PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 2, Pages 443–453
S 0002-9939(05)08036-6
Article electronically published on June 29, 2005

NON-COMMUTATIVE METRIC TOPOLOGY ON MATRIX STATE SPACE

WEI WU

(Communicated by David R. Larson)

Abstract. We present an operator space version of Rieffel's theorem on the agreement of the metric topology, on a subset of the Banach space dual of a normed space, from a seminorm with the weak*-topology. As an application we obtain a necessary and sufficient condition for the matrix metric from an unbounded Fredholm module to give the BW-topology on the matrix state space of the C^*-algebra. Motivated by recent results we formulate a non-commutative Lipschitz seminorm on a matrix order unit space and characterize those matrix Lipschitz seminorms whose matrix metric topology coincides with the BW-topology on the matrix state space.

1. Introduction

One basic idea in modern analysis is that C^*-algebras are non-commutative $C(K)$ spaces [7, 12]. The basic idea of non-commutative functional analysis is to study operator spaces as a generalization of Banach spaces [6, 18, 9]. Let $\mathcal{A}$ be a unital C^*-algebra. Connes has shown that an appropriate way to specify a Riemannian metric in this non-commutative situation is by means of a spectral triple [5]. This consists of a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, together with the generalized Dirac operator D on $\mathcal{H}$ satisfying certain conditions. The Lipschitz seminorm, L, is defined on the set $\mathcal{L}(\mathcal{A})$ of Lipschitz elements of $\mathcal{A}$ by the operator norm $L(a) = \|[D, a]\|$. From this, Connes defines a metric ρ on the state space of $\mathcal{A}$ by

$$\rho(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in \mathcal{L}(\mathcal{A}), L(a) \leq 1\}.$$

A natural question is when the metric topology from the metric ρ agrees with the underlying weak*-topology on the state space of $\mathcal{A}$.

In [16], Rieffel defined a metric on a subset of the Banach space dual in a very rudimentary Banach space setting and gave necessary and sufficient conditions under which the topology from the metric agrees with the weak*-topology. This problem has also been considered by Pavlović [15]. For an operator with domain contained in a unital C^*-algebra, range contained in a normed space and kernel $\mathbb{C}1$, she defined a metric on the state space of the C^*-algebra from the operator and then obtained the necessary and sufficient conditions for the metric to give the weak*-topology. The theory of operator spaces naturally leads to the problem of

Received by the editors June 6, 2003 and, in revised form, September 20, 2004.
2000 *Mathematics Subject Classification.* Primary 46L87, 58B30, 46L30.
Key words and phrases. BW-topology, generalized Dirac operator, matrix Lipschitz seminorm, matrix seminorm, matrix state space, operator space.

whether we may replace the Banach space by an operator space. The main purpose of this paper is to extend these results to the category of operator spaces.

The paper is organized as follows. We begin in section 2 with the explanation of terminology and notation. In section 3 we present the operator space version of Rieffel's theorem and in section 4 we use it to the matrix metrics from Dirac operators. Finally in section 5 we define a non-commutative version of Lipschitz seminorms on matrix order unit spaces and then use the results of the previous sections to determine when the metric topology from a matrix Lipschitz seminorm agrees with the BW-topology on the matrix state space.

2. Terminology and notation

All vector spaces are assumed to be complex throughout this paper. Given a vector space V, we let $M_{m,n}(V)$ denote the matrix space of all m by n matrices $v = [v_{ij}]$ with $v_{ij} \in V$, and we set $M_n(V) = M_{n,n}(V)$. If $V = \mathbb{C}$, we write $M_{m,n} = M_{m,n}(\mathbb{C})$ and $M_n = M_{n,n}(\mathbb{C})$, which means that we may identify $M_{m,n}(V)$ with the tensor product $M_{m,n} \otimes V$. We identify $M_{m,n}$ with the normed space $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^m)$. We use the standard matrix multiplication and *-operation for compatible scalar matrices, and 1_n for the identity matrix in M_n. There are two natural operations on the matrix spaces. For $v \in M_{m,n}(V)$ and $w \in M_{p,q}(V)$, the direct sum $v \oplus w \in M_{m+p,n+q}(V)$ is defined by letting $v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}$, and if we are given $\alpha \in M_{m,p}$, $v \in M_{p,q}(V)$ and $\beta \in M_{q,n}$, the matrix product $\alpha v \beta \in M_{m,n}(V)$ is defined by $\alpha v \beta = \left[\sum_{k,l} \alpha_{ik} v_{kl} \beta_{lj}\right]$.

