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THE BIG SLICE PHENOMENA IN M-EMBEDDED AND L-EMBEDDED SPACES

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ABSTRACT. We obtain sufficient conditions on an M-embedded or L-embedded space so that every nonempty relatively weakly open subset of its unit ball has norm diameter 2. We prove that, up to renorming, this holds for every Banach space containing c_0 and, as a consequence, for every proper M-ideal. The result obtained for L-embedded spaces can be applied to show that the above property is satisfied for every predual of an atomless real JBW*-triple. As a consequence, a characterization of the Radon-Nikodym property is obtained in this setting, showing that a predual of a real JBW*-triple E verifies the Radon-Nikodym property if, and only if, E is the l_∞-sum of real type I triple factors.

1. INTRODUCTION

The nonexistence of denting points in the unit ball of some function spaces has been the subject of several recent papers [13], [23]. A point x_0 in the sphere of a Banach space X, S_X, is a denting point of the unit ball in X, B_X, if there are slices, that is, subsets defined as

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > \|x^*\| - \alpha\},\ x^* \in X^*,\ \alpha \in \mathbb{R},$$

containing x_0, with diameter arbitrarily small. From [6], x_0 is a denting point of the unit ball of X if, and only if, x_0 is an extreme point in B_X and x_0 is a point of weak-norm continuity, that is, a point of continuity for the identity map from (B_X, w) onto (B_X, n), where w and n denote the weak and the norm topology, respectively. In particular, the existence of denting points in the unit ball of a Banach space X implies the existence of nonempty relatively weakly open subsets of the unit ball in X with diameter arbitrarily small. Then the extreme opposite property to the existence of denting points in the unit ball of a Banach space is that every nonempty relatively weakly open subset of the unit ball has diameter 2. This is the case, for example, for infinite-dimensional C^*-algebras [5], uniform algebras [20], non-hilbertizable real JB*-triples [4] and for some Banach spaces of vector valued functions and some spaces of operators [3].

The aim of this note is to study when every nonempty relatively weakly open subset of the unit ball of an M-embedded or L-embedded space has diameter 2. In Theorem 2.4, we obtain sufficient conditions in order to assure the above property

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in the M-ideals case, when only the original norms are considered, by improving the results in [23]. After this, it is shown in Proposition 2.6 that every Banach space containing c_0 can be equivalently renormed so that every nonempty relatively weakly open subset of its unit ball has diameter 2, and then the same is true for proper M-ideals.

The result for the L-embedded case is Theorem 2.8, where a sufficient condition to have diameter 2 for all nonempty relatively weakly open subsets of the unit ball of an L-embedded space is obtained. This condition works in the setting of preduals of real JBW*-triples and, as a consequence, we prove in Theorem 2.12 that every nonempty relatively weakly open subset of the unit ball of the predual of an atomless real JBW*-triple has diameter 2. Then the same holds for preduals of atomless Von Neumann algebras. Finally an easy characterization of the Radon-Nikodym property is given in Theorem 2.14, where it is shown that the predual of a real JBW*-triple E satisfies the Radon-Nikodym property if, and only if, E is the l_∞-sum of type I real triple factors and then, the predual B of a Von Neumann algebra A satisfies the Radon-Nikodym property if, and only if, B is the c_0-sum of trace class operators on a complex Hilbert space. The above characterizations of Radon-Nikodym property can be found in [8] and [2] for preduals of complex JBW*-triples and in [9] for preduals of Von Neumann algebras. Also the relation between L-embedded spaces and the Radon-Nikodym propety was studied in [17].

Finally, we also apply Theorem 2.8 in order to obtain that every nonempty relatively weakly open subset of the unit ball of L_1/H^1 has diameter 2, and Corollary 2.9, where L_1 and H^1 denote the classical Lebesgue space and Hardy space, respectively, on the unit interval $[0,1]$.

Let X be a real or complex Banach space. We denote by S_X, B_X, and X^* the unit sphere, the closed unit ball and the topological dual, respectively, of X. We denote by w the weak topology of X, and by w^* the weak* topology of X^*. Given a subspace M of X, we denote by M° the polar or annihilator subspace of M in X^*. An L-projection (resp. M-projection) on X is a linear projection p on X satisfying $\|x\| = \|p(x)\| + \|x - p(x)\|$ (resp. $\|x\| = max\{\|p(x)\|,\ \|x-p(x)\|\}$) for all $x \in X$. A subspace M of X is said to be an L-summand (resp. M-summand) of X if it is the range of an L-projection (resp. M-projection) on X, and an M-ideal of X if M° is an L-summand of X^*. X is said to be L-embedded (resp. M-embedded) whenever X is an L-summand (resp. M-ideal) of X^{**} (see [12]).

