1} s_e x s_e^*,$$

is well defined, contractive and completely positive [11]. For a finite graph E, the topological entropy $ht(\Phi_E)$ has been obtained as follows (see [2], [5], [19], or [11]).

Theorem 2.3. *Let E be a finite graph possibly with sinks and let A_E be the edge matrix of E. If E contains an infinite path, then*

$$ht(\Phi_E) = \log r(A_E),$$

where $r(A_E)$ is the spectral radius of A_E.

By $h_{top}(X)$ we denote the topological entropy of a compact space (X, T) together with a continuous map $T : X \to X$ (for a definition, see [23, Chapter 7]). Let E be a locally finite infinite graph with no sinks and X_E the locally compact shift space of (one-sided) infinite paths with the one point compactification $\bar{X}_E$. The first identity in the following theorem is shown for the doubly infinite path space of E by Gurevic [10]. See Remark 3.6(a) for a definition of the entropy $h(X_E)$ for a finite graph E.

Theorem 2.4 ([11, Theorem 4.4]). *Let E be a locally finite irreducible infinite graph. Then*

$$h_{top}(\bar{X}_E) = \sup_{E'} h(X_{E'}) \ \le \ ht(\Phi_E),$$

where the supremum is taken over all the finite (irreducible) subgraphs of E.

3. Main results

Throughout this section E will denote a locally finite infinite graph unless stated otherwise. For a path $\alpha \in E^*$, let α^0 be the set of vertices lying on $\alpha = \alpha_1 \cdots \alpha_n$, that is, $\alpha^0 = \{s(\alpha_1), r(\alpha_1), \dots, r(\alpha_n)\}$. For a fixed vertex v we consider the following subsets of finite paths E^n of length n:

(i) $E^n(v) = \{\alpha \in E^n \mid v \in \alpha^0\}$,
(ii) $E^n_s(v) = \{\alpha \in E^n \mid s(\alpha) = v\}$,
(iii) $E^n_s(v)^\star = \{\alpha \in E^n_s(v) \mid r(\alpha_i) \ne v,\ 1 \le i \le n\}$,
(iv) $E^n_l(v) = \{\alpha \in E^n \mid \alpha \text{ is a loop at } v\}$.

Similarly we can think of $E^n_r(v)$ and $E^n_r(v)^\star$.

Definition 3.1. Let E be a graph and $v \in E^0$. Put

$$h_l(E, v) = \limsup_n \frac{1}{n} \log |E^n_l(v)| \quad \text{and} \quad h_b(E, v) = \limsup_n \frac{1}{n} \log |E^n_s(v)|.$$

The *loop entropy* $h_l(E)$ and the *block entropy* $h_b(E)$ of E are defined by

$$h_l(E) := \sup_{v \in E^0} h_l(E, v) \quad \text{and} \quad h_b(E) := \sup_{v \in E^0} h_b(E, v).$$

If E is irreducible, $h_l(E,v)$ and $h_b(E,v)$ are independent of the choice of a vertex v [20]; hence $h_l(E) = h_l(E,v)$ and $h_b(E) = h_b(E,v)$ for any $v \in E^0$. Let ${}^tE$ denote the graph E with the direction of all edges reversed. Then $h_l(E) = h_l({}^tE)$ is immediate while $h_b(E) \neq h_b({}^tE)$ in general as we will see in Example 3.3.

We will use the following notation for the infinite series with coefficients from (i)-(iv) above:

(i)′ $E(v,z) := \sum |E^n(v)|z^n$,
(ii)′ $E_s(v,z) := \sum |E_s^n(v)|z^n$,
(iii)′ $E_s^\star(v,z) := \sum |E_s^n(v)^\star|z^n$,
(iv)′ $E_l(v,z) := \sum |E_l^n(v)|z^n$.

We denote the radius of convergence of the series $E_s^\star(v,z)$ by $R_{E_s^\star}$. Thus

$$R_{E_s^\star}^{-1} = \limsup_{n\to\infty} \left|E_s^n(v)^\star\right|^{1/n}.$$

Similarly $R_{E_r^\star}$ denotes the radius of convergence of $E_r^\star(v,z) := \sum |E_r^n(v)|^\star z^n$. As in [20, p.331], if $C_v^{(n)}$ is the number of sequences $vv_{i_1}\cdots v_{i_{n-1}}$ of vertices such that $v_j \neq v$ for $j = i_1, \ldots, i_{n-1}$, $|C_v^{(n)}| = |E_s^{n-1}(v)^\star|$ and so the radius of convergence of $E_s^\star(v,z)$ coincides with that (denoted by Q_0 in [20]) of $\sum_n C_v^{(n)} z^n$. The following is Lemma (3.1) of [20].