If V and W are vector spaces in duality, then it determines the matrix pairing $\langle\langle \cdot, \cdot \rangle\rangle : M_n(V) \times M_m(W) \longmapsto M_{nm}$, where $\langle\langle [v_{ij}], [w_{kl}] \rangle\rangle = [\langle v_{ij}, w_{kl} \rangle]$ for $[v_{ij}] \in M_n(V)$ and $[w_{kl}] \in M_m(W)$. Given an arbitrary vector space V, a *matrix gauge* $\mathcal{G} = (g_n)$ on V is a sequence of gauges $g_n : M_n(V) \longmapsto [0, +\infty]$ such that $g_{m+n}(v \oplus w) = \max\{g_m(v), g_n(w)\}$ and $g_n(\alpha v \beta) \leq \|\alpha\| g_m(v) \|\beta\|$ for any $v \in M_m(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$. A matrix gauge $\mathcal{G} = (g_n)$ is a *matrix seminorm* on V if for any $n \in \mathbb{N}, g_n(v) < +\infty$ for all $v \in M_n(V)$. If each g_n is a norm on $M_n(V)$, we say that $\mathcal{G}$ is a *matrix norm*. An *operator space* is a vector space together with a matrix norm. For other terminology and notation, we will follow [3] and [10].

3. Matrix metrics on matrix states

Let V be an operator space with the matrix norm $\|\cdot\| = (\|\cdot\|_n)$, and let W be a subspace of V. As in [16], we will not assume that W is closed. Suppose that $\mathcal{L} = (L_n)$ is a matrix seminorm on W and $\varphi_1^{(0)}$ is a bounded linear functional on the subspace $K_1 = \{a \in W : L_1(a) = 0\}$ with $\|\varphi_1^{(0)}\| = 1$. Set $\varphi_n^{(0)} = \underbrace{\varphi_1^{(0)} \oplus \cdots \oplus \varphi_1^{(0)}}_{n}$, $n \in \mathbb{N}$. Then $(\varphi_n^{(0)})$ is a sequence of completely bounded linear mappings $\varphi_n^{(0)} : K_1 \mapsto M_n$, with $\|\varphi_n^{(0)}\|_{cb} = 1$. Here $\|\varphi_n^{(0)}\|_{cb}$ is the completely bounded norm of $\varphi_n^{(0)}$. In the applications, K_1 will always be closed in V. Thus we assume that K_1 is closed in V. Denote $K_n = \{a \in M_n(W) : L_n(a) = 0\}$ for $n \in \mathbb{N}$, and let $\mathcal{K} = (K_n)$. The inequality $L_1(v_{ij}) \leq L_n([v_{ij}]) \leq \sum_{i,j=1}^n L_1(v_{ij})$ holds for $[v_{ij}] \in M_n(W)$ and $n \in \mathbb{N}$ [10]. So $K_n = M_n(K_1)$ for $n \in \mathbb{N}$.

