PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 1, Pages 129–138
S 0002-9939(05)08344-9
Article electronically published on August 16, 2005

STABILITY OF THE FIXED POINT PROPERTY IN HILBERT SPACES

EVA MARÍA MAZCUÑÁN-NAVARRO

(Communicated by Jonathan M. Borwein)

ABSTRACT. In this paper we prove that if X is a Banach space whose Banach-Mazur distance to a Hilbert space is less than $\sqrt{\frac{5+\sqrt{17}}{2}}$, then X has the fixed point property for nonexpansive mappings.

1. Introduction

Let $(X, \|.\|)$ be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* whenever $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$.

We shall say that a nonempty subset C of a Banach space has the *fixed point property for nonexpansive mappings* if each nonexpansive mapping $T : C \to C$ has a fixed point. A Banach space has the *fixed point property* (FPP for short) if each of its nonempty bounded closed convex subsets has the fixed point property for nonexpansive mappings, and the *weak fixed point property* ($wc-FPP$) if each of its nonempty weakly compact convex subsets has the fixed point property for nonexpansive mappings.

In 1971 Zizler [10] showed that every separable Banach space admits an equivalent renorming which is uniformly convex in every direction and consequently has the $wc-FPP$ [3]. In 1981 Alspach [2] proved that the separable Banach space $(L^1[0,1], \|.\|_1)$ lacks the $wc-FPP$. As a consequence of Zizler's and Alspach's results it was known that, in general, the $wc-FPP$ is not preserved under topological isomorphisms.

Despite of this fact, there are results establishing that the $wc-FPP$ extends from Banach spaces with nice geometric properties to Banach spaces whose Banach-Mazur distance to them is small enough. We recall that the Banach-Mazur distance between two isomorphic Banach spaces X and Y is

$$d(X,Y) = \inf\{\|U\|\|U^{-1}\| : U : X \to Y \text{ linear isomorphism}\}.$$

Given a Banach space X with the $wc-FPP$ (respectively the FPP), the problem of determining a number $k > 1$ such that any isomorphic Banach space Y with $d(X,Y) < k$ has the $wc-FPP$ (resp. FPP) is known as the stability problem of the $wc-FPP$ (resp. FPP). A wide survey of the recent developments in the theory of stability of the FPP can be found in Chapter 7 of [6].

Received by the editors December 17, 2003.
2000 *Mathematics Subject Classification.* Primary 47H10; Secondary 46B20.

In this paper we shall concentrate our attention on the stability problem of the FPP in Hilbert spaces. In [7] Pei-Kee Lin proved the following theorem.

Theorem 1.1. *Let H be a Hilbert space. If X is a Banach space isomorphic to H such that*

$$d(X,H) < \sqrt{\frac{5+\sqrt{13}}{2}},$$

then X has the FPP.

The aim of this paper is to prove that the conclusion of the above theorem remains true under the weaker hypothesis $d(X,H) < \sqrt{\frac{5+\sqrt{17}}{2}}$.

2. Preliminaries

Let $(X, \|.\|)$ be a Banach space, let C be a nonempty bounded closed convex subset of X and let $T : C \to C$ be a nonexpansive mapping.

It is known that there is a sequence (x_n) in C such that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. Such a sequence is called an *approximate fixed point sequence* (*afps* in short) for T.

If in addition C is weakly compact, then the family

$$\mathcal{F} = \{K \subset C : K \text{ nonempty, closed, convex, } T(K) \subset K\}$$

has a minimal element (with respect to set-inclusion). We shall say that a set K is *minimal for T* if it is a minimal element of $\mathcal{F}$.

A key result in order to establish the $wc - FPP$ is the *Goebel-Karlovitz Lemma* [5], [4].

Lemma 2.1 (Goebel-Karlovitz)**.** *Let K be a nonempty weakly compact convex subset of a Banach space $(X, \|.\|)$ which is minimal for a nonexpanxive mapping $T : K \to K$. If (x_n) is an afps for T in K, then*

$$\lim_{n\to\infty} \|x - x_n\| = \text{diam}(K)$$

for all $x \in K$.

We need to recall some facts about convergence over ultrafilters and the ultrapower of a Banach space. For details we refer to [1], [9] or Chapter 6 of [6].

