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A DEFORMATION OF QUATERNIONIC HYPERBOLIC SPACE

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Abstract. We construct a continuous family of new homogeneous Einstein spaces with negative Ricci curvature, obtained by deforming from the quaternionic hyperbolic space of real dimension 12. We give an explicit description of this family, which is made up of Einstein solvmanifolds which share the same algebraic structure (eigenvalue type) as the rank one symmetric space $\mathbf{H}H^3$. This deformation includes a continuous family of new homogeneous Einstein spaces with negative sectional curvature.

1. Introduction

The goal of this paper is the construction of new examples of homogeneous Einstein manifolds of negative Ricci curvature. The classical examples of Einstein manifolds of negative Ricci curvature are the symmetric spaces of non-compact type. Other examples can be found in [Al], [D], [GP-SV], [P-S], and [Wt]; most recently in Fanaï [F] and Lauret [L] (see [Hb] for more extensive references). The question of when a manifold carries an Einstein metric is still not well understood. As a system of PDE, the Einstein condition in its full generality is intractible, and examples are not plentiful. The strategy we use here for obtaining examples is via symmetries, considering homogeneous manifolds. It is known that for any homogeneous Einstein manifold M with negative Ricci curvature, M and its transitive Lie group G must be non-compact [Be, p. 190].

It was conjectured by Alekseevskii in 1975 [Al] that any homogeneous Einstein manifold with negative Ricci curvature has a maximal compact isotropy subgroup; the conjecture is still open. Under this assumption, the homogeneous space is a *Riemannian solvmanifold*, a solvable Lie group with a left-invariant metric. In fact, all known examples of homogeneous Einstein manifolds of negative Ricci curvature are Riemannian solvmanifolds of *Iwasawa type* (defined in Section 2).

In recent work of Jens Heber [Hb], he explores the questions of existence and uniqueness for Einstein solvmanifolds. Regarding uniqueness, he proves that any solvable Lie group S admits at most one left-invariant standard Einstein metric, up to homothety. Thus our approach is to deform the underlying algebraic structure rather than the metric. Regarding existence, Heber proves that the moduli space of

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Einstein solvmanifolds near a given one may have very large dimension. His computation of the dimension does not give explicit examples but rather shows that if there is one Einstein solvmanifold of a particular type, then one may be guaranteed a large moduli space. Building on Heber's results, the author and Gordon [GK] give an explicit description of the moduli space of the lowest-dimensional Einstein Carnot solvmanifold (dimension 10) for which a continuous family of Einstein solvmanifolds can exist.

In this paper we focus on the moduli space of Einstein solvmanifolds near the non-compact rank one symmetric space $\mathbf{H}H^3$, quaternionic hyperbolic space. Heber finds that the symmetric space $\mathbf{H}H^3$ sits in a 12-dimensional moduli space of nearby Einstein solvmanifolds which share the same algebraic structure. This is the simplest rank one symmetric space whose moduli space has positive dimension [Hb]. We give an explicit description of three intersecting two-dimensional families of Einstein solvmanifolds in the moduli space $\mathcal{M}$ of $\mathbf{H}H^3$.

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2. Preliminaries

A Riemannian solvmanifold, a solvable Lie group S together with a left-invariant Riemannian metric g, determines an inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{s}$ of S, and conversely, any inner product on $\mathfrak{s}$ gives rise to a unique left-invariant metric on S. We will say a Lie algebra endowed with an inner product is a metric Lie algebra. That is, a simply connected Riemannian solvmanifold is uniquely determined by a solvable metric Lie algebra. All solvmanifolds considered here will be of Iwasawa type.

Definition 2.1. Let (S, g) be a simply connected Riemannian solvmanifold, and let $\mathfrak{s}$ be the associated solvable metric Lie algebra. Write $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$. The metric Lie algebra $(\mathfrak{s}, \langle\ ,\ \rangle)$ and the solvmanifold (S, g) are said to be of *Iwasawa type* if the following conditions hold:

(i) $\mathfrak{a} = \mathfrak{n}^{\perp}$ is abelian.
(ii) For all $A \in \mathfrak{a}$, $\mathrm{ad}(A)$ is symmetric with respect to the inner product $\langle\ ,\ \rangle$ on $\mathfrak{s}$ and non-zero when A is non-zero.
(iii) For some $A \in \mathfrak{a}$, the restriction of $\mathrm{ad}(A)$ to $\mathfrak{n}$ is positive definite.

Two metric Lie algebras $(\mathfrak{s}, \langle\ ,\ \rangle)$ and $(\mathfrak{s}', \langle\ ,\ \rangle')$ are isomorphic if there exists a linear map $\tau : \mathfrak{s} \to \mathfrak{s}'$ which is both an inner product space isometry and a Lie algebra isomorphism. The condition for two Riemannian solvmanifolds of Iwasawa type to be isometric is a special case of Theorem 5.2 in [GW]:

Proposition 2.2. *Two simply connected solvmanifolds of Iwasawa type are isometric if and only if the associated metric Lie algebras are isomorphic.*

Definition 2.3. Define $U : \mathfrak{s} \times \mathfrak{s} \to \mathfrak{s}$ by $\langle U(X, Y), Z\rangle = \frac{1}{2}\langle [Z, X], Y\rangle + \frac{1}{2}\langle [Z, Y], X\rangle$. Set $H = \sum_i U(X_i, X_i)$, where $\{X_i\}$ is an orthonormal basis for $\mathfrak{s}$. Note that H lies in $\mathfrak{a}$, and for all $X \in \mathfrak{s}$, $\langle H, X\rangle = \mathrm{tr}\ \mathrm{ad}(X)$. ($H$ is the *mean curvature vector field* for the embedding of the nilradical N in S.)