Denote $GCS_n(V) = \{\varphi : \varphi$ is a completely bounded linear mapping from V to M_n, $\varphi = \varphi_n^{(0)}$ on K_1, and $\|\varphi\|_{cb} = 1\}$, and set $\mathcal{GCS}(V) = (GCS_n(V))$. We write $CB_n(V) = \{\varphi : \varphi$ is a completely bounded linear mapping from V to $M_n\}$, and $CB_n^1(V) = \{\varphi \in CB_n(V) : \|\varphi\|_{cb} \leq 1\}$, for $n \in \mathbb{N}$. Then $GCS_n(V)$ is a convex subset of $CB_n(V)$. Define $S : M_n(V^*) \mapsto CB_n(V)$ by $(S([\rho_{ij}]))(a) = [\rho_{ij}(a)]$, where $[\rho_{ij}] \in M_n(V^*)$ and $a \in V$. Then S is a linear isomorphism from $M_n(V^*)$ onto $CB_n(V)$. As a linear mapping from operator space $M_n(V^*)$ onto operator space $CB_n(V)$, S is a complete isometry [2, 8]. On $\mathcal{GCS}(V)$, we can define a sequence $\mathcal{D}_{\mathcal{L}} = (D_{L_n})$ of metrics by

$$D_{L_n}(\varphi, \psi) = \sup\{\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| : a \in M_r(W), L_r(a) \leq 1, r \in \mathbb{N}\},$$

for $\varphi, \psi \in GCS_n(V)$. (These may take the value $+\infty$.) We call $\mathcal{D}_{\mathcal{L}}$ the *matrix metric* from $\mathcal{L}$. We will refer to the topology on $\mathcal{GCS}(V)$ defined by $\mathcal{D}_{\mathcal{L}}$; that is, the topology on each $GCS_n(V)$ is induced by D_{L_n}, as the $\mathcal{D}_{\mathcal{L}}$*-topology* or the *matrix metric topology*. The natural topology on $\mathcal{GCS}(V)$ is the *BW-topology*; that is, the topologies on each $GCS_n(V)$ are induced by the BW-topology [1, 14]. We say that W *separates the points* of $\mathcal{GCS}(V)$ if given $\varphi, \psi \in GCS_n(V)$ and $\varphi \neq \psi$ for $n \in \mathbb{N}$ there is an $a \in W$ such that $\varphi(a) \neq \psi(a)$.

Proposition 3.1. *For $\varphi, \psi \in GCS_n(V)$,*

$$D_{L_n}(\varphi, \psi) = \sup\{\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| : a \in M_n(W), L_n(a) \leq 1\}.$$

Proof. Given $\varphi, \psi \in GCS_n(V)$, let $K_{L_n}(\varphi, \psi) = \sup\{\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| : a \in M_n(W), L_n(a) \leq 1\}$. Clearly, $K_{L_n}(\varphi, \psi) \leq D_{L_n}(\varphi, \psi)$. For any $a \in M_n(W)$, we have that $\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| \leq K_{L_n}(\varphi, \psi)L_n(a)$. Since $\mathcal{L}$ is a matrix gauge on W, we have that $\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| \leq K_{L_n}(\varphi, \psi)L_r(a)$ for all $a \in M_r(W)$ with $r \in \mathbb{N}$ arbitrary by Lemma 5.2 in [10]. So $D_{L_n}(\varphi, \psi) = \sup\{\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| : a \in M_r(W), L_r(a) \leq 1, r \in \mathbb{N}\} \leq K_{L_n}(\varphi, \psi)$. Therefore, $K_{L_n} = D_{L_n}$. □

Proposition 3.2. *$\mathcal{GCS}(V)$ is BW-compact.*

Proof. This follows immediately from the observation that $GCS_n(V) = \bigcap_{k \in K_1}\{\varphi \in CB_n^1(V) : \varphi(k) = \varphi_n^{(0)}(k)\}$ and the fact that $CB_n^1(V)$ is BW-compact [14]. □

Proposition 3.3. *If W separates the points of $\mathcal{GCS}(V)$, then the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{GCS}(V)$ is finer than the BW-topology.*

Proof. Let $\{\varphi_k\}$ be a sequence in $GCS_n(V)$ which converges to $\varphi \in GCS_n(V)$ for the metric D_{L_n}. Then for any $r \in \mathbb{N}$ and $a \in M_r(W)$ with $L_r(a) \leq 1$, we have $\lim_{k\to\infty} \|\langle\langle\varphi_k, a\rangle\rangle - \langle\langle\varphi, a\rangle\rangle\| = 0$. So, for every $r \in \mathbb{N}$ and $a \in M_r(W)$, $\{\langle\langle\varphi_k, a\rangle\rangle\}$ converges to $\langle\langle\varphi, a\rangle\rangle$ in the norm topology, and hence in the weak operator topology.

