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UPPER ESTIMATES FOR THE ENERGY OF SOLUTIONS OF NONHOMOGENEOUS BOUNDARY VALUE PROBLEMS

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Abstract. We establish upper bounds for the energy of critical levels of the functional associated to a perturbed superlinear elliptic boundary value problem. We show that the perturbed problem satisfies the estimates obtained by Bahri and Lions (1988) for the symmetric problem. We use these estimates to prove the existence of nonradial solutions to a radial elliptic boundary value problem. Our results fill a gap in an earlier paper by Aduén and Castro.

1. Introduction and statement of results

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 3$, and let $H \equiv H_0^1(\Omega)$ be the Sobolev space of square integrable functions in Ω having first-order partial derivatives in $L^2(\Omega)$. In [3] A. Bahri and P.L. Lions proved that for $p \in (2, \frac{2N-2}{N-2})$ and $f \in C^0(\overline{\Omega})$ the elliptic problem

$$(\wp) \qquad \begin{cases} -\Delta u = |u|^{p-2}\, u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a sequence of solutions $\{u_k\}$ which satisfy

$$J(u_k) \geq C_1 k^\gamma$$

for some positive constant C_1 and $\gamma = \frac{2p}{N(p-2)}$, where $J : H \to \mathbb{R}$ is the functional defined by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega fu.$$

The proofs in [3] are based on ideas developed in [2] combined with the semiclassical inequality due to M. Cwickel [6], E.H. Lieb [10] and Rosenbljum [11]. Actually, in [2] the existence of the u_k's was established for further restricted values of p. The arguments in [2] and [3] rely on the analysis of the symmetric case $f = 0$ for which J has a sequence (c_k) of critical values which satisfy

$$C_2 k^\gamma \leq c_k \leq C_3 k^\gamma. \tag{1.1}$$

If the boundary condition in $(\wp)$ is replaced by $u = u_0$, with $u_0 \in C^2(\overline{\Omega})$ and $\Delta u_0 = 0$, the existence of an unbounded sequence of solutions (u_k) was established

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by P. Bolle, N. Ghoussoub and H. Tehrani in [5] under the additional restriction $p < \frac{2N}{N-1}$. In this case problem ($\wp$) is equivalent to problem

$$(\wp') \qquad \begin{cases} -\Delta u = |u+u_0|^{p-2}(u+u_0)+f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the solutions of ($\wp'$) are the critical points of the functional $J : H \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u+u_0|^p - \int_\Omega fu.$$

Here we prove that (1.1) also holds for the nonsymmetric problem ($\wp'$). Namely, we prove the following.

Theorem 1.1. *If $u_0 = 0$ and $p < \frac{2N-2}{N-2}$, or if $u_0 \neq 0$ and $p < \frac{2N}{N-1}$, then J has an unbounded sequence of critical values $(\widetilde{c}_k)$ which satisfy*

$$\widetilde{c}_k \leq Ck^\gamma, \tag{1.2}$$

where C is a positive constant and $\gamma = 2p/N(p-2)$.

Theorem 1.1 fills a gap in the proof of Theorem 3 in [1] where (1.2) was used. As an application of the existence of exactly two radial solutions of problem ($\wp'$) with k nodal domains for large enough k (see proof of Theorem 3 in [1]) we obtain the following.

Theorem 1.2. *If Ω is a ball or an annulus, u_0 is a constant function, $f = 0$, and $p < \frac{2N}{N-1}$, then problem ($\wp'$) has infinitely many nonradial solutions.*

The reader is refered to [8, 9], and references therein, for recent results on the existence of nonsymmetric solutions for symmetric problems.

The proof of Theorem 1.1 requires some precise knowledge of the topology of the sublevel sets of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p$$

associated to the symmetric problem. Let

$$I^c = \{u \in H : I(u) \leq c\}$$

and let $\pi_k(I^c, I^0 \setminus \{0\})$ be the k-th relative homotopy group (or homotopy set if $k = 1$) with any base point in $I^0 \setminus \{0\}$ (for the definition of these groups see for example [12, Chapter 7]). We shall prove the following.

Theorem 1.3. *There exist $\alpha, \beta > 0$, depending only on Ω and p, such that for every $c \geq 0$ the homomorphism induced by the inclusion*

$$\pi_k(I^c, I^0 \setminus \{0\}) \to \pi_k(I^{\alpha c+\beta}, I^0 \setminus \{0\})$$

is trivial for all $k \geq 1$.