2. The main results

We begin with an elementary lemma which will be essential for the main results.

Lemma 2.1. *Let X be a Banach space such that every nonempty relatively weakly open subset of B_X has diameter 2. Then every nonempty relatively weakly open subset of $B_{X\oplus_\infty Y}$ has diameter 2, where Y is an arbitrary Banach space.*

Proof. We call $Z = X \oplus_\infty Y$ and let $P : Z \to X$ be the projection from Z onto X, which is weak open. It is clear that $B_Z = B_X \times B_Y$ and $\|P\| = 1$. Then if W is a weakly open subset of Z such that $W \cap B_Z \neq \emptyset$, one has that $V = P(W \cap B_Z)$ is a nonempty weak open set relative to B_X, and so $diam(V) = 2$. Hence $diam(W \cap B_Z) = 2$. □

The following is a w^* version of the above lemma. We omit the proof, since it is similar to the one above.

Lemma 2.2. *Let X be a Banach space. Assume that $X^{**} = Y^{\circ\circ} \oplus_\infty Z^\circ$, where Y is a closed subspace of X and Z is a closed subspace of X^*. Assume that every nonempty relatively $w^*(Y^{\circ\circ})$-open subset of $B_{Y^{\circ\circ}}$ has diameter 2. Then every nonempty relatively $w^*(X^{**})$-open subset of $B_{X^{**}}$ has diameter 2.*

For a Banach space X and a subspace Z of X^* given, we denote by $\sigma(X, Z)$ the weak topology on X endowed by the dual pair (X, Z), that is, the smallest vector topology on X such that every element of Z is a continuous map.

The following result shows that the size of many nonempty relatively weakly open subsets of the unit ball of an M-ideal have diameter 2.

Proposition 2.3. *Let X be a Banach space and let Y be a closed and proper subspace of X. Assume that Y is an M-ideal of X (that is, there is an L-projection from X^* onto some subspace Z of X^*, with kernel Y°). Then every nonempty relatively $\sigma(X, Z)$-open subset of B_X which intersects B_Y has diameter 2.*

Proof. Let U be a $\sigma(X, Z)$-open subset of X and assume that $U \cap B_Y \neq \emptyset$. Choose some $x_0 \in U \cap B_Y$.

As Y is a proper subspace of X, given $\varepsilon > 0$, there is an $x \in S_X$ such that $\|x + Y\| > 1 - \varepsilon$, where $x + Y$ denotes the class of the element x in the quotient X/Y.

By [22, Proposition 2.3], there exists a net $\{x_\alpha\}$ of elements of Y converging to x in the $\sigma(X, Z)$-topology and satisfying

$$\limsup_\alpha \|x_0 \pm (x - x_\alpha)\| \leq 1.$$

Then, for $0 < \lambda < 1$ given, we can choose α_0 so that $\lambda\|x_0 \pm (x - x_\alpha)\| \leq 1$, whenever $\alpha \geq \alpha_0$.

Furthermore, for λ close enough to 1, we can assume that $\lambda(x_0 \pm (x - x_\alpha) \in U$ for each $\alpha \geq \alpha_0$, since the net $\{\lambda(x_0 \pm (x - x_\alpha))\}$ converges to λx_0 in the $\sigma(X, Z)$-topology and $x_0 \in U \cap S_Y$.

Then $\lambda(x_0 \pm (x - x_\alpha)) \in U \cap B_X$, whenever $\alpha \geq \alpha_0$ and $0 < \lambda < 1$ is close enough to 1. Hence

$$diam(U \cap B_X) \geq 2\lambda\|x - x_\alpha\| \geq 2\lambda\|x + Y\| > 2\lambda(1 - \varepsilon)$$

whenever $\alpha \geq \alpha_0$ and $0 < \lambda < 1$ are close enough to 1. Now, it is enough to take the limit when λ tends to 1 and ε to 0, to obtain that U has diameter 2. □

The following is the main result in the M-ideals setting, which improves the results in [23], where only the nonexistence of strongly exposed points is deduced with an extra hypothesis.