Proposition 3.2 ([20])**.** *If E is an irreducible graph, then*

$$h_b(E) = \max\{\log\left(R_{E_s^\star}^{-1}\right),\, h_l(E)\}.$$

Note that if E is irreducible, then $h_b({}^tE) = \limsup \frac{1}{n}\log|E_r^n(v)|$ and so from the above proposition we have

$$h_b({}^tE) = \max\{\log\left(R_{E_r^\star}^{-1}\right),\, h_l(E)\}. \tag{1}$$

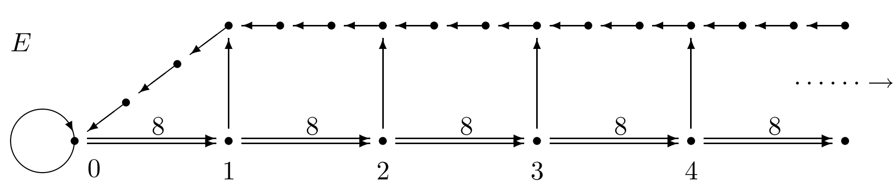
The following example shows that $h_b(E) \neq h_b({}^tE)$ in general.

Example 3.3. For each pair of positive real numbers $1 < p \leq q$, Salama [20] constructed an irreducible infinite graph $E_{p,q}$ with

$$h_l(E_{p,q}) = \log p \quad\text{and}\quad h_b(E_{p,q}) = \log q.$$

As mentioned in the proof of [20, Theorem (3.9)], $E_{p,q}$ may be constructed to be a (uniformly) locally finite graph using the idea in [20, Example (3.7)].

For example, the following graph $E := E_{2,8}$ satisfies $h_l(E) = \log 2$ and $h_b(E) = \log 8$. There are 8 edges from the vertex n to the vertex $n+1$ for each $n \geq 0$.



Now we first show that

$$\log\left(R_{E_r^\star}^{-1}\right) < h_l(E).$$

For a fixed vertex 0 we have

$$\begin{aligned} R_{E_r^\star}^{-1} &= \limsup_{n\to\infty} |E_r^n(0)^\star|^{1/n} \\ &= \limsup_{n\to\infty} \big|\{\alpha \in E_r^n(0) \mid s(\alpha_i) \neq 0, \text{ for } 1 \le i \le n\}\big|^{1/n}. \end{aligned}$$

Since

$$|E_r^{4k}(0)^\star| = 1 + 8^{k-1} + 8^{k-4} + 8^{k-7} + \cdots ,$$

it follows that

$$\limsup_{k\to\infty} |E_r^{4k}(0)^\star|^{\frac{1}{4k}} = 8^{1/4}.$$

But it is not hard to see that

$$\limsup_{n\to\infty} |E_r^n(0)^\star|^{\frac{1}{n}} = \limsup_{k\to\infty} |E_r^{4k}(0)^\star|^{\frac{1}{4k}}.$$

Hence $\log(R_{E_r^\star}^{-1}) = \log 8^{1/4} < \log 2 = h_l(E)$. By (1), $h_b({}^tE) = h_l(E) = \log 2$ and so we conclude that $h_b({}^tE) < h_b(E)$.

Lemma 3.4. *If E is an irreducible graph, then the value*

$$\limsup_{n\to\infty} \frac{1}{n} \log |E^n(v)|$$

is independent of the choice of a vertex v.

Proof. Let v, w be two vertices of E. Then there exist two paths $\mu \in E^k$, $\nu \in E^m$ with $s(\mu) = r(\nu) = v$, $s(\nu) = r(\mu) = w$ because E is irreducible. We assume that μ and ν have the smallest length, respectively. If $\alpha = \alpha_1\alpha_2\cdots\alpha_n \in E^n(v)$, then with $i_0 = \min\{i \mid s(\alpha_i) = v\}$ write $\alpha = \alpha'\alpha''$, where $\alpha' = \alpha_1\cdots\alpha_{i_0-1}$ and $\alpha'' = \alpha_{i_0}\cdots\alpha_n$ (if $i_0 = 1$, $\alpha = \alpha''$). Then the map

$$E^n(v) \to E^{n+k+m}(w), \quad \alpha = \alpha'\alpha'' \mapsto \alpha'\mu\nu\alpha''$$

is injective; hence $|E^n(v)| \le |E^{n+k+m}(w)|$ for each n. Therefore

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log |E^n(v)| &\le \limsup_{n\to\infty} \frac{1}{n} \log |E^{n+k+m}(w)| \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log |E^n(w)|. \end{aligned}$$

□

Proposition 3.5. *Let E be an irreducible graph and $v_0 \in E^0$.*

(a) *If E is finite, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |E_l^n(v_0)| = \limsup_{n\to\infty} \frac{1}{n} \log |E^n|.$$

In particular, $h_l(E) = h_b(E) = h_b({}^tE)$.