Proposition 2.4 ([Hb]). *Let (S, g) be an Einstein solvmanifold of Iwasawa type and let $H \in \mathfrak{s}$ be the mean curvature vector field. Then for some multiple λH of*

H, the eigenvalues of $\mathrm{ad}(\lambda H)|_{\mathfrak{n}}$ *are positive integers, and the distinct eigenvalues have no common divisors.*

Definition 2.5. Let $\mu_1 < \mu_2 < \cdots < \mu_m$ be the distinct eigenvalues of $\mathrm{ad}(\lambda H)|_{\mathfrak{n}}$ with multiplicities $d_1, d_2, \ldots, d_m$, resp. The *eigenvalue type* of the Einstein solvmanifold is the $2m$-tuple $(\mu_1, \ldots, \mu_m; d_1, \ldots, d_m)$.

The class of solvmanifolds of algebraic rank one, with eigenvalue type $(1, 2; d_1, d_2)$, is sometimes referred to as the Carnot solvmanifolds. This class of solvmanifolds includes the non-compact rank one symmetric spaces and all the harmonic manifolds of negative Ricci curvature constructed by Damek and Ricci [DR]. In what follows we describe a method, first given by Eberlein and Heber [EH], for classifying all Einstein solvmanifolds of this type.

2.1. **Constructing Carnot solvmanifolds.** Let $(\mathfrak{s}, \langle\ ,\ \rangle)$ be an arbitrary Einstein Carnot solvmanifold, of eigenvalue type $(1, 2; r, s)$, where the multiplicities of the eigenvalues are r and s. Let $\mathfrak{v}$ and $\mathfrak{z}$ be the eigenspaces of $\mathrm{ad}(\lambda H)|_{\mathfrak{n}}$ corresponding to the eigenvalues 1 and 2, respectively. The Iwasawa condition on $(\mathfrak{s}, \langle\ ,\ \rangle)$ guarantees $\mathrm{ad}(H)$ is symmetric; hence $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. The Jacobi identity implies that $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$ and that $\mathfrak{z}$ is in the center of $\mathfrak{n}$. Thus $\mathfrak{n}$ is either two-step nilpotent or abelian. We will see in Theorem 2.8 that $\mathfrak{n}$ must be two-step, not abelian, in order for the Einstein condition to hold.

Given any two-step metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$ with center $\mathfrak{z}$, and with $\mathfrak{v} = \mathfrak{z}^{\perp}$, we can define a linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ by $\langle j(Z)X, Y\rangle := \langle [X, Y], Z\rangle$ for X, $Y \in \mathfrak{v}$, $Z \in \mathfrak{z}$. Conversely, given any two finite-dimensional real inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ along with a linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$, we can define a metric Lie alegbra $\mathfrak{n}$ as the orthogonal direct sum of the inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$, declaring $\mathfrak{z}$ to be central in $\mathfrak{n}$, and defining the alternating bilinear bracket map $[\cdot, \cdot] : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{z}$ by $\langle [X, Y], Z\rangle := \langle j(Z)X, Y\rangle$. If j is non-zero, $\mathfrak{n}$ is two-step nilpotent.

We can extend this construction to a Riemannian solvmanifold (S, g) of algebraic rank one and eigenvalue type $(1, 2; r, s)$ where $r = \dim(\mathfrak{v})$ and $s = \dim(\mathfrak{z})$ as follows. Let $\mathfrak{a}$ be a one-dimensional inner product space and A a choice of unit vector in $\mathfrak{a}$. Define $\mathfrak{s}$ to be the orthogonal direct sum of $\mathfrak{a}$ and $\mathfrak{n}$. Give $\mathfrak{s}$ the unique Lie algebra structure for which $\mathfrak{n}$ is an ideal (the nilradical) and such that $\mathrm{ad}(A)|_{\mathfrak{v}} = \mathrm{Id}$ and $\mathrm{ad}(A)|_{\mathfrak{z}} = 2\,\mathrm{Id}$. Then $\mathfrak{s}$ is a metric solvable Lie algebra of rank one and eigenvalue type $(1, 2; r, s)$.

Definition 2.6. We call the associated simply-connected Riemannian solvmanifold (S, g) the solvmanifold *defined by the data triple* $(\mathfrak{v}, \mathfrak{z}, j)$. We say data triples $(\mathfrak{v}, \mathfrak{z}, j)$ and $(\mathfrak{v}', \mathfrak{z}', j')$ are *equivalent* if there exist orthogonal transformations α of $\mathfrak{v}$ and β of $\mathfrak{z}$ such that $j'(\beta(Z)) = \alpha \circ j(Z) \circ \alpha^{-1}$ for all $Z \in \mathfrak{z}$.