Denote $\phi(a, \xi, \eta)(\psi) = \langle\langle\psi, a\rangle\rangle\xi, \eta\rangle$ for $a \in M_r(W)$, $\psi \in GCS_n(V)$, $\xi, \eta \in \mathbb{C}^{rn}$ and $r, n \in \mathbb{N}$. The paragraph above shows that $\phi(a, \xi, \eta)(\varphi_k)$ converges to $\phi(a, \xi, \eta)(\varphi)$ for any $a \in M_r(W)$, $\xi, \eta \in \mathbb{C}^{rn}$ and $r \in \mathbb{N}$. Let $Q_{r,n} = \{\phi(a, \xi, \eta) : a \in M_r(W), \xi, \eta \in \mathbb{C}^{rn}\}$ and $Q_n = \bigcup_{r=1}^{\infty} Q_{r,n}$. Then Q_n is a linear space of BW-continuous functions on $GCS_n(V)$. Since W separates the points of $\mathcal{GCS}(V)$, Q_n separates the points of $GCS_n(V)$. For $a \in K_r$ and $\xi, \eta \in \mathbb{C}^{rn}$, we have that $\phi(a, \xi, \eta)(\varphi) = \langle\langle\varphi, a\rangle\rangle\xi, \eta\rangle = \langle\langle\varphi_n^{(0)}, a\rangle\rangle\xi, \eta\rangle$, where $\varphi \in GCS_n(V)$. So Q_n contains the constant functions. $GCS_n(V)$ BW-compact implies that Q_n determines the BW-topology of $GCS_n(V)$. Therefore, $\{\varphi_k\}$ converges to φ in the BW-topology. □

For $a \in M_n(W)$, define $\|a\|_n^\natural = \sup\{\|\langle\langle\varphi, a\rangle\rangle\| : \varphi \in GCS_n(V)\}$. Then we obtain a sequence $\|\cdot\|^\natural = (\|\cdot\|_n^\natural)$ of seminorms, and $\|a\|_n^\natural \leq \|a\|_n$ for $a \in M_n(W)$.

Proposition 3.4. *If $\|\cdot\|^\natural$ is a matrix seminorm on W, then*

$$\|a\|_n^\natural = \sup\{\|\langle\langle\varphi, a\rangle\rangle\| : \varphi \in GCS_r(V), r \in \mathbb{N}\}.$$

Proof. Denote $\|a\|_n^\flat = \sup\{\|\langle\langle\varphi, a\rangle\rangle\| : \varphi \in GCS_r(V), r \in \mathbb{N}\}$. Clearly, $\|a\|_n^\natural \leq \|a\|_n^\flat$. Given $\varphi \in GCS_r(V)$, if $r < n$, we choose a $\varphi_1 \in GCS_{n-r}(V)$ and then obtain that $\|\langle\langle\varphi, a\rangle\rangle\| \leq \max\{\|\langle\langle\varphi, a\rangle\rangle\|, \|\langle\langle\varphi_1, a\rangle\rangle\|\} = \left\|\begin{bmatrix} \langle\langle\varphi, a\rangle\rangle & 0 \\ 0 & \langle\langle\varphi_1, a\rangle\rangle \end{bmatrix}\right\| = \|\langle\langle\varphi \oplus \varphi_1, a\rangle\rangle\| \leq \|a\|_n^\natural$. When $r > n$, we have that $\|\langle\langle\varphi, a\rangle\rangle\| = \|\langle\langle\varphi, a \oplus 0_{r-n}\rangle\rangle\| \leq \|a \oplus 0_{r-n}\|_r^\natural = \|a\|_n^\natural$ since $\|\cdot\|^\natural$ is a matrix seminorm. Hence $\|a\|_n^\flat \leq \|a\|_n^\natural$, and this, with the previous inequality, gives that $\|a\|_n^\natural = \sup\{\|\langle\langle\varphi, a\rangle\rangle\| : \varphi \in GCS_r(V), r \in \mathbb{N}\}$. □