Throughout this paper $\mathcal{U}$ will denote a proper nontrivial ultrafilter on $\mathbb{N}$.

Recall that, if (Ω, τ) is a Hausdorff topological space and (x_n) is a sequence in Ω, it is said that $(\tau - \lim_{n,\mathcal{U}})x_n = x_0$ if for every $\tau-$neighborhood V of x_0 we have $\{n \in \mathbb{N} : x_n \in V\} \in \mathcal{U}$.

The *ultrapower* of the Banach space $(X, \|.\|)$—with respect to $\mathcal{U}$—is defined to be the quotient Banach space

$$\tilde{X} = \ell_\infty(X)/\mathcal{N}(X),$$

where

$$\ell_\infty(X) = \{(x_n) \in X^{\mathbb{N}} : \sup_{n\geq 1} \|x_n\| < \infty\}$$

and

$$\mathcal{N}(X) = \{(x_n) \in \ell_\infty(X) : \lim_{n,\mathcal{U}} \|x_n\| = 0\}.$$

The equivalence class of $(x_n) \in \ell_\infty(X)$ is denoted by $\widetilde{(x_n)}$.

If $\|.\|_{\tilde{X}}$ stands for the quotient norm of $\tilde{X}$ we have that

$$\|\widetilde{(x_n)}\|_{\tilde{X}} = \lim_{n,\mathcal{U}} \|x_n\|.$$

Whenever it does not lead to confusion, we will omit the subindex and still denote by $\|.\|$ the norm of the ultrapower.

Given a subset K of X, $\tilde{K}$ will denote the subset of $\tilde{X}$ defined by

$$\tilde{K} = \{\widetilde{(x_n)} : x_n \in K \text{ for each } n \in \mathbb{N}\},$$

and given a nonexpansive mapping $T : K \to K$, we will consider the associated nonexpansive mapping $\tilde{T} : \tilde{K} \to \tilde{K}$ defined by $\tilde{T}(\widetilde{(x_n)}) = \widetilde{(T(x_n))}$.

The following generalization of the Goebel-Karlovitz Lemma, due to P.K. Lin [8], is basic for establishing the $wc - FPP$ by using ultrapower methods.

Lemma 2.2 (Lin)**.** *Let K be a nonempty weakly compact convex subset of a Banach space $(X, \|.\|)$ which is minimal for a nonexpansive mapping $T : K \to K$. If $(\tilde{w}^n)$ is an afps for $\tilde{T}$ in $\tilde{K}$, then*

$$\lim_{n\to\infty} \|\tilde{w}^n - x\| = \operatorname{diam}(K)$$

for all $x \in K$.

In particular, if W is a nonempty closed convex and $\tilde{T}$-invariant subset of $\tilde{K}$, then

$$\sup\{\|\tilde{w} - x\| : \tilde{w} \in W\} = \operatorname{diam}(K)$$

for all $x \in K$.

Let us now consider a Hilbert space $(H, \|.\|)$. It is well known that any ultrapower $\tilde{H}$ of H is also a Hilbert space.

On the other hand, it is also well known that if (x_n) is a weakly null sequence in H, then for each $x \in H$ we have that

$$\limsup_{n\to\infty} \|x_n - x\|^2 = \limsup_{n\to\infty} \|x_n\|^2 + \|x\|^2 \tag{2.1}$$

and that the equality is also true if $\limsup_{n\to\infty}$ is replaced by $\liminf_{n\to\infty}$.

The following result shows that an analogous identity holds when dealing with weak convergence along ultrafilters.

Lemma 2.3. *Let (x_n) be a bounded sequence in a Hilbert space $(H, \|.\|)$ with $(w - \lim_{n,\mathcal{U}})x_n = 0$ and denote $\tilde{x} = \widetilde{(x_n)}$. Then, for any $x \in H$,*

$$\|\tilde{x} - x\|^2 = \|x\|^2 + \|\tilde{x}\|^2.$$

Proof. Let us denote by $\langle .,. \rangle$ the inner product of H.