(b) *If E is infinite, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |E^n(v_0)| \;=\; \max\{h_b(E), h_b({}^tE)\}.$$

Proof. (a) Let $E^0 = \{v_0, v_1, \dots, v_{k-1}\}$. Since E is irreducible there exist finite paths $\{\mu_i, \nu_i \mid 0 \le i \le k-1\}$ such that $s(\mu_i) = r(\nu_i) = v_0$, $r(\mu_i) = v_i = s(\nu_i)$. Suppose $|\mu_i| = m_i$, $|\nu_j| = l_j$. If $\alpha \in E^n$ is a path with $s(\alpha) = v_i$, $r(\alpha) = v_j$ then $\mu_i \alpha \nu_j \in E_l^{n+m_i+l_j}(v_0)$ is a loop at v_0. The map $\alpha \mapsto \mu_i \alpha \nu_j$ is not necessarily injective, but there exist at most k_0 paths in E^n that have the same image in $E_l^{n+m_i+l_j}(v_0)$ under the map, where $k_0 = \max_{i,j}\{m_i + l_j\}$. Hence we have

$$|E^n| \le k_0 \cdot \Big| \bigcup_{0 \le i,j \le k-1} E_l^{n+m_i+l_j}(v_0) \Big| \le k_0 k^2 \max_{i,j} |E_l^{n+m_i+l_j}(v_0)|.$$

On the other hand, for each n, there exists a $k_n \in \{0, \dots, k_0\}$ such that

$$|E_l^{n+k_n}(v_0)| = \max_{i,j} |E_l^{n+m_i+l_j}(v_0)|.$$

Then $|E^n| \le k_0 k^2 |E_l^{n+k_n}(v_0)|$ and it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log |E^n| \le \limsup_{n\to\infty} \frac{1}{n} \log |E_l^n(v_0)|.$$

(b) Note first that

$$\begin{aligned} |E^n(v_0)| &= \Big| \bigcup_{k=0}^{n} \{\alpha\beta \mid \alpha \in E_r^k(v_0)^\star,\ \beta \in E_s^{n-k}(v_0)\} \Big| \\ &= \sum_{k=0}^{n} \big|E_r^k(v_0)^\star\big| \, \big|E_s^{n-k}(v_0)\big| = \sum_{k=0}^{n} \big|({}^tE)_s^k(v_0)^\star\big| \, \big|E_s^{n-k}(v_0)\big|. \end{aligned}$$

Then

$$\begin{aligned} E(v_0, z) &= \sum_n \Big(\sum_{k=0}^{n} |({}^tE)_s^k(v_0)^\star| \, |E_s^{n-k}(v_0)| \Big) z^n \\ &= \Big(\sum_n |({}^tE)_s^n(v_0)^\star| z^n \Big) \Big(\sum_n |E_s^n(v_0)| z^n \Big) \\ &= ({}^tE)_s^\star(v_0, z) \cdot E_s(v_0, z), \end{aligned}$$

so that the radius of convergence R_E of $E(v_0, z)$ is equal to $\min\big\{R_{({}^tE)_s^\star},\, R_{E_s}\big\}$. Thus

$$R_E^{-1} = \max\big\{ R_{({}^tE)_s^\star}^{-1},\, R_{E_s}^{-1} \big\}.$$

But Proposition 3.2 gives

$$\log\big(R_{({}^tE)_s^\star}^{-1}\big) \le h_b({}^tE),$$

and also by definition $\log(R_{E_s}^{-1}) = h_b(E)$. Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log |E^n(v_0)| = \log(R_E^{-1}) \le \max\{h_b({}^tE),\, h_b(E)\}.$$

□

Remark 3.6. Let E be a finite graph with the irreducible components $\{E_i\}_{i=1}^s$ so that the *Perron eigenvalue* of the edge matrix A_E of E is $\lambda_E = \max_{1\le i\le s}\{\lambda_{E_i}\}$, where λ_{E_i} is the Perron eigenvalue of the edge matrix of the irreducible graph E_i (see [18, 4.4]).

(a) Assuming $\lambda_E = \lambda_{E_1}$ without loss of generality, we have from [18, Theorem 4.4.4] that

$$h(X_E) = \log \lambda_E = \log \lambda_{E_1},$$

where $h(X_E) = \lim_n \frac{1}{n}\log|E^n|$ is the *topological entropy* of X_E (or Σ_E, the two-sided edge shift space). See [18, Definition 4.1.1] or [14, p.23] for the definition of $h(X_E)$. Since $\log\lambda_{E_1} = h(X_{E_1})$ ([18, Theorem 4.3.1]) and $h(X_{E_1}) = h_l(E_1) = h_b(E_1)$ by Proposition 3.5(a) (E_1 is irreducible), we see that all the entropies $h_l(E)$, $h_b(E)$ and $h(X_E)$ are the same and equal to $\log\lambda_E$ because $\log\lambda_E = \log\lambda_{E_1} = h_l(E_1) \le h_l(E) \le h_b(E) \le h(X_E) = \log\lambda_E$.

(b) Since the eigenvalues of A_E are exactly the eigenvalues of the A_{E_i}, by [18, Lemma 4.4.3] it follows that $\log\lambda_E = \log r(A_E)$. Thus by Theorem 2.3, $ht(\Phi_E) = h_l(E) = h_b(E) = h(X_E)$ for any finite graph E which contains an infinite path (or a loop). If E has no infinite paths, $h(X_E) = -\infty$ while $ht(\Phi_E) \ge 0$.