In other words, given a map $\varphi : \mathbb{B}^k \to I^c$ such that $\varphi(\mathbb{S}^{k-1}) \subset I^0 \setminus \{0\}$, there exists a homotopy $\Theta : \mathbb{B}^k \times [0,1] \to I^{\alpha c+\beta}$ such that $\Theta(x,0) = \varphi(x)$ for all $x \in \mathbb{B}^k$ and $\Theta(\mathbb{S}^{k-1} \times [0,1] \cup \mathbb{B}^k \times \{1\}) \subset I^0 \setminus \{0\}$.

Here $\mathbb{B}^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$ is the unit ball and $\mathbb{S}^{k-1} = \{x \in \mathbb{R}^k : |x| = 1\}$ is the unit sphere in euclidean k-space $\mathbb{R}^k$.

This paper is organized as follows. In section 2 we derive Theorem 1.1 from Theorem 1.3 and apply it to prove Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.3.

2. Critical values of perturbed symmetric functionals

We start by recalling a critical point result due to Bolle, Ghoussoub and Tehrani.

Let X be an infinite-dimensional Hilbert space and let $\Phi : X \times [0,1] \to \mathbb{R}$ be a C^2-functional. We think of Φ as being a path of functionals

$$\Phi_t : X \to \mathbb{R}, \quad \Phi_t(u) = \Phi(u,t), \quad 0 \le t \le 1,$$

and denote by $\Phi_t'(u) = \frac{\partial}{\partial u}\Phi(u,t)$ the derivative of Φ_t. Assume that Φ has the following properties.

(**P1**) Every sequence $(u_n, t_n) \in X \times [0,1]$ such that $(\Phi_{t_n}(u_n))$ is bounded and $\|\Phi_{t_n}'(u_n)\| \to 0$ has a convergent subsequence.

(**P2**) For every $b \in \mathbb{R}$ there is a constant C such that

$$\left|\frac{\partial}{\partial t}\Phi(u,t)\right| \le C(\|\Phi_t'(u)\| + 1)(\|u\| + 1) \qquad \text{if } |\Phi_t(u)| \le b.$$

(**P3**) There exist two continuous functions $\theta_1, \theta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$, $\theta_1 \le \theta_2$, which are Lipschitz continuous on the second variable and such that

$$\theta_1(t, \Phi_t(u)) \le \frac{\partial}{\partial t}\Phi(u,t) \le \theta_2(t, \Phi_t(u)) \quad \text{if } \Phi_t'(u) = 0.$$

(**P4**) Φ_0 is even and, for every finite-dimensional subspace W of X,

$$\sup_{0 \le t \le 1} \Phi_t(w) \to -\infty \qquad \text{as } w \in W,\ \|w\| \to \infty.$$

Fix a sequence of linear subspaces $X_1 \subset \cdots \subset X_k \subset \cdots$ of X with $\dim X_k = k$, and define

$$c_k = \inf_{\varphi \in \Gamma} \sup_{x \in X_k} \Phi_0(\varphi(x))$$

where

$$\Gamma = \{\varphi \in C(X,X) : \varphi \text{ is odd and } \exists R > 0 \text{ such that } \varphi(x) = x \text{ for } \|x\| > R\}.$$

Let $\zeta_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, be defined by

$$\begin{cases} \zeta_i(0,s) = s, \\ \frac{\partial}{\partial t}\zeta_i(t,s) = \theta_i(t, \zeta_i(t,s)). \end{cases} \tag{2.1}$$

The following result was proved in [5] (see Proof of Theorem 2.2).

Theorem 2.1 (Bolle, Ghoussoub, Tehrani)**.** *Assume that Φ satisfies* (P1)-(P4)**.** *If $\zeta_2(1, c_k + \varepsilon) < \zeta_1(1, c_{k+1})$ for some $\varepsilon > 0$, then for every $\varphi \in \Gamma$ such that $\sup_{\varphi(X_k)} \Phi_0 < c_k + \varepsilon$ there is a critical value $\widetilde{c}_k$ of Φ_1 which satisfies*

$$\zeta_2(1, c_k) < \zeta_1(1, c_{k+1}) \le \widetilde{c}_k \le \zeta_2(1, \sup_{x \in X_{k+1}} \Phi_0(\varphi(x))). \tag{2.2}$$

Moreover, if the sequence

$$\left(\frac{c_{k+1} - c_k}{\max_{0 \le t \le 1} |\theta_1(t, c_{k+1})| + \max_{0 \le t \le 1} |\theta_2(t, c_k)| + 1} \right) \tag{2.3}$$

is unbounded, then $(\widetilde{c}_k)$ is unbounded.