Proposition 2.7 ([GK])**.** *The Riemannian solvmanifolds defined by data triples* $(\mathfrak{v}, \mathfrak{z}, j)$ *and* $(\mathfrak{v}', \mathfrak{z}', j')$ *are isometric if and only if* $(\mathfrak{v}, \mathfrak{z}, j)$ *and* $(\mathfrak{v}', \mathfrak{z}', j')$ *are equivalent.*

Theorem 2.8 ([EH])**.** *A Riemannian solvmanifold defined by the data triple* $(\mathfrak{v}, \mathfrak{z}, j)$ *as in Definition* 2.6 *is Einstein if and only if the following two conditions hold:*

(i) *The map* $j : \mathfrak{z} \to j(\mathfrak{z}) \subset \mathfrak{so}(\mathfrak{v})$ *is a linear isometry relative to the inner product on* $\mathfrak{so}(\mathfrak{v})$ *given by* $(A, B) = -\frac{1}{r}\,\mathrm{tr}(AB)$.
(ii) *For any orthonormal basis* $\{Z_i\}$ *of* $\mathfrak{z}$*, the operator* $\sum_{i=1}^{s} j(Z_i)^2$ *is scalar.*

Remark 2.9. Paired with condition (i), condition (ii) says that $\sum_i j(Z_i)^2 = -s\,\mathrm{Id}$.

Definition 2.10. Let W be an s-dimensional subspace of $\mathfrak{so}(r)$. We will say W is *uniform* if, relative to the standard inner product on $\mathfrak{so}(r)$ defined in Theorem 2.8, an orthonormal basis $\{\alpha_i\}$ of W satisfies $\sum_{i=1}^s \alpha_i^2 = c\,\mathrm{Id}$ for some c (necessarily $c = -s$). We will say two uniform subspaces of $\mathfrak{so}(r)$ are *equivalent* if they are conjugate by an orthogonal transformation of $\mathbf{R}^r$.

Corollary 2.11 ([GK]). *Up to scaling, isometry classes of Einstein solvmanifolds of algebraic rank one and eigenvalue type $(1, 2; r, s)$ are in one-to-one correspondence with equivalence classes of uniform s-dimensional subspaces of $\mathfrak{so}(r)$.*

3. The deformation

We start by describing the twelve-dimensional solvable metric Lie algebra corresponding to quaternionic hyperbolic space, $\mathbf{H}H^3$. We can write $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is the nilradical and $\mathfrak{a} = \mathfrak{n}^\perp$ is the one-dimensional Cartan subalgebra. The nilradical $\mathfrak{n}$ is two-step nilpotent; it decomposes into $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$, the center of $\mathfrak{n}$, is three-dimensional, and $\mathfrak{v}$, the orthogonal complement to $\mathfrak{z}$ in $\mathfrak{n}$, is eight-dimensional. Here, the j-map determined by the algebraic structure of $\mathfrak{n}$ is given by the natural representation of $\mathfrak{z} = \mathfrak{sp}(1)$ acting by left multiplication on $\mathfrak{v} = \mathbf{H}^2$ (identifying $\mathbf{R}^4$ with $\mathbf{H}$). A generator for $\mathfrak{a}$ is H, the mean curvature vector. Then $\mathfrak{v}$ and $\mathfrak{z}$ are the eigenspaces of $\mathrm{ad}(H)$, with eigenvalues 1 and 2, respectively, so that $\mathfrak{s}$ has eigenvalue type $(1, 2; 8, 3)$.

We will construct our examples of new Einstein solvmanifolds in the moduli space of quaternionic hyperbolic space $\mathbf{H}H^3$ by describing families of *inequivalent* uniform subspaces, obtained by continuously deforming from the uniform subspace corresponding to our $\mathfrak{s}$ above. Each subspace will determine a new Einstein Carnot solvmanifold in the moduli space.

Our building blocks will be five three-dimensional uniform subspaces of $\mathfrak{so}(8)$: $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. For each of these, we will give an orthonormal basis: $\{U_1, U_2, U_3\}$ for $\mathcal{U}$, $\{V_1, V_2, V_3\}$ for $\mathcal{V}$, $\{X_1, X_2, X_3\}$ for $\mathcal{X}$, $\{Y_1, Y_2, Y_3\}$ for $\mathcal{Y}$, and $\{Z_1, Z_2, Z_3\}$ for $\mathcal{Z}$. The bases are chosen so that for each i, as many i^{th} basis elements pairwise anticommute as possible. For example, U_i, X_i, and Y_i pairwise anticommute for $i = 1, 2, 3$. Thus for any $a, b, c \in \mathbf{R}$,

$$(aU_i + bX_i + cY_i)^2 = a^2U_i^2 + b^2X_i^2 + c^2Y_i^2.$$

Hence for $a^2+b^2+c^2 = 1$, the set $\{aU_i+bX_i+cY_i \mid i = 1, 2, 3\}$ gives an orthonormal basis for a uniform subspace of $\mathfrak{so}(8)$.