Let $\tilde{W} = W/K_1$ and $\tilde{V} = V/K_1$. Then $\mathcal{L}$ drops to a matrix norm on $\tilde{W}$ [10], which we denote by $\tilde{\mathcal{L}} = (\tilde{L}_n)$. But on $\tilde{W}$ and $\tilde{V}$ we also have the quotient matrix norm from $\|\cdot\|$ on V [9], which we denote by $\|\cdot\|^\sim = (\|\cdot\|_n^\sim)$. The image in $M_n(\tilde{W})$ of $a \in M_n(W)$ will be denoted by $\tilde{a}$. For each $n \in \mathbb{N}$, we let $L_n^t = \{a \in M_n(W) : L_n(a) \leq t\}$ and $B_n^t = \{a \in M_n(W) : L_n(a) \leq 1, \|a\|_n \leq t\}$ for $t > 0$, and set $\mathcal{L}^t = (L_n^t)$ and $\mathcal{B}^t = (B_n^t)$. We say that the image of a graded set $\mathbf{E} = (E_n)$ ($E_n \subseteq M_n(W)$) in $\tilde{\mathcal{W}}$ is *totally bounded* for $\|\cdot\|^\sim$ if each $\tilde{E}_n$ is totally bounded in $M_n(\tilde{\mathcal{W}})$ for $\|\cdot\|_n^\sim$.

Definition 3.5. Let V be a vector space, and let $\|\cdot\| = (\|\cdot\|_n)$ and $\|\cdot\|^\sharp = (\|\cdot\|_n^\sharp)$ be two matrix norms on V. We say that $\|\cdot\|$ and $\|\cdot\|^\sharp$ are *equivalent* if for any $n \in \mathbb{N}$, $\|\cdot\|_n$ is equivalent to $\|\cdot\|_n^\sharp$.

Proposition 3.6. *Suppose that $\|\cdot\| = (\|\cdot\|_n)$ and $\|\cdot\|^\sharp = (\|\cdot\|_n^\sharp)$ are two matrix norms on the vector space V. If $\|\cdot\|_1$ and $\|\cdot\|_1^\sharp$ are equivalent, then $\|\cdot\|$ and $\|\cdot\|^\sharp$ are equivalent.*

Proof. Since $\|\cdot\|_1$ and $\|\cdot\|_1^\sharp$ are equivalent, there are positive constants α and β such that $\alpha\|x\|_1^\sharp \leq \|x\|_1 \leq \beta\|x\|_1^\sharp$ for $x \in V$. Because $\|\cdot\|$ and $\|\cdot\|^\sharp$ are matrix norms, we have $\|x_{ij}\|_1 \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^n \|x_{ij}\|_1$ and $\|x_{ij}\|_1^\sharp \leq \|[x_{ij}]\|_n^\sharp \leq \sum_{i,j=1}^n \|x_{ij}\|_1^\sharp$ for $[x_{ij}] \in M_n(V)$ [18]. Hence $\alpha n^{-2}\|[x_{ij}]\|_n^\sharp \leq \|[x_{ij}]\|_n \leq n^2\beta\|[x_{ij}]\|_n^\sharp$. So $\|\cdot\|_n$ and $\|\cdot\|_n^\sharp$ are equivalent. Therefore, $\|\cdot\|$ and $\|\cdot\|^\sharp$ are equivalent. □