We shall apply this result to the path of functionals

$$I_t(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u + tu_0|^p - t\int_\Omega fu, \qquad u \in H_0^1(\Omega).$$

Note that $I_0 = I$ and $I_1 = J$. We write

$$\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{1/2}, \qquad |u|_p = \left(\int_\Omega |u|^p\right)^{1/p}$$

for the norm in the Sobolev space $H_0^1(\Omega)$ of square integrable functions in Ω having first-order partial derivatives in $L^2(\Omega)$, and the norm in $L^p(\Omega)$ respectively. The following corollary of Theorem 1.3 will be proved in the following section.

Corollary 2.2. *There are constants $\alpha, \beta > 0$, depending only on Ω and p, with the following property: For every pair of finite-dimensional subspaces $V \subset W$ of $H_0^1(\Omega)$ with $\dim W = \dim V + 1$, every odd map $\varphi : V \to H_0^1(\Omega)$ and every $R > 0$ such that $\varphi(v) = v$ if $\|v\| \geq R$, there is an $\widetilde{R} \geq R$ and an odd map $\widetilde{\varphi} : W \to H_0^1(\Omega)$ which satisfies:*

(i) $\widetilde{\varphi}(v) = \varphi(v)$ *for every $v \in V$,*
(ii) $\widetilde{\varphi}(w) = w$ *for every $w \in W$ with $\|w\| \geq \widetilde{R}$,*
(iii) $\max_{w\in W} I(\widetilde{\varphi}(w)) \leq \alpha \max_{v\in V} I(\varphi(v)) + \beta$.

We apply this result to prove Theorem 1.1.

Proof of Theorem 1.1. Bolle, Ghoussoub and Tehrani [5] showed that I_t satisfies properties (P1)-(P4) with $\theta_2(t,s) = A(s^2+1)^{1/4} = -\theta_1(t,s)$ if $u_0 \neq 0$. If $u_0 = 0$ it is easy to see that I_t satisfies these properties with $\theta_2(t,s) = A(s^2+1)^{1/2p} = -\theta_1(t,s)$. Let $X_k \subset H_0^1(\Omega)$ be the space spanned by the first k Dirichlet eigenfunctions of $-\Delta$ and let

$$c_k = \inf_{\varphi\in\Gamma} \sup_{u\in X_k} I(\varphi(u))$$

where $\Gamma = \{\varphi \in C(H_0^1(\Omega), H_0^1(\Omega)) : \varphi$ is odd, $\varphi(u) = u$ for $\|u\|$ large enough$\}$. Using the estimate

$$C_1 k^\gamma \leq c_k \tag{2.4}$$

with $\gamma = 2p/N(p-2)$ [3, 13], Bolle, Ghoussoub and Tehrani [5] showed that the sequence (2.3) is unbounded, provided $u_0 = 0$ and $p < \frac{2N-2}{N-2}$, or $u_0 \neq 0$ and $p < \frac{2N}{N-1}$. Hence J has an unbounded sequence of critical values in those cases. If $\zeta_2(1, c_k + \varepsilon) < \zeta_1(1, c_{k+1})$ for some $0 < \varepsilon < 1$, choose $\varphi \in \Gamma$ such that $\sup I(\varphi(X_k)) < c_k + \varepsilon$ and apply Corollary 2.2 to obtain an odd map $\widetilde{\varphi} : X_{k+1} \to H_0^1(\Omega)$ such that $\widetilde{\varphi}(u) = \varphi(u)$ for $u \in X_k$, $\widetilde{\varphi}(u) = u$ for $\|u\| > R$, and

$$\sup I(\widetilde{\varphi}(X_{k+1})) \leq \alpha(c_k + \varepsilon) + \beta < \alpha c_k + \delta.$$

By Tietze's extension theorem $\widetilde{\varphi}$ can be extended to an odd map $\widetilde{\varphi} : H_0^1(\Omega) \to H_0^1(\Omega)$ which satisfies $\widetilde{\varphi}(u) = u$ for $\|u\| > R$. It follows from inequality (2.2) in Theorem 2.1 that J has a critical value $\widetilde{c}_k$ which satisfies

$$\widetilde{c}_k \leq \zeta_2(1, \sup I(\widetilde{\varphi}(X_{k+1}))) \leq \zeta_2(1, \alpha c_k + \delta).$$