Let E be an irreducible infinite graph and let $\mathcal{D}_E$ be the commutative C^*-subalgebra of $C^*(E)$ generated by the projections $\{p_\alpha = s_\alpha s_\alpha^* \mid \alpha \in E^*\}$. Then $\mathcal{D}_E = \overline{\operatorname{span}}\{p_\alpha \mid \alpha\in E^*\}$ and the map

$$w : \mathcal{D}_E \to C_0(X_E),\ w(p_\alpha) = \chi_{[\alpha]},$$

is a C^*-isomorphism such that $w(\Phi_E|_{\mathcal{D}_E})w^{-1} = \sigma_E^*$ [11]. Here $\chi_{[\alpha]}$ is the characteristic function on the cylinder set $[\alpha] = \{\beta\in X_E \mid \beta = \alpha\beta'\}$ which is both open and closed, and $\sigma_E^* : C_0(X_E)\to C_0(X_E)$ is the $*$-homomorphism induced by the shift map σ_E on X_E, that is, $\sigma_E^*(f) = f\circ\sigma_E$ for $f\in C_0(X_E)$. By Remark 2.2(a), $ht(\Phi_E|_{\mathcal{D}_E}) = ht(\sigma_E^*)$. But $ht(\sigma_E^*) = ht(\widetilde{\sigma_E^*})$ by Remark 2.2(b) and $ht(\widetilde{\sigma_E^*}) = h_{top}(\bar{X}_E)$ by [8, Proposition 1.2]. On the other hand, $h_{top}(\bar{X}_E) = \sup_{E'\subset E} h(X_{E'})$ is proved in [10], where the supremum is taken over all the finite subgraphs E' of E (or equivalently, over all the irreducible finite subgraphs). If $\sup_{E'\subset E} h(X_{E'}) < \infty$, $h_l(E) = \sup_{E'\subset E} h(X_{E'})$ is known (see [18, p.465]). If $\sup_{E'\subset E} h(X_{E'}) = \infty$, clearly $h_l(E) = \infty$ since $h(X_{E'}) = h_l(E')$ for any finite graph E' (Remark 3.6(a)) and $h_l(E') \le h_l(E)$. Thus $h_{top}(\bar{X}_E) = \sup_{E'\subset E} h(X_{E'}) = h_l(E)$ always holds for a locally finite irreducible infinite graph E. Hence we have

$$ht(\Phi_E|_{\mathcal{D}_E}) = h_l(E). \tag{2}$$

Put

$$\mathcal{A}_E := \overline{\operatorname{span}}\{\, s_\alpha s_\beta^* \mid \alpha,\beta\in E^*,\ |\alpha| = |\beta|\,\}.$$

Then $\mathcal{A}_E$ is a Φ_E-invariant AF C^*-subalgebra of $C^*(E)$ with $\mathcal{D}_E \subset \mathcal{A}_E$; hence it follows from (2) that

$$h_l(E) \le ht(\Phi_E|_{\mathcal{A}_E}). \tag{3}$$

Lemma 3.7. *Let v be a vertex of an irreducible graph E with at least two vertices and let $n \geq 1$. Then the elements in the set*

$$\omega(n,v) = \left\{ s_\alpha s_\beta^* \,\middle|\, r(\alpha) = r(\beta) = v,\ |\alpha| = |\beta| \leq n \right\}$$

are linearly independent.

Proof. We prove the assertion by induction on n. For $n = 1$, suppose

$$x = \sum_{\substack{e,f \in E^1 \\ r(e)=r(f)=v}} \lambda_{ef} s_e s_f^* + \lambda_0 p_v = 0.$$

If e_0 and f_0 are edges with $r(e_0) = r(f_0) = v$ and either $s(e_0) \neq v$ or $s(f_0) \neq v$, then $s_{e_0}^* p_v s_{f_0} = 0$; hence

$$0 = s_{e_0}^* x s_{f_0} = \lambda_{e_0 f_0} (s_{e_0}^* s_{e_0})(s_{f_0}^* s_{f_0}) = \lambda_{e_0 f_0} p_v;$$

thus $\lambda_{e_0 f_0} = 0$. Similarly, $\lambda_{ef} = 0$ if e and f are loops at v and $e \neq f$. Then x becomes

$$x = \sum_{e \in E_l^1(v)} \lambda_{ee} s_e s_e^* + \lambda_0 p_v = 0.$$

By irreducibility of E and the assumption that $|E^0| > 1$, there exists an edge f with $s(f) = v, r(f) \neq v$. Then $s_f s_f^* x = \lambda_0 s_f s_f^* = 0$, so that $\lambda_0 = 0$ and we have $x = \sum_{e \in E_l^1(v)} \lambda_{ee} s_e s_e^* = 0$. Since the projections $\{s_e s_e^* \mid e \in E_l^1(v)\}$ are mutually orthogonal, it follows that $\lambda_{ee} = 0$ for each $e \in E_l^1(v)$.