We have two other combinations of anticommuting basis elements: V_i, X_i, and Y_i, and then U_j, X_j, and Z_j. Each linear combination of basis elements V_j, X_j, Y_j (resp. U_j, X_j, Z_j) will span a uniform subspace. (For each such linear combination, it is clear that (a, b, c) and $(-a, -b, -c)$ determine the same uniform subspace.) We will show we have three trios, $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{V}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Z}$, each giving a family of uniform subspaces parametrized by $\mathbf{R}P^2$ or a finite quotient. These three surfaces intersect in quotients of great circles. This gives an explicit description of a subset of the moduli space of Einstein solvmanifolds near $\mathbf{H}H^3$.

Remark 3.1. We note that $\mathcal{U}$ (described below) is the uniform subspace corresponding to our initial Einstein solvmanifold: the symmetric space $\mathbf{H}H^3$. Two of our three families contain $\mathcal{U}$ within their deformation. Since the rank one symmetric space has strictly negative sectional curvature, by continuity we can conclude that for an open set of nearby uniform subspaces containing $\mathcal{U}$, the sectional curvature of the corresponding solvmanifolds will also be negative. Two of our three families include continuous sets of negatively curved Einstein solvmanifolds.

From [GK], we know there exist exactly two nonconjugate uniform subspaces of dimension 3 in $\mathfrak{so}(4)$. They arise from the identification $\mathbf{R}^4 = \mathbf{H}$, by left and right multiplication of the purely imaginary unit quaternions i, j and k:

$$U = \operatorname{span}\{L_i, L_j, L_k\} \quad \text{and} \quad V = \operatorname{span}\{R_i, L_j, L_i\}.$$

In $\mathfrak{so}(4)$, we take the following ordered basis for U:

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and for V:

$$V_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Using the isometric embedding $\tau : \mathfrak{su}(4) \hookrightarrow \mathfrak{so}(8)$ and viewing $\mathfrak{so}(4)$ as a natural subalgebra of $\mathfrak{su}(4)$, we see that $\tau(\mathfrak{so}(4))$ is $\Delta\mathfrak{so}(4) \subset \mathfrak{so}(4) \oplus \mathfrak{so}(4) \subset \mathfrak{so}(8)$. Let $\mathcal{U} := \tau(U)$ be the three-dimensional subspace of $\mathfrak{so}(8)$ with basis given by $\tau(U_i)$ for $i = 1, 2, 3$. Similarly, $\mathcal{V} := \tau(V)$ is the three-dimensional subspace spanned by the $\tau(V_i)$'s. (For simplicity, we will make the identification $U_i = \tau(U_i)$ and $V_i = \tau(V_i)$.) One easily checks that both $\mathcal{U}$ and $\mathcal{V}$ are uniform subspaces of $\mathfrak{so}(8)$.

Next we describe $\mathcal{X}$, a subspace of $\mathfrak{so}(8)$ in $\tau(\mathfrak{su}(4))$, orthogonal to $\tau(\mathfrak{so}(4))$. Let

$$X_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Our subspace $\mathcal{X}$ is spanned by $\tau(\sqrt{-1}X_1)$, $\tau(\sqrt{-1}X_2)$, and $\tau(\sqrt{-1}X_3)$. (Again, we make the identification $X_i = \tau(\sqrt{-1}X_i)$.) Notice as the purely imaginary diagonal matrices with trace zero, $\mathcal{X}$ is the Cartan subalgebra of $\mathfrak{su}(4)$. One can check that the $\mathcal{X}$ is uniform and that the corresponding basis vectors of $\mathcal{U}$ and $\mathcal{X}$ and of $\mathcal{V}$ and $\mathcal{X}$ anticommute. That is, $U_iX_i = -X_iU_i$ and $V_iX_i = -X_iV_i$, for each $i = 1, 2, 3$.

Next we have $\mathcal{Y}$: another subspace in $\tau(\mathfrak{su}(4))$, orthogonal to $\mathcal{X}$ and to $\tau(\mathfrak{so}(4))$ (hence orthogonal to $\mathcal{U}$ and $\mathcal{V}$). For each of

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

let $\mathcal{Y}$ be the subspace of $\mathfrak{so}(8)$ spanned by $\tau(\sqrt{-1}Y_1)$, $\tau(\sqrt{-1}Y_2)$, and $\tau(\sqrt{-1}Y_3)$. (As before, we will denote these by Y_1, Y_2 and Y_3, for simplicity.) Notice $\mathcal{Y}$ is uniform and the basis vectors of $\mathcal{Y}$ anticommute with the corresponding basis vectors in $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{V}$.

Finally, we describe $\mathcal{Z}$: a subspace in $\mathfrak{so}(8)$, orthogonal to $\tau(\mathfrak{su}(4))$. The subspace $\mathcal{Z}$ is spanned by $Z_1 = \begin{pmatrix} 0 & U_2 \\ U_2 & 0 \end{pmatrix}$, $Z_2 = \begin{pmatrix} 0 & U_3 \\ U_3 & 0 \end{pmatrix}$, and $Z_3 = \begin{pmatrix} 0 & U_1 \\ U_1 & 0 \end{pmatrix}$ (where here the U_i's are in $\mathfrak{so}(4)$). $\mathcal{Z}$ is uniform and the basis vectors of $\mathcal{Z}$ anticommute with corresponding basis vectors in $\mathcal{X}$ and $\mathcal{U}$.