Since, for any $n \in \mathbb{N}$, $\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\langle x, x_n\rangle$, taking limits over the ultrafilter, we have that

$$\begin{aligned} \|\tilde{x} - x\|^2 &= \lim_{n,\mathcal{U}} \|x_n - x\|^2 = \lim_{n,\mathcal{U}} \|x_n\|^2 + \|x\|^2 - 2\lim_{n,\mathcal{U}} \langle x, x_n\rangle \\ &= \|\tilde{x} - x\|^2 + \|x\|^2 - 2\lim_{n,\mathcal{U}} \langle x, x_n\rangle. \end{aligned}$$

So, the proof is finished if we see that $\lim_{n,\mathcal{U}}\langle x, x_n\rangle = 0$, that is, for any $\varepsilon > 0$ $\{n \in \mathbb{N} : |\langle x, x_n\rangle| < \varepsilon\} \in \mathcal{U}$.

Let $\varepsilon > 0$. Consider the function $f_x : H \to \mathbb{R}$ defined by $f_x(y) := \langle x, y\rangle$. Since $f_x \in H^*$, the set $V := \{y \in H : |f_x(y) - f_x(0)| < \varepsilon\} = \{y \in H : |\langle x, y\rangle| < \varepsilon\}$

is a neighbourhood of 0 for the weak topology. Consequently, provided that $(w-\lim_{n,\mathcal{U}})x_n = 0$, we have that $\{n \in \mathbb{N} : x_n \in V\} = \{n \in \mathbb{N} : |\langle x, x_n\rangle| < \varepsilon\} \in \mathcal{U}$, as required. □

3. The stability result

Let $(H, \|.\|)$ be a Hilbert space and let $|.|$ be an equivalent norm in H satisfying that, for any $x \in H$,

$$\|x\| \leq |x| \leq B\|x\|.$$

The aim is to prove that if $B < \sqrt{\frac{5+\sqrt{17}}{2}}$, then $(H, |.|)$ has the FPP. Therefore, by Theorem 1.1, we can assume $B \geq \sqrt{\frac{5+\sqrt{13}}{2}}$.

In $\tilde{H}$, the ultrapower of H with respect to $\mathcal{U}$, we shall consider the norms associated to $\|.\|$ and $|.|$, that is, the norms respectively defined by $\|\tilde{x}\| = \lim_{n,\mathcal{U}} \|x_n\|$ and $|\tilde{x}| = \lim_{n,\mathcal{U}} |x_n|$ for $\tilde{x} = \widetilde{(x_n)} \in \tilde{H}$.

In the statement of the following lemmas we shall assume that K is a nonempty weakly compact convex subset of H, with $\operatorname{diam}(K) = 1$ and which is minimal for a $|.|$-nonexpansive mapping $T : K \to K$.

Let (x_n) be an afps for T in K and set $\tilde{x} = \widetilde{(x_n)}$. Let $r \in (0, 1)$. Associated to these elements, we define the subset of $\tilde{K}$

$$\tilde{W}(\tilde{x}, r) = \{\tilde{w} \in \tilde{K} : |\tilde{w} - \tilde{x}| \leq r, D_{|.|}(\tilde{w}) \leq 1 - r\},$$

where $D_{|.|}(\tilde{w}) = \lim_{n,\mathcal{U}} \lim_{m,\mathcal{U}} |w_n - w_m|$, if $\tilde{w} = \widetilde{(w_n)}$.

It is not difficult to check that $(1-r)\tilde{x} + rx \in \tilde{W}(\tilde{x}, r)$ for any $x \in K$, and that $\tilde{W}(\tilde{x}, r)$ is a closed convex $\tilde{T}$-invariant set.

Let $\tilde{w} = \widetilde{(w_n)} \in \tilde{W}(\tilde{x}, r)$. Since K is weakly compact, we can consider $w_0 = (w - \lim_{n,\mathcal{U}})w_n$, and we have

$$|\tilde{w} - w_0| \leq D_{|.|}(\tilde{w}) \leq 1 - r. \tag{3.1}$$

On the other hand, by the Goebel-Karlovitz lemma, $\lim_{n\to\infty} |x_n - x_0| = 1$ and therefore

$$|\tilde{w} - w_0 - (\tilde{x} - x_0)| \geq |\tilde{x} - x_0| - |\tilde{w} - w_0| \geq 1 - (1 - r) = r,$$

so

$$\|\tilde{w} - w_0 - (\tilde{x} - x_0)\| \geq \frac{r}{B}. \tag{3.2}$$

We also define the number

$$D(K) = \inf\{\limsup_{n\to\infty} \|y_n - y\| : (y_n) \text{ is an afps for } T \text{ in } K, y_n \xrightarrow{w} y\}$$

(note that $D(K) \leq \operatorname{diam}(K) < +\infty$).