By definition of ζ_2 (see (2.1)),

$$|s - \zeta_2(t,s)| \leq C_1 |\theta_2(s)| \leq C_2(s^2+1)^{1/4}.$$

On the other hand, Bahri and Lions [3] showed that

$$c_k \leq C_2 k^\gamma$$

with $\gamma = 2p/N(p-2)$. It follows that

$$\widetilde{c}_k \leq Ck^\gamma$$

for some positive constant C. □

Remark 2.3. Note that, since $\zeta_2(t,c)$ is nondecreasing in t, inequalities (2.2) and (2.4) immediately imply that

$$C' k^\gamma \leq \widetilde{c}_k$$

for some positive constant C'.

Proof of Theorem 1.2. From the proof of Theorem 3 in [1] it follows that problem $(\wp')$ has at most two radial solutions with k interior nodal hypersurfaces, for each k large enough. Theorem 1.1 above, together with Theorem 1 in [1] (or Theorem 4 in [7]), imply that $(\wp')$ has infinitely many nonradial solutions. □

3. Extensions of odd maps with linear energy estimates

The purpose of this section is to prove Theorem 1.3. We split the proof into several lemmas.

Lemma 3.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Then there exists $a \in \mathbb{R}$ with the following properties:*
(i) *There exists $x' \in \mathbb{R}^{N-1}$ such that $(x', a) \in \Omega$.*
(ii) *If $(x', b) \in \Omega$ and $b \geq a$, then $(x', t) \in \Omega$ for all $a \leq t \leq b$.*

Proof. Let $e_N = (0, ..., 0, 1) \in \mathbb{R}^N$, let $M = \max\{x \cdot e_N : x \in \overline{\Omega}\}$ and let $K = \{x \in \overline{\Omega} : x \cdot e_N = M\}$. Then K is a compact subset of $\partial\Omega$. Let $\nu : \partial\Omega \to \mathbb{R}^N$ be the outer unit normal vector field, and let $\mathcal{O} = \{x \in \partial\Omega : \nu(x) \cdot e_N > 0\}$. Then $\mathcal{O}$ is an open neighborhood of K in $\partial\Omega$ and, since K is compact, there is an $a < M$ such that the set $A = \{x \in \partial\Omega : x \cdot e_N \geq a\} \subset \mathcal{O}$. Thus, for every $(x', t) \in A$ there exists $\varepsilon > 0$ such that $(x', s) \notin \Omega$ if $t < s < t + \varepsilon$ and $(x', s) \in \Omega$ if $t - \varepsilon < s < t$. It follows that, for every $(x', t) \in A$,

$$(\{x'\} \times [a, M]) \cap \partial\Omega = \{(x', t)\} \quad \text{and} \quad \{x'\} \times [a, t) \subset \Omega$$

as claimed. □

Set

$$I^\#(u) = 2\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

Lemma 3.2. *There exists an even continuous function $\tau : H_0^1(\Omega) \to [0, \infty)$ with the following properties:*
(i) $I([(1-s) + s\tau(u)]\,u) \leq I(u)$ *for every $u \in H_0^1(\Omega)$, $0 \leq s \leq 1$.*
(ii) *If* $I^\#(u) \leq 0$, *then* $\tau(u) = 1$.
(iii) *If* $2I(u) \leq \max_{t \geq 0} I(tu)$, *then* $I^\#(\tau(u)u) \leq 0$.
(iv) $I^\#(\tau(u)u) \leq \max\{\alpha I(u), 0\}$ *with* $\alpha = 2^{(3p-2)/(p-2)}$.