We conclude this section with the observation that while $\mathcal{U}$ and $\mathcal{V}$ share a two-dimensional subspace, $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are pairwise orthogonal, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$ are pairwise orthogonal, and $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Z}$ are pairwise orthogonal.

4. The three families

4.1. **The first family.** Now, take any triple (p, q, r) with $p^2 + q^2 + r^2 = 1$. Let

$$W(p, q, r) = \operatorname{span}\{D_i = pU_i + qX_i + rY_i \mid i = 1, 2, 3\}.$$

Since the subspaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ are pairwise orthogonal, the set $\{D_1, D_2, D_3\}$ is an orthonormal basis for $W(p, q, r)$, and since U_i, X_i and Y_i pairwise anticommute for $i = 1, 2, 3$,

$$\sum_i D_i^2 = \sum_i (pU_i + qX_i + rY_i)^2 = p^2 \sum_i U_i^2 + q^2 \sum_i X_i^2 + r^2 \sum_i Y_i^2 = -3\,\mathrm{Id}\,.$$

Hence $W(p, q, r)$ is a uniform subspace. Clearly $(-p, -q, -r)$ and (p, q, r) determine the same subspace. We will show that two triples (p, q, r) and (p', q', r') for which $(p', q', r') \neq (\pm p, \pm q, \pm r)$ determine non-conjugate subspaces. This will verify that we indeed have a family of Einstein solvmanifolds parametrized by $\mathbf{R}P^2$ or possibly a finite quotient of $\mathbf{R}P^2$.

Proposition 4.1. *The set of isometry classes of Einstein solvmanifolds corresponding to $W(p, q, r)$ defined above is parametrized by a finite quotient of $\mathbf{R}P^2$.*

We begin with the following helpful lemma.

Lemma 4.2. *Let $W(p, q, r)$ and $W(p', q', r')$ be two uniform subspaces generated by vectors $D_i = pU_i + qX_i + rY_i$ and $D'_j = p'U_j + q'X_j + r'Y_j$ respectively, as above. Assume $q \neq 0$. If for some orthogonal map P, $PD'_jP^t = xD_1 + yD_2 + zD_3$, then two of x, y, and z must equal zero. That is, conjugation must take basis vectors to basis vectors.*

Proof. For D'_j as defined above, we have ${D'_j}^2 = -\mathrm{Id}$, for $j = 1, 2, 3$. For any $P \in \mathrm{SO}(8)$, PD'_jP^t has the same property: $(PD'_jP^t)^2 = -\mathrm{Id}$ for $j = 1, 2, 3$. Let $xD_1 + yD_2 + zD_3$ be any vector in $W(p, q, r)$. Then $(xD_1 + yD_2 + zD_3)^2 = -\mathrm{Id}$ if and only if $x^2 + y^2 + z^2 = 1$ (necessary since P is orthogonal) and

$$xy(D_1D_2 + D_2D_1) + xz(D_1D_3 + D_3D_1) + yz(D_2D_3 + D_3D_2) = 0. \tag{4.1}$$

We will see this occurs exactly when two of x, y, and z are zero. When we write

$$xy(D_1D_2 + D_2D_1) + xz(D_1D_3 + D_3D_1) + yz(D_2D_3 + D_3D_2) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix},$$

A has diagonal entries

$$A_{11} = -2q^2(xy + xz + yz), \qquad A_{22} = 2q^2(xy - xz + yz),$$
$$A_{33} = 2q^2(-xy + xz + yz), \qquad A_{44} = 2q^2(xy + xz - yz).$$

If $q \neq 0$, then via the diagonal entries of A we see that $xy = xz = yz = 0$ is necessary; hence two of x, y and z must be zero. From Equation (4.1) it is easy to see that this is also sufficient. □

Remark 4.3. If we consider the case when basis vectors D_i are obtained using the trio $\mathcal{U}, \mathcal{X}$ and $\mathcal{Z}$, the analogous statement holds, with an identical proof. In the case obtained using the trio $\mathcal{V}, \mathcal{X}$, and $\mathcal{Y}$, we will prove that the statement holds with no condition on q.

Proof. Now to complete the proof of Proposition 4.1, we let $W(p,q,r)$ be a subspace of our family. The three generators for $[W(p,q,r), W(p,q,r)]$ have scalar squares,

$$[D_1, D_2]^2 = [D_1, D_3]^2 = -4(1-q^2)\,\mathrm{Id} \quad \text{and} \quad [D_2, D_3]^2 = -4(1-q^2)(1-r^2)\,\mathrm{Id},$$

and these are invariant under conjugation. If $q \neq 0$, then we know by Lemma 4.2 that in order for $W(p,q,r)$ to be conjugate to $W(p',q',r')$, conjugation must take D_i to $\pm D'_j$. Hence since $-4(1-q^2)$ must occur twice, we must have $-4(1-q^2) = -4(1-{q'}^2)$. This proves $q' = \pm q$ if $W(p,q,r)$ and $W(p',q',r')$ are conjugate. From this we can conclude $-4(1-q^2)(1-r^2) = -4(1-{q'}^2)(1-{r'}^2)$, which requires $r' = \pm r$. Since $p^2 = 1-q^2-r^2$ and ${p'}^2 = 1-{q'}^2-{r'}^2$, we have shown that if $q \neq 0$ and if $W(p,q,r)$ and $W(p',q',r')$ are conjugate, then $(p',q',r') = (\pm p, \pm q, \pm r)$.