Lemma 3.1. *Let (x_n) be an afps for T in K and denote $\tilde{x} = \widetilde{(x_n)}$. Let $r \in (0, 1)$. For each $x \in K$ and each $\varepsilon > 0$ there exists $\tilde{w} \in \tilde{W}(\tilde{x}, r)$ such that $|\tilde{w} - x| \geq 1 - \varepsilon$ and $\|\tilde{w} - x\| \geq D(K) - \varepsilon$.*

Proof. Suppose that there exist $x \in K$ and $\varepsilon > 0$ for which, if $\tilde{w} \in \tilde{W}(\tilde{x}, r)$ satisfies $|\tilde{w} - x| \geq 1 - \varepsilon$, then $\|\tilde{w} - x\| < D(K) - \varepsilon$.

Since $\tilde{W}(\tilde{x}, r)$ is closed convex and $\tilde{T}$-invariant, it contains an afps for $\tilde{T}$, $(\widetilde{w}^m)$, that is, a sequence for which, if $\widetilde{w}^m = \widetilde{(w_n^m)}$, then

$$\lim_{m\to\infty} \lim_{n,\mathcal{U}} \|w_n^m - Tw_n^m\| = 0. \tag{3.3}$$

From Lin's Lemma,

$$\lim_{m\to\infty} |\tilde{w}^m - x| = 1,$$

so there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, $|\tilde{w}^m - x| \geq 1 - \varepsilon$.

From the previous inequality, according to our assumption, for all $m \geq m_0$, we must have

$$\lim_{n,\mathcal{U}} \|w_n^m - x\| = \|\tilde{w}^m - x\| < D(K) - \varepsilon. \tag{3.4}$$

Fix $k \in \mathbb{N}$. By (3.3) we can find m_k, as large as desired. In particular $m_k \geq m_0$, for which

$$\lim_{n,\mathcal{U}} \|w_n^{m_k} - T(w_n^{m_k})\| < \frac{1}{k},$$

and so $A_k = \{n \in \mathbb{N} : \|w_n^{m_k} - T(w_n^{m_k})\| < \frac{1}{k}\} \in \mathcal{U}$.

On the other hand, by (3.4), the set $B_k = \{n \in \mathbb{N} : \|w_n^{m_k} - x\| < D(K) - \varepsilon\}$ is also a member of $\mathcal{U}$. Therefore, $A_k \cap B_k$ is an infinite set, and the existence of $n_k \in A_k \cap B_k$ as large as desired is thus guaranteed.

According to this reasoning, we can construct two strictly increasing sequences of natural numbers, (n_k) and (m_k), for which, defining $z_k := w_{n_k}^{m_k}$, we have that, for any $k \geq 1$,

$$\|z_k - Tz_k\| < \frac{1}{k} \text{ and } \|z_k - x\| < D(K) - \varepsilon.$$

Hence, the sequence (z_k) so defined, is an afps for T satisfying that, for each $k \geq 1$,

$$\|z_k - x\| < D(K) - \varepsilon.$$

We can assume, by passing to subsequences if necessary, that (z_k) converges weakly, say $z_k \xrightarrow{w} z$.

Then so, by (2.1), we obtain that

$$\begin{aligned} \limsup_{k\to\infty} \|z_k - z\| &\leq \limsup_{k\to\infty} \|z_k - x\| \\ &\leq D(K) - \varepsilon < D(K). \end{aligned}$$

But this contradicts the definition of $D(K)$, so the proof is complete. □

In the proof of the next result we will use the following generalization of the parallelogram law: Given $x, y \in H$ and given $\lambda \in (0,1)$, we have the identity

$$\begin{aligned} &\|\lambda x + (1-\lambda)y\|^2 \\ &= \langle \lambda x + (1-\lambda)y, \lambda x + (1-\lambda)y \rangle \\ &= \lambda^2\|x\|^2 + 2\lambda(1-\lambda)\langle x, y\rangle + (1-\lambda)^2\|y\|^2 \\ &= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 + \lambda(1-\lambda)(2\langle x, y\rangle - \langle x, x\rangle - \langle y, y\rangle) \\ &= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \end{aligned}$$

Lemma 3.2. *The inequality*

$$D(K)^2 \geq \frac{B^2-1}{B^2(B^2-2)}$$

holds.