Now suppose that the assertion is true for $n - 1$. If

$$x = \sum_{\substack{|\alpha|=|\beta| \leq n \\ r(\alpha)=r(\beta)=v}} \lambda_{\alpha\beta} s_\alpha s_\beta^* = 0,\ \ \lambda_{\alpha\beta} \in \mathbb{C},$$

then for an edge $e \in E^1$ we have

$$0 = s_e^* x s_e = \sum_{\substack{\alpha = e\alpha' \\ \beta = e\beta'}} \lambda_{\alpha\beta} s_e^* s_\alpha s_\beta^* s_e = \sum_{|\alpha'|=|\beta'| \leq n-1} \lambda_{(e\alpha')(e\beta')} s_{\alpha'} (s_{\beta'})^*.$$

Note that the elements $s_{\alpha'}(s_{\beta'})^*$ appearing in the sum are distinct. Thus by the induction hypothesis, one sees that $\lambda_{(e\alpha')(e\beta')} = 0$. But the edge e was arbitrary, and so we conclude that the coefficients $\lambda_{\alpha\beta}$ are all zero. □

Using the same idea as in the proof of [4, Proposition 2.6] one can prove the following, which is stated in [2] without a proof in the case where $\{\omega_\lambda\}$ is an increasing sequence. We provide a proof only for the reader's convenience.

Proposition 3.8. *Let $\Phi : A \to A$ be a contractive cp map of an exact C^*-algebra A. If $\{\omega_\lambda\}_{\lambda \in \Lambda}$ is a net (partially ordered by inclusion) of finite subsets in A such that the linear span of $\bigcup_{\lambda, l \in \mathbb{Z}^+} \Phi^l(\omega_\lambda)$ is dense in A, then*

$$ht(\Phi) = \sup_\lambda ht(\Phi, \omega_\lambda).$$

Proof. Let $\omega = \{a_1, a_2, \ldots, a_m\}$ be a finite subset in A and $\delta > 0$. Then there exists a $\lambda \in \Lambda$ and $p \in \mathbb{N}$ such that if $\bigcup_{0 \leq l \leq p} \Phi^l(\omega_\lambda) = \{x_1, \cdots, x_k\}$, then

$$\|a_i - \sum_{i,j} \lambda_{ij} x_j\| < \delta$$

for some $\lambda_{ij} \in \mathbb{C}$. Put $C := \max_{i,j} |\lambda_{ij}|$. Choose $(\phi, \psi, B) \in CPA(id, A)$ with $rank(B) = rcp(\omega_\lambda \cup \cdots \cup \Phi^{p+n}(\omega_\lambda), C^{-1}\delta)$. Then for $0 \le l \le p+n$,

$$\begin{aligned}
&\|\psi \circ \phi(\Phi^l(a_i)) - \Phi^l(a_i)\| \\
\le& \|\psi \circ \phi(\Phi^l(a_i) - \Phi^l(\sum \lambda_{ij}x_j)\| \\
&+ \|\psi \circ \phi(\Phi^l(\sum \lambda_{ij}x_j)) - \Phi^l(\sum \lambda_{ij}x_j)\| + \|\Phi^l(\sum \lambda_{ij}x_j) - \Phi^l(a_i)\| \\
=& 2\delta + \|\sum_{i,j} \lambda_{ij}(\psi \circ \phi(\Phi^l(x_j)) - \Phi^l(x_j))\| \\
\le& 2\delta + \max_{i,j} |\lambda_{ij}| \cdot C^{-1}\delta = 3\delta.
\end{aligned}$$

Thus for any $n \in \mathbb{N}$,

$$rcp(\omega \cup \cdots \cup \Phi^{p+n}(\omega), 3\delta) \le rcp(\omega_\lambda \cup \cdots \cup \Phi^{p+n}(\omega_\lambda), C^{-1}\delta),$$

which implies that

$$ht(\Phi, \omega, 3\delta) \le ht(\Phi, \omega_\lambda, C^{-1}\delta).$$

Therefore we have $ht(\Phi, \omega) \le ht(\Phi, \omega_\lambda)$. □

The AF algebra $\mathcal{A}_E$ contains Φ_E-invariant AF subalgebras $\mathcal{A}_E(v)$, $v \in E^0$,

$$\mathcal{A}_E(v) := \overline{\operatorname{span}}\{\, s_\alpha s_\beta^* \mid r(\alpha) = r(\beta) = v,\ |\alpha| = |\beta|\,\}.$$

We show that the topological entropy of the restriction map $\Phi_E|_{\mathcal{A}_E(v)}$ has an upper bound $h_b({}^tE)$ which might be strictly smaller than the upper bound for $ht(\Phi_E|_{\mathcal{A}_E})$ given in Theorem 3.10.

Proposition 3.9. *Let E be an irreducible infinite graph. Then for each $v \in E^0$,*

$$ht(\Phi_E|_{\mathcal{A}_E(v)}) \le h_b({}^tE).$$

Proof. Let $A_n(v)$ be the C^*-subalgebra of $\mathcal{A}_E(v)$ generated by $\omega(n, v)$. Then from

$$s_\alpha s_\beta^* \cdot s_\mu s_\nu^* = \begin{cases} s_{\alpha\mu'} s_\nu^*, & \text{if } \mu = \beta\mu', \\ s_\alpha s_{\nu\beta'}^*, & \text{if } \beta = \mu\beta', \\ 0, & \text{otherwise,} \end{cases}$$

we see that $A_n(v) = \operatorname{span}(\omega(n, v))$ is finite dimensional.