Consider the case $q = 0$. We have $D_i = pU_i + rY_i$ and $p^2 + r^2 = 1$. For this case, we examine the algebraic structure of $W(p,q,r)$ more closely. Observe that $\langle [D_i, D_j], D_k \rangle = 2p(p^2 + r^2)$ for (i,j,k) a cyclic permutation of $(1,2,3)$. Since $q = 0$, we have $\langle [D_i, D_j], D_k \rangle = 2p$. By orthogonality, for any $P \in \mathrm{SO}(8)$, clearly $2p = \langle [PD_iP^t, PD_jP^t], PD_kP^t \rangle$. Suppose $PD_iP^t = x_iD'_1 + y_iD'_2 + z_iD'_3$. Then if (i,j,k) is a cyclic permutation of $(1,2,3)$,

$$2p = \langle [PD_iP^t, PD_jP^t], PD_kP^t \rangle = 2p'((x_i, y_i, z_i) \times (x_j, y_j, z_j)) \cdot (x_k, y_k, z_k)$$

where on the right-hand side we have the triple scalar product in $\mathbf{R}^3$ of the vectors (x_i, y_i, z_i), (x_j, y_j, z_j), and (x_k, y_k, z_k). Since conjugation takes orthonormal vectors to orthonormal vectors, $((x_i, y_i, z_i) \times (x_j, y_j, z_j))$ is parallel to (x_k, y_k, z_k). Thus the right-hand side is $\pm 2p'$, and we conclude $p = \pm p'$. Since $r^2 = 1 - p^2$ and ${r'}^2 = 1 - {p'}^2$, $r = \pm r'$. Now, in both the $q = 0$ case and the $q \neq 0$ case, we have shown that $W(p',q',r')$ is not conjugate to $W(p,q,r)$ except possibly if $(p',q',r') = (\pm p, \pm q, \pm r)$. Hence we have a family of uniform subspaces parametrized by $\mathbf{R}P^2$ or a finite quotient. □

Remark 4.4. We have not found a conjugacy between $W(p,q,r)$ and, for example, $W(-p,q,r)$, but we have not ruled out the possibility.

4.2. **The second family.** Consider the subspaces made up from the trio $\mathcal{U}, \mathcal{X}$, and $\mathcal{Z}$ (replacing $\mathcal{Y}$ from the previous family with $\mathcal{Z}$). We start with any triple (p,q,t) such that $p^2 + q^2 + t^2 = 1$. Let

$$W(p,q,t) = \operatorname{span}\{D_i = pU_i + qX_i + tZ_i \mid i = 1,2,3\}.$$

By the same argument as before, $W(p,q,t)$ is a uniform subspace. We will show that two triples (p,q,t) and (p',q',t') for which $(p',q',t') \neq (\pm p, \pm q, \pm t)$ determine non-conjugate subspaces. This will verify that we have a second family of Einstein solvmanifolds parametrized by $\mathbf{R}P^2$ or possibly a finite quotient of $\mathbf{R}P^2$.

Proposition 4.5. *The set of isometry classes of Einstein solvmanifolds corresponding to $W(p,q,t)$ defined above is parametrized by a finite quotient of $\mathbf{R}P^2$.*

Proof. Here we will consider the following two invariants: for (i, j, k) any cyclic permutation of $(1, 2, 3)$, we have $\langle [D_i, D_j], D_k\rangle = 2p^3$. We first consider the case $q \neq 0$, where conjugation of any basis vector is another basis vector. Suppose for P an orthogonal map, $PD_iP^t = \pm D'_j$, a generator for $W(p', q', t')$,

$$\pm 2{p'}^3 = \langle [PD_iP^t, PD_jP^t], PD_kP^t\rangle = \langle [D_i, D_j], D_k\rangle = 2p^3.$$

This shows $p' = \pm p$.

As in the previous family, the three generators for $[W(p, q, t), W(p, q, t)]$ again have scalar squares:

$$[D_1, D_2]^2 = [D_1, D_3]^2 = [D_2, D_3]^2 = -4(p^2 + t^4)\,\mathrm{Id}\,.$$

If (p, q, t) and (p', q', t') determine conjugate subspaces, we would necessarily have $-4(p^2 + t^4) = -4({p'}^2 + {t'}^4)$. Substituting ${p'}^2 = p^2$ into this invariant, we conclude that $t' = \pm t$. Since $q^2 = 1 - p^2 - t^2$, it follows that $q' = \pm q$.