Theorem 2.4. *Let X be a Banach space and let Y be a closed and proper subspace of X. Assume that Y is an M-ideal of X (that is, there is an L-projection on X^* onto some subspace Z of X^*, with kernel Y°). If B_Z is weak-$*$ dense in B_{X^*}, then every nonempty relatively weakly open subset of B_X and B_Y has diameter 2.*

Proof. As B_Z is weak-$*$ dense in B_{X^*}, then $\|x\| = \sup_{z \in B_Z} |z(x)|\ \forall x \in X$, and so the norm of X is $\sigma(X, Z)$-lower semi-continuous.

Let U be a nonempty relatively weakly open subset of B_Y. Then, there are $z_1, \cdots, z_n \in Z$ and $y_0 \in B_Y$ such that

$$U_0 = \{y \in B_Y : |z_i(y - y_0)| < 1, 1 \leq i \leq n\} \subset U,$$

since the $\sigma(X,Z)$-topology on Y is just the weak topology of Y.

Setting $V=\{x\in B_X : |z_i(x-y_0)|<1, 1\leq i\leq n\}$, we have that V is a nonempty relatively $\sigma(X,Z)$ open subset of B_X intersecting B_Y. By Proposition 2.3, we obtain that $diam(V)=2$. Now, we claim that U_0 is dense in the topological space $(V,\sigma(X,Z))$.

Indeed, $X^{**}=Y^{\circ\circ}\oplus_\infty Z^\circ$; hence every $x\in B_X$ can be written as $x=u+v$ with $u\in B_{Y^{\circ\circ}}$ and $v\in B_{Z^\circ}$. There exists a net of elements $y_\alpha\in B_Y$ which converges to u in the weak-$*$ topology. Hence, for every $z\in Z$, we have $z(x)=(u+v)(z)=u(z)=\lim_\alpha z(y_\alpha)$. Now it is clear that the assumption $|z_i(x-y_0)|<1$, $1\leq i\leq n$, implies that, for some α_0 and $\alpha\geq\alpha_0$, $|z_i(y_\alpha-y_0)|<1$, $1\leq i\leq n$. That proves the $\sigma(X,Z)$-density of U_0 in V. Moreover, as the norm of X is $\sigma(X,Z)$-lower semi-continuous, we have that $diam(U_0)=2$ and so, $diam(U)=2$. Then we have proved that every nonempty relatively weakly open subset of B_Y has diameter 2.

As every nonempty relatively weak-$*$ open subset of $B_{Y^{**}}$ contains a nonempty relatively weakly open subset of B_Y, and B_Y is weak-$*$ dense in $B_{Y^{**}}$, we deduce that every nonempty relatively weak-$*$ open subset of $B_{Y^{**}}$ also has diameter 2. Now, as Y is an M-ideal of X, we have $X^*=Y^\circ\oplus_1 Z$ and then $X^{**}=Y^{\circ\circ}\oplus_\infty Z^\circ$, where, $\oplus_1$ and $\oplus_\infty$ denote the ℓ_1 and ℓ_∞ sum, respectively. Hence, by Lemma 2.2, we have that every nonempty relatively weak-$*$ open subset of $B_{X^{**}}$ has diameter 2, and then every nonempty relatively weakly open subset of B_X also has diameter 2, since from the weak-$*$ density of B_X in $B_{X^{**}}$, every nonempty relatively weakly open subset of B_X is weak-$*$ dense in some nonempty relatively weak-$*$ open subset of $B_{X^{**}}$ and the norm in X^{**} is weak-$*$ lower semi-continuous. □

For M-embedded Banach spaces X, we have $X^{***}=X^*\oplus_1 X^\circ$, and so we have automatically the weak-$*$ density of B_{X^*} in $B_{X^{***}}$. Then we obtain the following

Corollary 2.5. *Let X be a nonreflexive M-embedded Banach space, and let Y be a closed subspace of X^{**} containing X. Then every nonempty relatively weakly open subset of B_Y has diameter* 2.

Proof. If Y is a closed subspace of X^{**} containing X, then X is an M-ideal of Y. In fact, if p is the L-projection in X^{***} with kernel X° and image X^*, we identify Y^* with the quotient X^{***}/Y° and define $\pi: Y^*\to Y^*$ by $\pi(x^{***}+Y^\circ)=p(x^{***})+Y^\circ$. Now π is an L-projection whose kernel is the anihilator of X in Y^*.