Proof. Consider three parameters $\lambda \in (0,1)$, $r \in (0,1)$ and $\varepsilon \in (0,r)$.

By the definition of $D(K)$, K contains a weakly convergent afps for T, (x_n), satisfying that, if x_0 is its weak limit, then

$$\limsup_{n\to\infty} \|x_n - x_0\| \leq D(K) + \varepsilon,$$

so, setting $\tilde{x} = \widetilde{(x_n)}$, we have

$$\|\tilde{x} - x_0\| \leq D(K) + \varepsilon. \tag{3.5}$$

Applying Lemma 3.1, we can guarantee the existence of $\tilde{w} \in \tilde{W}(\tilde{x}, r)$ with

$$|\tilde{w} - x_0| \geq 1 - \varepsilon \tag{3.6}$$

and

$$\|\tilde{w} - x_0\| \geq D(K) - \varepsilon. \tag{3.7}$$

Put $\tilde{w} = \widetilde{(w_n)}$. Since K is weakly compact, we can consider $w_0 = (w{-}\lim_{n,\mathcal{U}})w_n$.

From (3.1) and (3.6), we obtain that

$$\begin{aligned} |w_0 - x_0| &= |(\tilde{w} - x_0) - (\tilde{w} - w_0)| \\ &\geq |\tilde{w} - x_0| - |\tilde{w} - w_0| \\ &\geq (1-\varepsilon) - (1-r) = r - \varepsilon \end{aligned}$$

and consequently

$$\|w_0 - x_0\| \geq \frac{r-\varepsilon}{B}. \tag{3.8}$$

Since $\tilde{W}(\tilde{x}, r)$ is convex, $\lambda\tilde{w} + (1-\lambda)((1-r)\tilde{x} + rx_0) \in \tilde{W}(\tilde{x}, r)$. Moreover, $(w - \lim_{n,\mathcal{U}})(\lambda w_n + (1-\lambda)((1-r)x_n + rx_0)) = \lambda w_0 + (1-\lambda)x_0$, so from (3.2),

$$\begin{aligned} &\|\lambda(\tilde{w} - w_0) + (1-\lambda)(1-r)(\tilde{x} - x_0) - (\tilde{x} - x_0)\| \\ &= \|(\lambda\tilde{w} + (1-\lambda)((1-r)\tilde{x} + rx_0)) - (\lambda w_0 + (1-\lambda)x_0) - (\tilde{x} - x_0)\| \geq \frac{r}{B}. \end{aligned} \tag{3.9}$$

Now, since

$$(w - \lim_{n,\mathcal{U}})(\lambda(w_n - w_0) + (1-\lambda)(1-r)(x_n - x_0) - (x_n - x_0)) = 0,$$

by Lemma 2.3, we have

$$\begin{aligned} &\|\lambda\tilde{w} + (1-\lambda)((1-r)\tilde{x} + rx_0) - \tilde{x}\|^2 \\ &= \|\lambda(\tilde{w} - w_0) + (1-\lambda)(1-r)(\tilde{x} - x_0) - (\tilde{x} - x_0)\|^2 + \lambda^2\|w_0 - x_0\|^2, \end{aligned}$$

so, from (3.8) and (3.9), we obtain

$$\|\lambda\tilde{w} + (1-\lambda)((1-r)\tilde{x} + rx_0) - \tilde{x}\|^2 \geq \frac{1}{B^2}(r^2 + \lambda^2(r-\varepsilon)^2). \tag{3.10}$$