Turning to the case $q = 0$, where we no longer know that conjugation must take basis vectors to basis vectors, our first invariant is still the key. Suppose $PD_iP^t = x_iD'_1 + y_iD'_2 + z_iD'_3$, for some orthogonal map $P \in \mathrm{SO}(8)$, for $i = 1, 2, 3$. Then

$$2p^3 = \langle [PD_iP^t, PD_jP^t], PD_kP^t\rangle = 2{p'}^3((x_i, y_i, z_i) \times (x_j, y_j, z_j)) \cdot (x_k, y_k, z_k)$$

where again on the right-hand side we have the triple scalar product in $\mathbf{R}^3$ of the vectors (x_i, y_i, z_i), (x_j, y_j, z_j), and (x_k, y_k, z_k). Since conjugation takes orthonormal vectors to orthonormal vectors, $((x_i, y_i, z_i) \times (x_j, y_j, z_j))$ is parallel to (x_k, y_k, z_k). Thus the right-hand side is $\pm 2{p'}^3$ and we see that $p' = \pm p$. As in the case $q \neq 0$, we now use the second invariant to conclude that $t' = \pm t$. Thus $W(p', q', t')$ is not conjugate to $W(p, q, t)$ except possibly if $(p', q', t') = (\pm p, \pm q, \pm t)$. Hence our second family of uniform subspaces is parametrized by $\mathbf{R}P^2$ or a finite quotient. □

We note that in order for an element of the first family to be conjugate to an element of the second family, we would need the generators of $[W(p, q, r), W(p, q, r)]$ to all have the same eigenvalue, and this is possible only if $r = 0$, in which case the uniform subspace $W(p, q, 0)$ is itself in the second family, with $t = 0$.

4.3. **The third family.** Now we consider the subspaces made up from the trio $\mathcal{V}, \mathcal{X}$, and $\mathcal{Y}$. For any triple (q, r, s) with $q^2 + r^2 + s^2 = 1$, let

$$W(q, r, s) = \mathrm{span}\{D_i = qX_i + rY_i + sV_i \mid i = 1, 2, 3\}.$$

By the same argument as in the previous families, each $W(q, r, s)$ is a uniform subspace. We will show that two triples (q, r, s) and (q', r', s') for which $(q', r', s') \neq (\pm q, \pm r, \pm s)$ determine non-conjugate subspaces.

Proposition 4.6. *The set of isometry classes of Einstein solvmanifolds corresponding to $W(q, r, s)$ defined above is parametrized by $\mathbf{R}P^2$ or a finite quotient of $\mathbf{R}P^2$.*

We first prove a lemma analogous to Lemma 4.2.

Lemma 4.7. *Let $W(q, r, s)$ and $W(q', r', s')$ be two uniform subspaces generated by vectors D_i and D'_j respectively, as above. If for some orthogonal map P, $PD'_jP^t = xD_1 + yD_2 + zD_3$, then $PD'_jP^t = \pm D_i$.*

Proof. For D'_j as defined above, ${D'_j}^2 = -\operatorname{Id}$. Thus for any $P \in \mathrm{SO}(8)$, $(PD'_jP^t)^2 = -\operatorname{Id}$, as well. For a generic element in $W(q, r, s)$, its square $(xD_1 + yD_2 + zD_3)^2$ is $-\operatorname{Id}$ if and only if $xy(D_1D_2 + D_2D_1) + xz(D_1D_3 + D_3D_1) + yz(D_2D_3 + D_3D_2) = 0$, and $x^2 + y^2 + z^2 = 1$. We will show this occurs if and only if two of x, y, and z are zero. We compute

$$xy(D_1D_2 + D_2D_1) + xz(D_1D_3 + D_3D_1) + yz(D_2D_3 + D_3D_2) = 2\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

where the diagonal entries of A are the same as in Lemma 4.2, proving if $q \neq 0$, two of x, y, z vanish. If $q = 0$,

$$A = -\begin{bmatrix} s^2xz & r^2yz & 0 & s^2xy \\ r^2yz & s^2xz & s^2xy & 0 \\ 0 & s^2xy & -s^2xz & r^2yz \\ s^2xy & 0 & r^2yz & -s^2xz \end{bmatrix}.$$

The first row of A vanishes only if $xz = 0$, $yz = 0$, and $xy = 0$. Thus two of x, y, and z must be zero. □

Proof. We are now ready to prove Proposition 4.6. By Lemma 4.7, we know conjugation can take basis vectors only to basis vectors. In this family, each D_i has two four-dimensional invariant subspaces. An invariant subspace of D_2 intersected with an invariant subspace of either D_1 or D_3 will be two-dimensional, whereas an invariant subspace of D_1 intersects an invariant subspace of D_3 in either a one- or three-dimensional subspace. The dimensions of intersections of eigenspaces must be preserved under conjugation. Therefore, if $W(q, r, s)$ and $W(q', r', s')$ are conjugate, D_2 is necessarily conjugated to $\pm D'_2$. From this, we conclude that the eigenvalues of $[D_1, D_3]^2$ and $[D'_1, D'_3]^2$ must agree:

$$-4r^2 \operatorname{Id} = [D_1, D_3]^2 = [D'_1, D'_3]^2 = -4{r'}^2 \operatorname{Id}. \tag{4.2}$$