Finally, as B_{X^*} is weak-$*$ dense in $B_{X^{***}}$, given $x^{***}\in X^{***}$ with $\|x^{***}+Y^\circ\|\leq 1$, we choose $y^\circ\in Y^\circ$ such that $\|x^{***}+y^\circ\|\leq 1$, and then there exists a net $\{z_\lambda\}$ of elements of B_{X^*} converging to $x^{***}+y^\circ$ in the weak-$*$ topology on X^{***}. Then $\{z_\lambda+Y^\circ\}$ is a net of elements of $B_{\pi(Y^*)}$ converging to $x^{***}+Y^\circ$ in the weak-$*$ topology on Y^*. Then we have proved that the closed unit ball of $\pi(Y^*)$ is weak-$*$ dense in B_{Y^*}. It is enough to apply Theorem 2.4 to obtain that every nonempty relatively weakly open subset of B_Y has diameter 2. □

Note that, since the property of being M-embedded is hereditary and stable by quotients, the same result is true when Y is a nonreflexive closed subspace of X or a nonreflexive quotient of X.

In particular, from the above corollary, we deduce that the unit closed ball of every closed and nonreflexive subspace of an M-embedded space has no continuity points and so has no strongly exposed points, and the same is true for nonreflexive quotients of an M-embedded spaces. Roughly speaking, this fact shows that every subspace and every quotient of an M-embedded space fails the Radon-Nikodym property in a very strong way, if it is not reflexive. As a consequence of the above corollary, it is worth mentioning some interesting examples. As c_0 is an M-embedded space, not only every infinite-dimensional subspace or quotient of c_0 verifies the conclusion of Corollary 2.5, but also every subspace of ℓ_∞ containing c_0. If H is a Hilbert space, and $K(H)$ and $L(H)$ stand for the space of compact operators on H and the space of all bounded operators on H, respectively, it is well known that $K(H)$ is an M-embedded space. Again, not only every subspace or quotient of $K(H)$ satisfies the conclusion of Corollary 2.5, but also every subspace of $L(H)$ containing $K(H)$, since $K(H)^{**}$, the bidual space of $K(H)$, is isometrically isomorphic to $L(H)$.

As the failure of the Radon-Nikodym property in nonreflexive M-embedded spaces is well known, since every nonreflexive M-embedded space contains an isomorphic copy of c_0, it is natural to ask for the behavior of relatively weakly open subsets of the unit ball of a Banach space containing c_0-copies. Of course, not every Banach space containing c_0-copies lacks a point of continuity in its unit ball. For this, it is enough to consider $X = c_0 \oplus_1 \ell_1$. It is easy to see that $(0, e_1)$ is a denting point of B_X, where e_1 denotes the first vector of the unit vector basis in ℓ_1. However the following result shows that, up to renorming, the above situation cannot happen.

Proposition 2.6. *Let X be a Banach space containing a subspace isomorphic to c_0. Then there exists an equivalent norm in X so that every nonempty relatively weakly open subset of the new unit ball has diameter* 2.

Proof. As the conclusion is isomorphic in nature, we can suppose that X contains a subspace Y isometric to c_0. Now, by [12, Proposition II.2.10], there exists an equivalent norm $|\ |$ on X which agrees with the original norm on Y so that Y becomes M-ideal in X.

Then we have $(X, |\ |^*)^{**} = Z \oplus_\infty Y^{\circ\circ}$ for some subspace Z of X^{**}. Finally, as $Y^{\circ\circ}$ is isometric to ℓ_∞ and every nonempty relatively weakly open subset of B_{ℓ_∞} has diameter 2, it is enough to apply Lemma 2.1 to obtain that every nonempty relatively weakly open subset of $(B_X, |\ |)$ has diameter 2. □

As every proper M-ideal, that is, an M-ideal which is not an M-summand, contains an isomorphic copy of c_0, we deduce the following

Corollary 2.7. *Let X be a proper M-ideal of a superspace Y. Then there is an equivalent norm in Y so that every nonempty relatively weakly open subset of the new unit ball of X and Y has diameter* 2.

Now we pass to study the size of nonempty relatively weakly open subsets of the unit ball of an L-embedded space. The result is the following

Theorem 2.8. *Let X be an L-embedded Banach space, that is, $X^{**} = X \oplus_1 Z$ for some subspace Z of X^{**}. If B_Z is weak-$*$ dense in $B_{X^{**}}$, then every nonempty relatively weakly open subset of B_X has diameter* 2.