Since $\{\omega(n, v)\}_n$ is an increasing sequence of finite subsets in $\mathcal{A}_E(v)$ such that the linear span of $\bigcup_n \omega(n, v)$ is dense in $\mathcal{A}_E(v)$, by Proposition 3.8 it suffices to show that

$$ht(\Phi_E, \omega(n, v)) \le h_b({}^tE), \quad n \in \mathbb{N}.$$

Set $E_l^*(v) := \bigcup_{k \ge 0} E_l^k(v)$ and $r(n) := |\bigcup_{k=0}^n E_r^k(v)|$. Fix $n_0 \in \mathbb{N}$, and define a map $\phi : \omega(n_0, v) \to M_{r(n_0)}$ by

$$\phi(s_\alpha s_\beta^*) = \sum_{\substack{|\alpha\gamma| \le n_0 \\ \gamma \in E_l^*(v)}} e_{(\alpha\gamma)(\beta\gamma)},$$

where $\{e_{\mu\nu}\}$ are the standard matrix units of the matrix algebra $M_{r(n_0)}$. Since the elements in $\omega(n_0, v)$ are linearly independent by Lemma 3.7, one can extend the map to the linear map $\phi : A_{n_0}(v) \to M_{r(n_0)}$. Now we show that ϕ is in fact a $*$-isomorphism. To prove that it is a $*$-homomorphism, we only need to see that

$$\phi((s_\alpha s_\beta^*)(s_\mu s_\nu^*)) = \phi(s_\alpha s_\beta^*)\phi(s_\mu s_\nu^*).$$

If $\beta = \mu\beta'$, then $s_\alpha s_\beta^* s_\mu s_\nu^* = s_\alpha (s_{\nu\beta'})^*$ and

$$\begin{aligned}\phi(s_\alpha s_\beta^*)\phi(s_\mu s_\nu^*) &= \sum_{\substack{|\alpha\gamma|\le n_0\\ \gamma\in E_l^*(v)}} e_{(\alpha\gamma)(\mu\beta'\gamma)} \cdot \sum_{\substack{|\mu\delta|\le n_0\\ \delta\in E_l^*(v)}} e_{(\mu\delta)(\nu\delta)}\\ &= \sum_{\substack{|\alpha\gamma|\le n_0\\ \gamma\in E_l^*(v)}} e_{(\alpha\gamma)(\nu\beta'\gamma)} = \phi(s_\alpha(s_{\nu\beta'})^*) = \phi(s_\alpha s_\beta^* s_\mu s_\nu^*).\end{aligned}$$

If $\mu = \beta\mu'$, a similar proof works. Otherwise, we have $\phi((s_\alpha s_\beta^*)(s_\mu s_\nu^*)) = 0 = \phi(s_\alpha s_\beta^*)\phi(s_\mu s_\nu^*)$. In order to show that ϕ is injective, let $\phi(\sum_{\alpha,\beta} \lambda_{\alpha\beta}\, s_\alpha s_\beta^*) = 0$. Then

$$\sum_{\alpha,\beta} \lambda_{\alpha\beta}\phi(s_\alpha s_\beta^*) = \sum_{\alpha,\beta} \lambda_{\alpha\beta}\big(\sum_{\substack{|\alpha\gamma|\le n_0\\ \gamma\in E_l^*(v)}} e_{(\alpha\gamma)(\beta\gamma)}\big) = 0.$$

But the vectors $\sum_{\substack{|\alpha\gamma|\le n_0\\ \gamma\in E_l^*(v)}} e_{(\alpha\gamma)(\beta\gamma)}$ $(r(\alpha) = r(\beta) = v,\ |\alpha| = |\beta| \le n_0)$ are linearly independent in $M_{r(n_0)}$. In fact, if $A := \sum_{\alpha,\beta} \lambda_{\alpha\beta}\big(\sum_{\substack{|\alpha\gamma|\le n_0\\ \gamma\in E_l^*(v)}} e_{(\alpha\gamma)(\beta\gamma)}\big) = 0$, then $e_{vv}Ae_{vv} = \lambda_{vv}e_{vv} = 0$, that is, $\lambda_{vv} = 0$, and for any $\alpha, \beta \in E_r^1(v)$, $e_{\alpha\alpha}Ae_{\beta\beta} = \lambda_{\alpha\beta}e_{\alpha\beta} = 0$; hence $\lambda_{\alpha\beta} = 0$. Repeating the process one has $\lambda_{\alpha\beta} = 0$ for any $\alpha, \beta \in \bigcup_{k=0}^{n_0} E_r^k(v)$. Therefore $\sum_{\alpha,\beta} \lambda_{\alpha\beta}\, s_\alpha s_\beta^* = 0$, and the map ϕ is injective. The surjectivity of ϕ follows from $\dim(A_{n_0}(v)) = r(n_0)^2$. We simply write ϕ for $\phi : A_{n_0+l}(v) \to M_{r(n_0+l)}$ $(l \ge 0)$, and $\bar\phi$ for its contractive cp extension to $\mathcal{A}_E(v)$ that exists by Arveson's extension theorem.