Hence, $r^2 = {r'}^2$, or equivalently, $q^2 + s^2 = {q'}^2 + {s'}^2$. Looking more closely at $[W(q, r, s), W(q, r, s)]$, we see

$$\langle [D_1, D_2], [D_2, D_3] \rangle = -4r^2(qr + s^2). \tag{4.3}$$

Up to sign, this also will be preserved by conjugation. Other invariants we can use are the eigenvalues of the other generators of $[W(q, r, s), W(q, r, s)]$:

$$[D_1, D_2]^2 = -4(r^2 + q^2s^2) \operatorname{Id}, \tag{4.4}$$

$$[D_2, D_3]^2 = -4(qr + s^2)^2 \operatorname{Id}. \tag{4.5}$$

(Here $\langle [W(q, r, s), W(q, r, s)], W(q, r, s) \rangle = 0$ for any triple (q, r, s); the invariant which we used before is no help.) We have two cases: either conjugation maps D_1 to $\pm D'_1$ (and D_3 to $\pm D'_3$), or D_1 is mapped to $\pm D'_3$ (and D_3 to $\pm D'_1$). Since $W(q, r, s) = W(-q, -r, -s)$, we may assume that $r = r'$.

Case 1. Assume D_1 is conjugated to $\pm D'_1$ and D_3 to $\pm D'_3$. From Equation (4.4), we know $r^2 + q^2s^2 = {r'}^2 + {q'}^2{s'}^2$. Thus $(qs)^2 = (q's')^2$. Pairing this with Equation (4.2), we obtain $(q + s)^2 = (q' \pm s')^2$ and $(q - s)^2 = (q' \mp s')^2$. The solutions are $(q', r', s') = (\pm q, r, \pm s)$ or $(q', r', s') = (\pm s, r, \pm q)$. We see we already have a family parametrized by $\mathbf{R}P^2$ or a finite quotient.

When we compare with Equation (4.5), we see that when $(q', r, s') = (\pm q, r, \pm s)$, we must have $(s^2 + qr)^2 = ({s'}^2 + q'r)^2$. If $s \neq 0$, $r \neq 0$, it follows that $q'r' = +qr$, i.e., the subspace $W(-q, r, s)$ cannot be conjugate to $W(q, r, \pm s)$ if $s \neq 0$. Comparing

$(q', r', s') = (\pm s, r, \pm q)$ with Equation (4.5), we see that either $qr + s^2 = +(q^2 \pm sr)$ or $qr + s^2 = -(q^2 \pm sr)$. The equation $qr + s^2 = +(q^2 \pm sr)$ is equivalent to $(q + s)(q - s) = r(q \mp s)$. When $q' = +s$, then our equation holds only if our original triple satisfies $s = q$ or $r = q + s$. When $q' = -s$, then it holds only if our original triple satisfies $s = -q$ or $r = q - s$.

Next, $qr + s^2 = -(q^2 \pm sr)$ can hold only if

$$q = \frac{1}{2r}\left((r^2 - 1) \pm \sqrt{(1 - r^2)(3r^2 - 1)}\right)$$

and

$$s = \pm\frac{1}{2r}\sqrt{2(1 - r^2)(r^2 \pm \sqrt{(1 - r^2)(3r^2 - 1)})}$$

(real when $r^2 \geq \frac{1}{3}$).

Case 2. Assume instead D_1 is conjugated to $\pm D_3'$ and D_3 to $\pm D_1'$. Using the eigenvalues of the generators of $[W(q, r, s), W(q, r, s)]$, we have equalities $r^2 + q^2 s^2 = (q'r + {s'}^2)^2$ and $(qr + s^2)^2 = r^2 + {q'}^2 {s'}^2$. Combining these with Equation (4.3), we obtain

$$r^2 + q^2 s^2 = (q'r + {s'}^2)^2 = (qr + s^2)^2 = r^2 + {q'}^2 {s'}^2.$$

This is a more restrictive set of equations than we had in Case 1. By the argument above, we know that either $(q', r, s') = (\pm q, r, \pm s)$ or $(q', r, s') = (\pm s, r, \pm q)$. We will show that only $(q', r, s') = (\pm q, r, \pm s)$ is possible. When we substitute ${q'}^2 = s^2$ and ${s'}^2 = q^2$, we need a real simultaneous solution to $(qr + s^2)^2 = r^2 + q^2 s^2$ and $(q^2 \pm sr)^2 = (qr + s^2)^2$. Solving, we get $(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}})$ or $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$ (assuming $r > 0$); hence $q^2 = {q'}^2$ and $s^2 = {s'}^2$. □

If we could conjugate a uniform subspace of the first or second family to one in the third family, then by Lemma 4.7, conjugation maps the basis vectors to basis vectors. But only in the third family of uniform subspaces do we have the property that two basis vectors (D_1 and D_3) have invariant subspaces which have odd-dimensional intersections. Thus, elements of the first or second family which are not already in the third family cannot be conjugated into the third family.

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