Proof. For every $v \in H_0^1(\Omega)$ with $\|v\| = 1$, define $0 < t_v^- < \widehat{t}_v < t_v^+ < T_v < \infty$ as follows:

$$\begin{aligned} I(\widehat{t}_v v) &= \max_{t \geq 0} I(tv), \\ 2I(tv) &\geq \max_{t \geq 0} I(tv) \Longleftrightarrow t \in [t_v^-, t_v^+], \\ 2(T_v)^2 &= \frac{1}{p} |v|_p^p (T_v)^p. \end{aligned}$$

For $t \geq 0$ set

$$\rho(tv) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_v^-, \\ \widehat{t}_v (t - t_v^-)/(\widehat{t}_v - t_v^-) & \text{if } t_v^- \leq t \leq \widehat{t}_v, \\ (T_v - \widehat{t}_v)(t - \widehat{t}_v)/(t_v^+ - \widehat{t}_v) + \widehat{t}_v & \text{if } \widehat{t}_v \leq t \leq t_v^+, \\ T_v & \text{if } t_v^+ \leq t \leq T_v, \\ t & \text{if } T_v \leq t \end{cases}$$

and, for $u = tv \in H_0^1(\Omega)$ with $\|v\| = 1$, $t \geq 0$, set

$$\tau(u) = \frac{\rho(tv)}{t}.$$

Then $\rho(tv) \leq t$ if $0 \leq t \leq \widehat{t}_v$, and $\rho(tv) \geq t$ if $\widehat{t}_v \leq t$. Therefore (i) holds. Property (ii) follows immediately from the definition. If $2I(tv) \leq \max_{s \geq 0} I(su) = I(\widehat{t}_v v)$, then either $0 \leq t \leq t_v^-$ or $t_v^+ \leq t$. In the first case $\rho(tv) = 0$. In the second case $\rho(tv) \geq T_v$. Hence, in both cases,

$$2 \|\tau(u) u\|^2 = 2 \|\rho(tv) v\|^2 \leq \frac{1}{p} |\rho(tv) v|_p^p = \frac{1}{p} |\tau(u) u|_p^p.$$

This proves (iii). Finally, it is easy to see that

$$\max_{t \geq 0} I^{\#}(tu) = 2^{2p/(p-2)} \max_{t \geq 0} I(tu).$$

Hence

$$I^{\#}(\tau(u) u) \leq 2^{(3p-2)/(p-2)} I(u) \quad \text{if} \quad \max_{t \geq 0} I(tu) \leq 2I(u).$$

This, combined with (iii), yields (iv). □

For any subspace V of $H_0^1(\Omega)$ and any $R > 0$ we write

$$B_R V = \{v \in V : \|v\| \leq R\}, \qquad S_R V = \{v \in V : \|v\| = R\}.$$

Lemma 3.3. *Let V be a finite-dimensional subspace of $H_0^1(\Omega)$ and let $R > 0$ be such that $I(v) \leq 0$ for those $v \in V$ with $\|v\| \geq R$. Then, for every map $\varphi : B_R V \to I^c$ such that $c \geq 0$ and $\varphi(S_R V) \subset I^0 \setminus \{0\}$, there exists a homotopy $\Theta : B_R V \times [0,1] \to I^c$ such that*

(i) $\Theta(v, 0) = \varphi(v)$ *for every* $v \in B_R V$,
(ii) $\Theta(z, 1) = z$ *for every* $z \in S_R V$,
(iii) $\Theta(S_R V \times [0,1]) \subset I^0 \setminus \{0\}$.

Proof. Since $I^0 \setminus \{0\}$ is homotopy equivalent to the unit sphere in $H_0^1(\Omega)$, it is contractible. Hence there is a homotopy $\Psi : S_R V \times [0,1] \to I^0 \setminus \{0\}$ with $\Psi(z, 0) = \varphi(z)$ and $\Psi(z, 1) = z$. Define $\vartheta : B_{R+1} V \to I^c$ as follows:

$$\vartheta(v) = \begin{cases} \varphi(v) & \text{if } \|v\| \leq R, \\ \Psi(R \frac{v}{\|v\|}, \|v\| - R) & \text{if } R \leq \|v\| \leq R + 1. \end{cases}$$

Now define $\Theta : B_R V \times [0,1] \to I^c$ by

$$\Theta(v,t) = \vartheta\left(\frac{R+t}{R}v\right).$$

It is easy to see that Θ has the desired properties. □

Lemma 3.4. *There exist $\alpha, \beta > 0$, depending only on Ω and p, with the following property: Given a finite-dimensional subspace V of $H_0^1(\Omega)$, a map $\varphi : V \to H_0^1(\Omega)$ and an $R > 0$ such that $\varphi(v) = v$ if $\|v\| \geq R$, there exist a map $\psi : V \times [0,\infty) \to H_0^1(\Omega)$ and an $R' \geq R$ which satisfy:*
(i) $\psi(v,0) = \varphi(v)$ *for every* $v \in V$,
(ii) $\psi(v,r) \in I^0 \setminus \{0\}$ *if* $\|v\| \geq R'$ *or* $r \geq R'$,
(iii) $I(\psi(v,r)) \leq \alpha \max\{I(\varphi(v)), 0\} + \beta$ *for every* $v \in V$, $r \geq 0$.