Proof. Let U be a nonempty relatively weakly open subset of B_X. As X is infinite dimensional, there is an $x_0 \in U \cap S_X$ and then there exist $f_1, \cdots, f_n \in X^*$ such that

$$U_0 = \{x \in B_X : |f_i(x - x_0)| < 1,\ 1 \leq i \leq n\} \subset U.$$

Let $V = \{x^{**} \in B_{X^{**}} : |f_i(x^{**} - x_0)| < 1,\ 1 \leq i \leq n\}$. Then V is a nonempty relatively weak-$*$ open subset of $B_{X^{**}}$ such that $x_0 \in U_0 \subset V$.

From the weak-$*$ density of B_Z in $B_{X^{**}}$ we can choose a net $\{z_\lambda\}$ of elements in B_Z converging to x_0 in the weak-$*$ topology of X^{**}, hence there is a λ_0 such that $z_\lambda \in V$, whenever $\lambda \geq \lambda_0$. Furthermore, as the norm of X^{**} is weak-$*$ lower semi-continuous, we have $\liminf_\lambda \|z_\lambda\| \geq \|x_0\| = 1$. Then, given $\varepsilon > 0$ then there is a $\mu \geq \lambda_0$ such that $\|z_\mu\| > 1 - \varepsilon$, and $z_\mu \in V$. Now, we deduce

$$diam(V) \geq \|x_0 - z_\mu\| = \|x_0\| + \|z_\mu\| > 1 + 1 - \varepsilon = 2 - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude that $diam(V) = 2$.

Finally, from the weak-$*$ density of B_X in $B_{X^{**}}$ we obtain that U_0 is relatively dense in the topological space V endowed with the weak-$*$ topology of X^{**}. Now, the weak-$*$ lower semicontinuity of the norm in X^{**} allows us to assure that $diam(U) \geq diam(U_0) = diam(V) = 2$. □

In order to show examples where Theorem 2.8 works, we denote by H^1 and L_1 the Hardy space and the Lebesgue space on the interval $[0, 1]$. Also, H^1_0 stands for the subspace of H^1 of functions in H^1 vanishing at 0. From [1, p. 27], the unit ball of L_1/H^1_0 lacks extreme points, and it is well known that L_1/H^1_0 is an L-embedded space. Then we can apply Theorem 2.8 as in Theorem 2.12 to obtain the following

Corollary 2.9. *Every nonempty relatively weakly open subset of the unit ball of L_1/H^1_0 h as diameter 2.*

The same is true for L_1/H^1 instead L_1/H^1_0 since they are isometric.

Let X be a Banach space. For u in S_X, we define the **roughness of X at** u, $\eta(X, u)$, by the equality

$$\eta(X, u) \ := \ \limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|} \ .$$

We remark that the absence of roughness of X at u (i.e., $\eta(X, u) = 0$) is nothing other than the Fréchet differentiability of the norm of X at u [10, Lemma I.1.3]. Given $\epsilon > 0$, the Banach space X is said to be **ϵ-rough** if, for every u in S_X, we have $\eta(X, u) \geq \epsilon$. We say that X is **rough** whenever it is ϵ-rough for some $\epsilon > 0$, and **extremely rough** whenever it is 2-rough.

Assume that X is a Banach space such that every nonempty relatively weakly open subset of B_X has diameter 2. Then, by [10, Proposition I.1.11], the dual X^* of X (resp. the predual X_* of X, if this exists) is extremely rough.

Then, we have the following consequences (see Theorem 2.12 and Corollary 2.13 for part ii)):

Corollary 2.10. *The following Banach spaces are extremely rough:*

i) *The dual of any nonreflexive M-embedded space.*
ii) *The real atomless JBW*-triples, and so the atomless Von Neumann algebras.*

Corollary 2.11. *Every Banach space X containing an isomorphic copy of c_0 can be equivalently renormed so that X^* becomes extremely rough.*

In order to show a new application of Theorem 2.8 we introduce some notation.

We recall that a complex JB*-triple is a complex Banach space X with a continuous triple product $\{\cdots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables and conjugate-linear in the middle variable, and satisfies:

(1) For all $x \in X$, the map $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
(2) $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ for all $a, b, x, y, z \in X$.
(3) $\|\{xxx\}\| = \|x\|^3$ for all $x \in X$.