By the generalized parallelogram law, the identities

$$\begin{aligned}&\|\lambda\tilde{w}+(1-\lambda)((1-r)\tilde{x}+rx_0)-\tilde{x}\|^2\\ &= \|\lambda(\tilde{w}-\tilde{x})+(1-\lambda)r(x_0-\tilde{x})\|^2\\ &= \lambda\|\tilde{w}-\tilde{x}\|^2+(1-\lambda)r^2\|\tilde{x}-x_0\|^2-\lambda(1-\lambda)\|\tilde{w}-(1-r)\tilde{x}-rx_0\|^2\end{aligned}$$

and

$$\begin{aligned}&\|\tilde{w}-(1-r)\tilde{x}-rx_0\|^2\\ &= \|(1-r)(\tilde{w}-\tilde{x})+r(\tilde{w}-x_0)\|^2\\ &= (1-r)\|\tilde{w}-\tilde{x}\|^2+r\|\tilde{w}-x_0\|^2-r(1-r)\|\tilde{x}-x_0\|^2\end{aligned}$$

hold, which put together lead to

$$\begin{aligned}&\|\lambda\tilde{w}+(1-\lambda)((1-r)\tilde{x}+rx_0)-\tilde{x}\|^2\\ &= \lambda(1-(1-\lambda)(1-r))\|\tilde{w}-\tilde{x}\|^2\\ &\quad +(1-\lambda)r(r+\lambda(1-r))\|\tilde{x}-x_0\|^2-\lambda(1-\lambda)r\|\tilde{w}-x_0\|^2.\end{aligned}$$

Taking into account (3.5), (3.7), (3.10) and the fact that, provided that $\tilde{w}\in\tilde{W}(\tilde{x},r)$, we have $\|\tilde{w}-\tilde{x}\|\le|\tilde{w}-\tilde{x}|\le r$, we get

$$\begin{aligned}\frac{r^2+\lambda^2(r-\varepsilon)^2}{B^2} &\le \lambda(1-(1-\lambda)(1-r))r^2\\ &+ (1-\lambda)r(r+\lambda(1-r))(D(K)+\varepsilon)^2\\ &\quad -\lambda(1-\lambda)r(D(K)-\varepsilon)^2.\end{aligned}$$

Letting ε go to 0, we obtain

$$\frac{r^2(1+\lambda^2)}{B^2}\le\lambda(1-(1-\lambda)(1-r))r^2+(1-\lambda)^2r^2D(K)^2,$$

from where, simplifying r^2 first and then taking limits as $r\to 0$,

$$\frac{1+\lambda^2}{B^2}\le\lambda^2+(1-\lambda)^2D(K)^2,$$

so

$$D(K)^2\ge\frac{\frac{1+\lambda^2}{B^2}-\lambda^2}{(1-\lambda)^2}.$$

For $\lambda=\frac{1}{B^2-1}$ (note that $\lambda\in(0,1)$ since we are assuming $B\ge\sqrt{\frac{5+\sqrt{13}}{2}}>\sqrt{2}$), we finally get

$$D(K)^2\ge\frac{B^2-1}{B^2(B^2-2)}.$$

□

Lemma 3.3. *Let (x_n) be an afps for T in K such that $x_n\stackrel{w}{\to}x_0$ and denote $\tilde{x}:=\widetilde{(x_n)}$. Then, for any $r\in(0,1)$, the inequality*

$$\sup\{\|\tilde{w}-x_0\|^2:\tilde{w}\in\tilde{W}(\tilde{x},r)\}\le\frac{(1-r)^2}{2}+r^2\Big(1-\frac{1}{B^2}\Big)$$

holds.

Proof. Let $\tilde{w} = \widetilde{(w_n)}$ be a member of $\tilde{W}(\tilde{x}, r)$. Since K is weakly compact we can consider $w_0 = (w - \lim_{n,\mathcal{U}})w_n$.