Proof. Let a be as in Lemma 3.1. We may assume without loss of generality that $a = 0$. Let $\tau : H_0^1(\Omega) \to \mathbb{R}$ be as in Lemma 3.2. We write $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \equiv \mathbb{R}^N$. Given $u \in H_0^1(\Omega)$ and $0 \leq s \leq 2$ define

$$u_s(x) = \begin{cases} [(1-s) + s\tau(u)]u(x) & \text{if } 0 \leq s \leq 1, \\ \tau(u)u(x', sx_N) & \text{if } x_N \geq 0,\ 1 \leq s \leq 2, \\ \tau(u)u(x', x_N) & \text{if } x_N \leq 0,\ 1 \leq s \leq 2. \end{cases}$$

Here $u(x',t) = 0$ if $(x',t) \notin \Omega$. Then $u_s \in H_0^1(\Omega)$ for every $0 \leq s \leq 2$, and

$$\int_\Omega |\nabla u_s|^2 \leq s \int_\Omega |\nabla(\tau(u)u)|^2 \quad \text{and} \quad \int_\Omega |u_s|^p \geq s^{-1} \int_\Omega |\tau(u)u|^p \qquad \text{if } 1 \leq s \leq 2.$$

Hence

$$sI(u_s) \leq \frac{s^2}{2} \int_\Omega |\nabla(\tau(u)u)|^2 - \frac{1}{p} \int_\Omega |\tau(u)u|^p \qquad \text{if } 1 \leq s \leq 2.$$

This inequality and Lemma 3.2 yield

$$I(u_s) \leq \begin{cases} I(u) & \text{if } 0 \leq s \leq 1, \\ \max\{I^\#(\tau(u)u), 0\} & \text{if } 1 \leq s \leq 2. \end{cases} \tag{3.1}$$

Let $\Omega_2 = \{(x', \frac{1}{2}x_N) : x \in \Omega,\ x_N \geq 0\} \cup \{x \in \Omega : x_N \leq 0\}$. Since $\Omega_2 \subsetneq \Omega$ because of Lemma 3.1, we may choose $\omega \in C_c^\infty(\Omega \setminus \Omega_2)$, $\omega \neq 0$, with

$$\int_\Omega |\nabla \omega|^2 = \int_\Omega |\omega|^p. \tag{3.2}$$

Given a map $\varphi : V \to H_0^1(\Omega)$ and an $R > 0$ such that $\varphi(v) = v$ if $\|v\| \geq R$, define $\psi : V \times [0,\infty) \to H_0^1(\Omega)$ by

$$\psi(v,r) = \begin{cases} \varphi(v)_r, & v \in V,\ 0 \leq r \leq 2, \\ \varphi(v)_2 + (r-2)\omega, & v \in V,\ 2 \leq r. \end{cases}$$

Then ψ is an extension of φ, and (3.1) yields

$$I(\psi(v,r)) \leq \begin{cases} I(\varphi(v)) & \text{if } 0 \leq r \leq 1, \\ \max\{I^\#(\tau(\varphi(v))\varphi(v)), 0\} & \text{if } 1 \leq r \leq 2, \\ \max\{I^\#(\tau(\varphi(v))\varphi(v)), 0\} + I((r-2)\omega) & \text{if } 2 \leq r. \end{cases} \tag{3.3}$$

The third inequality follows because $\varphi(v)_2$ and ω have disjoint supports. Hence, Lemma 3.2 yields

$$I(\psi(v,r)) \leq \alpha \max\{I(\varphi(v)), 0\} + \beta \quad \text{for every } v \in V,\ r \geq 0,$$

with $\alpha = 2^{(3p-2)/(p-2)}$ and $\beta = I(\omega)$. Finally, let $R' > R$ be such that

$$\begin{cases} I(v) \le 0, \quad \tau(v) = 1, \quad \text{and} \quad I^{\#}(v) \le -I(\omega) & \text{if } v \in V, \ \|v\| \ge R', \\ I((r-2)\omega) \le -\max_{v \in V} I^{\#}(\tau(\varphi(v))\varphi(v)) & \text{if } r \ge R'. \end{cases}$$