We also recall that a bounded linear operator T on a complex Banach space X is said to be hermitian if $\|exp(irT)\| = 1$ for every real number r.

Following [14], we define **real JB^*-triples** as norm-closed real subtriples of complex JB^*-triples. Here, by a **subtriple** we mean a subspace which is closed under triple products of its elements. A **triple ideal** of a real or complex JB^*-triple X is a subspace M of X such that $\{XXM\} + \{XMX\} \subseteq M$. **Real JBW^*-triples** where first introduced as those real JB^*-triples which are dual Banach spaces in such a way that the triple product becomes separately w^*-continuous (see [14, Definition 4.1 and Theorem 4.4]). Later, it was shown in [18] that the requirement of separate w^*-continuity of the triple product is superabundant.

Finally, we recall that an element x of a real JB*-triple E is said to be a tripotent if $\{xxx\} = x$. Given x, y tripotents in E, we say that x and y are orthogonal if $\{uuv\} = 0$ and we say that $x \geq y$ if $x - y$ and y are orthogonal tripotents. Then a minimal tripotent will be a tripotent which is minimal in the partial order defined above.

Examples of real JB^*-triples are the spaces $L(H, K)$, for arbitrary real, complex, or quaternionic Hilbert spaces H and K, under the triple product $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$. Also, the corresponding spaces of all symmetric, $S(H)$, and skew, $A(H)$, bounded linear operators on H can be considered real JB^*-triples. The above examples become particular cases of those arising by considering either the so-called complex Cartan factors (regarded as real JB^*-triples) or real forms of complex Cartan factors [16]. We recall that **real forms** of a complex Banach space X are defined as the real closed subspaces of X of the form $X^\tau := \{x \in X : \tau(x) = x\}$, for some conjugation (i.e., conjugate-linear isometry of period two) on X. We note that, if X is a complex JB^*-triple, then every real form of X is a real JB^*-triple (since conjugations on X preserve triple products [15]). Among complex Cartan factors, the so-called **complex spin factors** become especially relevant for our present approach. They are built from an arbitrary complex Hilbert space $(H, (\cdot|\cdot))$ of hilbertian dimension ≥ 3 by taking a conjugation σ on H, and then by defining the triple product and the norm by

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all x, y, z in H. Following [19], we say that a real JB^*-triple is a **generalized real spin factor** if it is either a complex spin factor (regarded as a real JB^*-triple) or a real form of a complex spin factor.

It is well known that every complex JBW*-triple E has a unique isometric predual V, and this is an L-embedded space. This is also the case for preduals of real JBW*-triples as done in [4].

Now we are ready to show the application of Theorem 2.8.

Theorem 2.12. *Let A be a real JBW*-triple and let A_* be its predual. If A is atomless, that is, A lacks minimal tripotentes, then every nonempty relatively weakly open subset of B_{A_*} has diameter* 2.

Proof. As A is atomless, according to [21, Corollary 2.1], we deduce that B_{A_*} lacks extreme points. Now, A_* is an L-embedded space by [4, Proposition 2.2]; then we have $A^* = A_* \oplus_1 Z$, for some subspace Z of A^*. Let us see that B_Z is weak*-dense in B_{A^*}.

As B_{A_*} lacks extreme points and the set of extreme points of B_{A^*} is the union of the sets of extreme point of B_{A_*} and B_Z, we obtain that $ext(B_{A^*}) = ext(B_Z)$, where $ext(K)$ denotes the set of extreme points of K. Now, the Krein-Milman theorem applied to B_{A^*} gives us that $B_{A^*} = \overline{co}^{w^*}(ext(B_Z))$, and then the desired conclusion.

Finally, it is enough to apply Theorem 2.8 to finish the proof. □

In the setting of C^*-algebras, the concept of mininal tripotents is exactly the well-known notion of minimal projections. As a Von Neumann algebra is also a real JBW*-triple, we obtain the following

Corollary 2.13. *Let A be an atomless Von Neumann algebra, that is, A lacks minimal projections, and A_* stands for its predual. Then every nonempty relatively weakly open subset of B_{A_*} has diameter* 2.

Finally, we show a characterization of Radon-Nikodym property in the setting of the preduals of real JBW^*-triples and, as a consequence, in the setting of the preduals of Von Neumann algebras, too.

