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MULTIPLICITIES OF REPRESENTATIONS IN SPACES OF MODULAR FORMS

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Abstract. This paper shows that for a given irreducible representation ρ of Γ/Γ_1, the two functions $\dim(M_k(\Gamma_1,\rho))$ and $\dim(S_k(\Gamma_1,\rho))$ of k are almost linear functions.

Let Γ and Γ_1 be two Fuchsian subgroups of $\mathrm{SL}_2(\mathbb{Q})$ of the first kind, with Γ_1 a normal subgroup of Γ of index μ. Let $\mathbb{H}^*$ denote the extended upper half-plane as in Shimura [4] and put $Y = \Gamma\backslash\mathbb{H}^*$ and $X = \Gamma_1\backslash\mathbb{H}^*$. Let $\Omega(X)$ and $\Omega(Y)$ be the fields of meromorphic functions of X and Y respectively. Then $A_0(\Gamma) \cong \Omega(Y)$ and $A_0(\Gamma_1) \cong \Omega(X)$ in a natural way, where $A_k(*)$ denotes the space of meromorphic modular forms for $*$ of weight k. We define a representation π_k of Γ on $A_k(\Gamma_1)$ by the formula

$$\pi_k(\gamma)(f) = f|[\gamma^{-1}]_k \qquad (\gamma \in \Gamma, \quad f \in A_k(\Gamma_1)),$$

where the notation $[*]_k$ is as in [4]. The representation π_k factors through Γ/Γ_1 to give a representation —also denoted π_k—of the latter group. The space $M_k(\Gamma_1)$ of holomorphic modular forms of weight k for Γ_1 and the subspace $S_k(\Gamma_1)$ of cusp forms of weight k for Γ_1 are both stable under π_k, and the resulting representations of Γ/Γ_1 on $M_k(\Gamma_1)$ and $S_k(\Gamma_1)$ will be denoted ρ_k and σ_k respectively.

Henceforth, ρ denotes an irreducible complex representation of Γ/Γ_1. If $-I \in \Gamma$, then the value of ρ on the coset of $-I$ in Γ/Γ_1 is a scalar by Schur's lemma, and we say that ρ is *even* or *odd* according to whether the scalar is 1 or -1. If $-I \in \Gamma_1$, then ρ is automatically even.

If π is any finite-dimensional complex representation of Γ/Γ_1, then we write $\langle\rho,\pi\rangle$ for the multiplicity of ρ in π. For example, $\langle\rho,\rho_{\mathrm{reg}}\rangle = \dim\rho$, where ρ_{reg} is the regular representation of Γ/Γ_1.

Theorem. *Fix an irreducible complex representation ρ of Γ/Γ_1, and put*

$$c = \frac{1}{4\pi}\int_{\Gamma\backslash\mathbb{H}} \frac{dxdy}{y^2}.$$

If $-I \notin \Gamma$, then

$$\lim_{k\to\infty} \frac{\langle\rho,\rho_k\rangle}{k\langle\rho,\rho_{\mathrm{reg}}\rangle} = \lim_{k\to\infty} \frac{\langle\rho,\sigma_k\rangle}{k\langle\rho,\rho_{\mathrm{reg}}\rangle} = c.$$

If $-I \in \Gamma$, then the same assertion holds provided k runs through positive integers of the same parity as ρ.

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Proof. We shall prove our assertion only for ρ_k. The proof for σ_k is similar.

For any positive integer k, let $i(k) = (-1)^k$. If $-I \in \Gamma_1$, we assume k is always even.

Let $\mathcal{S}$ be the set of the irreducible representations of Γ/Γ_1 with the same parity as k if $-I \in \Gamma$, and the set of all irreducible representations of Γ/Γ_1 otherwise.

Let p be a non-cusp point of Y which has μ different liftings $p_1, ..., p_\mu$ on X, and let $\tilde{p}_j$ be a lifting of p_j in $\mathbb{H}^*$ $(1 \leq j \leq \mu)$. By the Riemann-Roch theorem we know that there exists an $f \in A_{i(k)}(\Gamma_1)$ such that f has poles only at cusps and $f(\tilde{p}_1) = 1$, $f(\tilde{p}_j) = 0$ for all $j = 2, \ldots, \mu$. For any $\alpha \in \Gamma/\Gamma_1$, put $f_\alpha = f|[\alpha]_{i(k)}$. For a fixed sufficiently large even n_0, we may choose $\varphi = \varphi_{n_0} \in S_{n_0}(\Gamma)$ such that φf_α $(\alpha \in \Gamma/\Gamma_1)$ are all cusp forms. Let W denote the $\mathbb{C}$-linear subspace of $M_{i(k)+n_0}(\Gamma_1)$ spanned by $\{\varphi f_\alpha | \alpha \in \Gamma/\Gamma_1\}$. Clearly, W is stable under $\pi_{i(k)+n_0}$. It is easy to see that

$$W \cong \bigoplus_{\rho \in \mathcal{S}} \langle \rho, \rho_{\mathrm{reg}} \rangle \rho$$

since the two-hand sides have the same trace.

When $k \geq i(k) + n_0$, the choice of f ensures that the natural map

$$W \otimes_{\mathbb{C}} M_{k-i(k)-n_0}(\Gamma) \to M_k(\Gamma_1)$$

is injective. Hence,

$$\frac{\langle \rho, \rho_k \rangle}{k \langle \rho, \rho_{\mathrm{reg}} \rangle} \geq \frac{\dim(M_{k-i(k)-n_0}(\Gamma))}{k}. \tag{1}$$

For ρ and k as in the theorem, in [4] Shimura showed that

$$\lim_{k \to \infty} \frac{\dim(M_{k-i}(\Gamma))}{k} = c \tag{2}$$

and

$$\lim_{k \to \infty} \frac{\dim(M_k(\Gamma_1))}{k} = c\mu. \tag{3}$$

From (1) and (2) we easily deduce that

$$\liminf_{k \to \infty} \frac{\langle \rho, \rho_k \rangle}{k} \geq c \langle \rho, \rho_{\mathrm{reg}} \rangle. \tag{4}$$

It is obvious that (3) is equivalent to

$$\lim_{k \to \infty} \sum_{\rho \in \mathcal{S}} \dim(\rho) \frac{\langle \rho, \rho_k \rangle}{k} = \sum_{\rho \in \mathcal{S}} \dim(\rho)\ c \langle \rho, \rho_{\mathrm{reg}} \rangle.$$

Combining this equality with (4), we get

$$\lim_{k \to \infty} \frac{\langle \rho, \rho_k \rangle}{k} = c \langle \rho, \rho_{\mathrm{reg}} \rangle,$$

as desired. □

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