Since $\varphi(v) = v$ if $\|v\| \ge R' \ge R$, and since $I(t\omega) \le I(\omega)$ for every $t \ge 0$ by (3.2), it follows that

$$I(\psi(v,r)) \le 0 \quad \text{if} \quad \|v\| \ge R' \quad \text{or} \quad r \ge R'.$$

□

Proof of Theorem 1.3. Let $\sigma : \mathbb{B}^k \to I^c$ be a map such that $\sigma(\mathbb{S}^{k-1}) \subset I^0 \setminus \{0\}$. Composing this map with a homeomorphism $\vartheta : B_R V \cong \mathbb{B}^k$ and applying Lemma 3.3, we may assume that $\sigma \circ \vartheta$ is the restriction to $B_R V$ of a map $\varphi : V \to I^c$ defined on a k-dimensional subspace V of $H_0^1(\Omega)$ which satisfies $\varphi(v) = v$ if $\|v\| \ge R$. Let $\psi : V \times [0,\infty) \to H_0^1(\Omega)$ and an $R' \ge R$ be as in Lemma 3.4. Define $\Theta : B_R V \times [0,1] \to H_0^1(\Omega)$ by

$$\Theta(v,s) = \begin{cases} \varphi([(1-2s) + 2s\frac{R'}{R}]v) & \text{if } \ 0 \le s \le \frac{1}{2}, \\ \psi(\frac{R'}{R}v, (2s-1)R') & \text{if } \ \frac{1}{2} \le s \le 1. \end{cases}$$

Then $\Theta(v,0) = \varphi(v)$ for all $v \in B_R V$, $\Theta(S_R V \times [0,1] \cup B_R V \times \{1\}) \subset I^0 \setminus \{0\}$, and

$$I(\Theta(v,s)) \le \alpha c + \beta \qquad \text{for all } (v,s) \in B_R V \times [0,1]$$

as claimed. □

Proof of Corollary 2.2. For the given φ and R, let $\psi : V \times [0,\infty) \to H_0^1(\Omega)$ and $R' > R$ be as in Lemma 3.4. Fix $e \in W$ orthogonal to V with $\|e\| = 1$, and extend φ to the half space $W^+ = \{v + re : v \in V, \, r \ge 0\}$ by

$$\overline{\varphi}(v + re) = \psi(v,r) \qquad \text{if } v \in V, \ r \ge 0.$$

Then there exists $R'' \ge R'$ such that $\overline{\varphi}(w) \in I^0 \setminus \{0\}$ for all $w \in W^+$ with $\|w\| = R''$. Since $I^0 \setminus \{0\}$ is homotopy equivalent to the unit sphere in $H_0^1(\Omega)$, it is contractible. Hence, there is a homotopy

$$\Psi : \{w \in W^+ : \|w\| = R''\} \times [0,1] \to I^0 \setminus \{0\}$$

such that $\Psi(w,0) = \overline{\varphi}(w)$, $\Psi(w,1) = w$, and $\Psi(v,t) = v$ for $v \in V$, $t \in [0,1]$. Let $\widetilde{R} = R'' + 1$ and define

$$\widetilde{\varphi}(w) = \begin{cases} \overline{\varphi}(w) & \text{if } \ w \in W^+, \ \|w\| \le R'', \\ \frac{\|w\|}{R''}\Psi(R''\frac{w}{\|w\|}, \|w\| - R'') & \text{if } \ w \in W^+, \ \ R'' \le \|w\| \le \widetilde{R}, \\ w & \text{if } \ w \in W^+, \ \ \widetilde{R} \le \|w\|, \\ -\widetilde{\varphi}(-w) & \text{if } \ -w \in W^+. \end{cases}$$

Since φ is odd, $\widetilde{\varphi}$ is well-defined and it is, by definition, an odd extension of φ to W which satisfies $\widetilde{\varphi}(w) = w$ if $\|w\| \ge \widetilde{R}$. Note that $ru \in I^0 \setminus \{0\}$ if $u \in I^0 \setminus \{0\}$ and $|r| \ge 1$. Hence $I(\widetilde{\varphi}(w)) \le 0$ if $\|w\| \ge R''$ and, by Lemma 3.4,

$$I(\widetilde{\varphi}(v + re)) = I(\psi(v,|r|)) \le \alpha \max\{I(\varphi(v)), 0\} + \beta \qquad \text{if } \|v + re\| \le R'',$$

as claimed. □

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