Theorem 2.14. *Let A be real JBW^*-triple and A_* its predual. Then:*

i) *A_* satisfies the Radon-Nikodym property if, and only if, A is purely atomic, that is, A is the weak-$*$ closed linear span of its minimal tripotents. Furthermore, in this case, A is the ℓ_∞-sum of weak-$*$ closed simple ideals which are either finite-dimensional, infinite-dimensional generalized real spin factors or of the form $L(H,K)$, $S(H)$ or $A(H)$ for some real, complex or quaternionic infinite-dimensional Hilbert spaces H and K.*

ii) *A_* fails the Radon-Nikodym property if, and only if, A_* can be equivalently renormed so that every nonempty relatively weakly open subset of B_{A_*} has diameter* 2.

iii) *A_* satisfies the Radon-Nikodym property if, and only if, A_* satisfies the Krein-Milman property.*

Proof. i) Assume that A_* verifies the Radon-Nikodym property. By [21, Theorem 3.6], we have $A = B \oplus_\infty C$, where B and C are weak* closed real triple ideals of A, such that B is purely atomic and C is atomless. By the above corollary $C = 0$, since the Radon-Nikodym property is hereditary. Then $A = B$ is purely atomic.

Now, in order to describe the preduals of real JBW* triples satisfiyng the Radon-Nikodym property, assume that A is purely atomic. We denote by $\hat{A} = A \oplus iA$ the complexification of A. By [21], $\hat{A}$ is a purely atomic complex JBW* triple and then, by [11], $\hat{A}$ is the ℓ_∞ sum of type I Cartan factors, that is, the ℓ_∞ sum of w*-closed simple ideals which are either finite-dimensional, infinite-dimensional complex spin factors or of the form $L(H,K)$, $S(H)$ or $A(H)$ for some complex Hilbert spaces H and K. Taking into account that the conjugation τ preserves the triple product and is w*-continuous, it is enough to apply [16] to deduce that A is the ℓ_∞ sum of w*-closed simple ideals which are either finite-dimensional, infinite-dimensional generalized real spin factors or of the form $L(H,K)$, $S(H)$ or $A(H)$ for some real, complex or quaternionnic Hilbert spaces H and K. Finally, as the Radon-Nikodym property is stable by ℓ_1 sums and the preduals of the above spaces satisfy the Radon-Nikodym property (see [7]) we deduce that A_* verifies the Radon-Nikodym property.

ii) If A_* fails the Radon-Nikodym property, as in the above paragraph, we set $A = B \oplus_\infty C$. Now, as B and C are w^*-closed real triple ideals of A, we have $A_* = D \oplus_1 E$, where E is the predual of the atomless real JBW^*-triple C. By the above corollary, every nonempty relatively weakly open subset of B_E has diameter 2. Now it is enough to apply Lemma 2.1, to see that A_* can be equivalently renormed so that every nonempty relatively weakly open subset of B_{A_*} has diameter 2. The converse implication is trivial.

iii) It is well known that every Banach space satisfying the Radon-Nikodym property also verifies the Krein-Milman property. In order to prove the converse, assume that A_* fails the Radon-Nikodym property. Then, by i), A is not purely atomic. Now, as in ii), $A = B \oplus_\infty C$, where B and $C \neq 0$ are weak* closed real triple ideals of A, such that B is purely atomic and C is atomless. Then, $A_* = D \oplus_1 E$, where E is the predual of the atomless real JBW^*-triple C. By [21, Corollary 2.1], we deduce that B_E lacks extreme points and hence A_* fails the Krein-Milman property. □

Corollary 2.15. *Let A be a Von Neumann algebra and A_* its predual. Then:*

i) *A_* satisfies the Radon-Nikodym property if, and only if, A is purely atomic. Furthermore, in this case, there exists $\{H_i\}$ a family of infinite-dimensional complex Hilbert spaces, such that $A = \ell_\infty - \sum_i L(H_i)$ and $A_* = \ell_1 - \sum_i N(H_i)$, where $N(H)$ denotes the space of all nuclear operators on H.*

ii) *A_* fails the Radon-Nikodym property if, and only if, A_* can be equivalently renormed so that every nonempty relatively weakly open subset of B_{A_*} has diameter 2.*

iii) *A_* satisfies the Radon-Nikodym property if, and only if, A_* satisfies the Krein-Milman property.*

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