For each $n \in \mathbb{N}$ and $0 \le l \le n-1$, note that

$$\bigcup_{l=0}^{n-1} \Phi_E^l(\omega(n_0, v)) \subseteq \mathrm{span}(\omega(n_0 + n - 1, v)).$$

Then the element

$$(\bar\phi,\, \psi := \phi^{-1}, M_{r(n_0+n-1)}) \in CPA(id, \mathcal{A}_E(v))$$

satisfies $\psi \circ \bar\phi|_{\omega(n_0+n-1,v)} = id_{\omega(n_0+n-1,v)}$. Thus for each $\delta > 0$,

$$rcp(id, \omega(n_0 + n - 1, v), \delta) \le r(n_0 + n - 1),$$

and so

$$\begin{aligned}ht(\Phi_E|_{\mathcal{A}_E(v)}, \omega(n_0, v), \delta) &\le \limsup_{n\to\infty} \frac{1}{n} \log(r(n_0 + n - 1))\\ &= \limsup_{n\to\infty} \frac{1}{n} \log(r(n))\\ &= \limsup_{n\to\infty} \frac{1}{n} \log \Big| \bigcup_{k=0}^{n} E_r^k(v) \Big|\\ &= h_b({}^tE).\end{aligned}$$

For the last equality, note that if $k \le n$, then $\big| E_r^k(v) \big| \le \big| E_r^n(v) \big|$; hence

$$\Big| \bigcup_{k=0}^{n} E_r^k(v) \Big| \le (n + 1) \cdot \big| E_r^n(v) \big|.$$

□

The following theorem gives an upper bound for $ht(\Phi_E|_{\mathcal{A}_E})$.

Theorem 3.10. *Let E be an irreducible infinite graph and let $\mathcal{A}_E$ be the AF subalgebra of $C^*(E)$ generated by the partial isometries $\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, |\alpha| = |\beta|\}$. Then*

$$ht(\Phi_E|_{\mathcal{A}_E}) \le \max\{h_b({}^tE), h_b(E)\}.$$

Proof. Let $E^0 = \{v_1, v_2, \cdots\}$. For each $n_0 \in \mathbb{N}$ and $n_1 \in \mathbb{Z}^+ = \{0\} \cup \mathbb{N}$, put

$$\omega(n_0, n_1) := \Big\{ s_\alpha s_\beta^* \,\Big|\, \alpha, \beta \in E^{n_1}, r(\alpha) = r(\beta) \in \{v_1, \cdots, v_{n_0}\} \Big\},$$
$$\omega_\Sigma(n_0, n_1) := \Big\{ \sum s_{\alpha_i} s_{\beta_i}^* \,\Big|\, s_{\alpha_i} s_{\beta_i}^* \in \omega(n_0, n_1) \Big\}.$$

Note that $\omega_\Sigma(n_0, n_1)$ is not the linear span of $\omega(n_0, n_1)$. Then $\{\omega_\Sigma(n_0, n_1) \mid n_0 \in \mathbb{N}, n_1 \in \mathbb{Z}^+\}$ is a net of finite subsets in $\mathcal{A}_E$ which is partially ordered by inclusion. In fact, given two finite sets $\omega_\Sigma(n_0, n_1)$, $\omega_\Sigma(m_0, m_1)$ $(n_1 \le m_1)$, one may write each element $s_\alpha s_\beta^* \in \omega(n_0, n_1)$ as

$$s_\alpha s_\beta^* = s_\alpha (\sum_{|\mu| = m_1 - n_1} s_\mu s_\mu^*) s_\beta^* = \sum s_{\alpha\mu} (s_{\beta\mu})^* \in \omega_\Sigma(m_2, m_1),$$

where $m_2 > \max\{n_0, m_0\}$ is an integer large enough so that $r(\alpha\mu) \in \{v_1, \cdots, v_{m_2}\}$ for any $\alpha\mu$ appearing in the last sum. Then clearly $\omega_\Sigma(n_0, n_1) \cup \omega_\Sigma(m_0, m_1)$ is contained in $\omega_\Sigma(m_2, m_1)$.

Since the linear span of the set $\bigcup_{n_0, n_1, n} \Phi_E^n(\omega_\Sigma(n_0, n_1))$ is dense in $\mathcal{A}_E$, by Proposition 3.8, we show that for each finite set $\omega_\Sigma(n_0, n_1)$,

$$ht(\Phi_E, \omega_\Sigma(n_0, n_1)) \le \max\{h_b({}^tE), h_b(E)\}.$$

If $s_\alpha s_\beta^* \in \omega(n_0, n_1)$, $r(\alpha) = r(\beta) = v$, then for $l \le n - 1$,