Given that $(w - \lim_{n,\mathcal{U}})(w_n - w_0 - (x_n - x_0)) = 0$, an application of Lemma 2.3 leads to

$$\|\tilde{w} - \tilde{x}\|^2 = \|w_0 - x_0\|^2 + \|\tilde{w} - w_0 - (\tilde{x} - x_0)\|^2. \tag{3.11}$$

So, taking into account (3.2) and that

$$\|\tilde{w} - \tilde{x}\| \leq |\tilde{w} - \tilde{x}| \leq r,$$

we obtain

$$\|w_0 - x_0\|_2^2 \leq r^2\Big(1 - \frac{1}{B^2}\Big). \tag{3.12}$$

Since $(w - \lim_{n,\mathcal{U}})(w_n - w_0) = 0$, again applying Lemma 2.3, we have that, for all $m \geq 1$,

$$\|\tilde{w} - w_m\|^2 = \|w_0 - w_m\|^2 + \|\tilde{w} - w_0\|^2,$$

so, taking limits along the ultrafilter, we get

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{U}} \|w_n - w_m\|^2 = 2\|\tilde{w} - w_0\|^2$$

and consequently

$$\begin{aligned} \|\tilde{w} - w_0\|^2 &= \frac{1}{2} \lim_{n,\mathcal{U}} \lim_{m,\mathcal{U}} \|w_n - w_m\|^2 \\ &\leq \frac{1}{2} D_{|.|}(\tilde{w})^2 \leq \frac{(1-r)^2}{2}. \end{aligned} \tag{3.13}$$

By Lemma 2.3 once more,

$$\|\tilde{w} - x_0\|^2 = \|w_0 - x_0\|^2 + \|\tilde{w} - w_0\|^2.$$

Inequalities (3.12) and (3.13), along with this last equation, lead to

$$\|\tilde{w} - z_0\|^2 \leq r^2\Big(1 - \frac{1}{B^2}\Big) + \frac{(1-r)^2}{2}$$

which completes the proof. □

Theorem 3.4. *If $B < \sqrt{\frac{5+\sqrt{17}}{2}}$, then $(H, |.|)$ has the FPP.*

Proof. Assume that $(H, |.|)$ does not have the FPP. Then, by an standard argument, we can suppose that there exists a weakly compact convex subset K of H, with $\operatorname{diam}_{|.|}(K) = 1$, which is minimal for a $|.|$-nonexpansive mapping $T : K \to K$ and contains a weakly null afps for T, (x_n).

From Lemmas 3.1, 3.2 and 3.3, we have that, for any $r \in (0, 1)$,

$$\frac{B^2 - 1}{B^2(B^2 - 2)} \leq D(K)^2 \leq \sup\{\|\tilde{w}\|^2 : \tilde{w} \in \tilde{W}(\tilde{x}, r)\} \leq \frac{(1-r)^2}{2} + r^2(1 - \frac{1}{B^2}).$$

In particular, for $r = \frac{B^2}{3B^2 - 2}$, it follows that

$$\frac{B^2 - 1}{B^2(B^2 - 2)} \leq \frac{B^2 - 1}{3B^2 - 2},$$

from where

$$B^4 - 5B^2 + 2 \geq 0$$

and in consequence

$$B \geq \sqrt{\frac{5+\sqrt{17}}{2}},$$

which is the contradiction derived from assuming that $(H, |.|)$ does not have the FPP. □

Corollary 3.5. *Let H be a Hilbert space. If X is a Banach space such that*

$$d(X, H) < \sqrt{\frac{5+\sqrt{17}}{2}},$$

then X has the FPP.

Proof. Provided that $d(X, H) < \sqrt{\frac{5+\sqrt{17}}{2}}$, we can find $B > 1$ such that

$$d(X, H) < B < \sqrt{\frac{5+\sqrt{17}}{2}}.$$

By the definition of the Banach-Mazur distance, there exists a linear isomorphism $U : X \to H$ for which

$$\|U\|\|U^{-1}\| < B.$$

The formula

$$|x| = \|U\|\|U^{-1}x\|$$

defines an equivalent norm in H satisfying

$$\|x\| \leq |x| \leq B\|x\|$$

for all $x \in H$.

Since $B < \sqrt{\frac{5+\sqrt{17}}{2}}$, due to the previous results, we know that $(H, |.|)$ has the FPP. From this, it is easy to conclude that X also enjoys the FPP. □

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Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjassot, Valencia, Spain

E-mail address: `Eva.M.Mazcunan@uv.es`

Current address: Departamento de Matemáticas, Escuela de Ingenierías Industrial e Informática, Universidad de León, Campus de Vegazana, 24071 León, Spain

E-mail address: `dememn@unileon.es`