$$\Phi_E^l(s_\alpha s_\beta^*) = \sum_{|\mu| = l} s_{\mu\alpha} s_{\mu\beta}^* = \sum_{|\mu| = l} s_{\mu\alpha} (\sum_{\substack{|\nu| = n - l \\ s(\nu) = v}} s_\nu s_\nu^*) s_{\mu\beta}^* = \sum_{\substack{|\mu\alpha\nu| = n + n_1 \\ |\mu| = l}} s_{\mu\alpha\nu} (s_{\mu\beta\nu})^*,$$

because $p_v = \sum_{\substack{|\nu| = n - l \\ s(\nu) = v}} s_\nu s_\nu^*$. Hence one sees that

$$\bigcup_{i=0}^{n-1} \Phi_E^i(\omega_\Sigma(n_0, n_1)) \subseteq \Big\{ \sum_{|\mu\alpha\nu| = n + n_1} s_{\mu\alpha\nu} (s_{\mu\beta\nu})^* \,\Big|\, s_\alpha s_\beta^* \in \omega(n_0, n_1) \Big\}.$$

Since the set $\{s_\mu s_\nu^* \mid \mu, \nu \in \bigcup_{i=1}^{n_0} E^{n_1 + n}(v_i)\}$ forms a matrix unit, it generates the C^*-subalgebra of $\mathcal{A}_E$ which is isomorphic to M_{k_n}, where $k_n = |\bigcup_{i=1}^{n_0} E^{n_1 + n}(v_i)|$. Let

$$\rho_n : \mathrm{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in \bigcup_{i=1}^{n_0} E^{n_1 + n}(v_i)\} \to M_{k_n}$$

be a $*$-isomorphism with the inverse ρ^{-1}. Then by Arveson's extension theorem ρ extends to a contractive cp map $\bar{\rho} : \mathcal{A}_E \to M_{k_n}$, so that we obtain an element $(\bar{\rho}, \rho^{-1}, M_{k_n}) \in CPA(id, \mathcal{A}_E)$ such that $\|\rho^{-1} \circ \bar{\rho}(x) - x\| = 0$ if

$$x \in \bigcup_{i=0}^{n-1} \Phi_E^i(\omega_\Sigma(n_0, n_1)) \subseteq \mathrm{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in \bigcup_{i=1}^{n_0} E^{n_1 + n}(v_i)\}.$$

Hence

$$rcp\big(\bigcup_{i=0}^{n-1} \Phi_E^i(\omega_\Sigma(n_0,n_1)),\, \delta\big) \,\leq\, k_n$$

holds for any $\delta > 0$. Thus

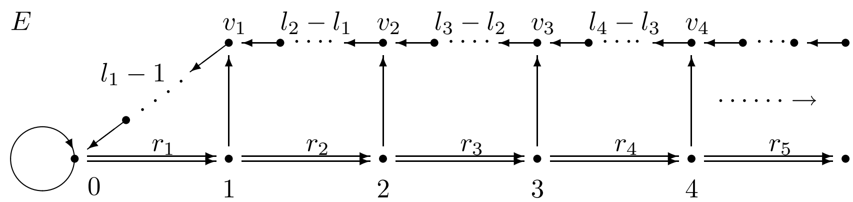
$$ht(\Phi_E, \omega_\Sigma(n_0,n_1)) \,\leq\, \limsup_{n\to\infty} \frac{1}{n}\log(k_n).$$

On the other hand, the irreducibility of E implies that there is an N such that $|E^{n_1+n}(v_i)| \leq |E^{n_1+n+N}(v_1)|$ for $1 \leq i \leq n_0$. Hence $k_n = |\bigcup_{i=1}^{n_0} E^{n_1+n}(v_i)| \leq n_0|E^{n_1+n+N}(v_1)|$. Therefore

$$\limsup_{n\to\infty} \frac{1}{n}\log k_n \leq \limsup_{n\to\infty} \frac{1}{n}\log |E^n(v_1)|,$$

and the assertion then follows from Proposition 3.5(b). □

Example 3.11. Let $E := E_{\{r_n\},\{l_n\}}$ be Salama's infinite irreducible graph (see [20]). We assume here that $l_n + 1 \leq l_{n+1}$ for each n. There are r_k edges from the vertex $k-1$ to k, and there is only one path (of length $l_k - l_{k-1}$) from the vertex v_k to v_{k-1}.



Note that for each n, $|E_r^n(0)^\star| \leq |E_s^n(0)^\star|$, which then implies by Proposition 3.2 that

$$h_b({}^tE) \leq h_b(E).$$

Thus from Theorem 3.10, we have

$$ht(\Phi_E|_{\mathcal{A}_E}) \leq h_b(E).$$

In particular, if $E_p := E_{p,p}$ $(p > 1)$ is an irreducible infinite graph of Salama satisfying $h_l(E_p) = h_b(E_p) = \log p$, by (3) we have

$$ht(\Phi_{E_p}|_{\mathcal{A}_{E_p}}) = \log p.$$

Remark 3.12. After the paper had been submitted, the authors found a meaningful lower bound for $ht(\Phi_E|_{\mathcal{A}_{E(v)}})$ (see Proposition 3.9) and a better upper bound for $ht(\Phi_E|_{\mathcal{A}_E})$ in [12].

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