Theorem 3.7. *Let V be an operator space with the matrix norm $\|\cdot\| = (\|\cdot\|_n)$, and let W be a subspace of V, which is not necessarily closed. Suppose that $\mathcal{L} = (L_n)$ is a matrix seminorm on W with closed (in V) kernels $K_n = \{a \in M_n(W) : L_n(a) = 0\}$ ($n \in \mathbb{N}$) and $(\varphi_n^{(0)})$ is a sequence of completely bounded linear mappings $\varphi_n^{(0)} : K_1 \mapsto M_n$, with $\varphi_n^{(0)} = \underbrace{\varphi_1^{(0)} \oplus \cdots \oplus \varphi_1^{(0)}}_{n}$ and $\|\varphi_n^{(0)}\|_{cb} = 1$. Denote $GCS_n(V) = \{\varphi : \varphi$ is a completely bounded linear mapping from V to M_n, $\varphi = \varphi_n^{(0)}$ on K_1, and $\|\varphi\|_{cb} = 1\}$ and $\|a\|_n^\natural = \sup\{\|\langle\langle\varphi, a\rangle\rangle\| : \varphi \in GCS_n(V)\}, a \in M_n(W)$. Set $\mathcal{GCS}(V) = (GCS_n(V))$ and $\|\cdot\|^\natural = (\|\cdot\|_n^\natural)$. We assume that W separates the points of $\mathcal{GCS}(V)$.*

(1) *Suppose that* $\|\cdot\|^\natural$ *is a matrix norm on* W *and that it is equivalent to the matrix norm* $\|\cdot\|$. *If the* $\mathcal{D}_{\mathcal{L}}$*-topology on* $\mathcal{GCS}(V)$ *agrees with the BW-topology, then the image of* $\mathcal{L}^1$ *in* $\tilde{W}$ *is totally bounded for* $\|\cdot\|^\sim$.

(2) *If the image of* L_1^1 *in* $\tilde{W}$ *is totally bounded for* $\|\cdot\|_1^\sim$, *then the* $\mathcal{D}_{\mathcal{L}}$*-topology on* $\mathcal{GCS}(V)$ *agrees with the BW-topology.*

Proof. (1) If $\|\cdot\|^\natural$ is equivalent to $\|\cdot\|$, then $\|\cdot\|_1^\natural$ is equivalent to $\|\cdot\|_1$. When $n=1$, $GCS_1(V)$ is just the space S in [16]. Since the D_{L_1}-topology agrees with the BW-topology on $GCS_1(V)$, the image of L_1^1 in $\tilde{W}$ is totally bounded for $\|\cdot\|_1^\sim$ by Theorem 1.8 in [16] and Proposition 3.1. So given $\epsilon>0$, there exist elements $a_1,\cdots,a_k\in L_1^1$ such that $\tilde{L}_1^1\subseteq\bigcup_{j=1}^k\left\{\tilde{a}\in\tilde{W}:\|\tilde{a}-\tilde{a}_j\|_1^\sim<\epsilon\right\}$. For a matrix gauge $\mathcal{G}=(g_n)$ on a vector space Y and $v=[v_{ij}]\in M_n(Y)$, we have the constraint $g_1(v_{ij})\le g_n(v)\le\sum_{i,j=1}^n g_1(v_{ij})$ [10]. From the first inequality, we get that $L_n^1\subseteq M_n(L_1^1)$.

Denote $X_n=\{a\in M_n(W):a=[a_{ij}],a_{ij}\in\{a_1,\cdots,a_k\}\}$. Then X_n is a finite set for every $n\in\mathbb{N}$. Now for $b=[b_{ij}]\in L_n^1$, there exists an $a=[a_{ij}]\in X_n$ such that $\|\tilde{b}_{ij}-\tilde{a}_{ij}\|_1^\sim<\epsilon$. So $\|\tilde{b}-\tilde{a}\|_n^\sim\le\sum_{i,j=1}^n\|\tilde{b}_{ij}-\tilde{a}_{ij}\|_1^\sim<n^2\epsilon$. Therefore, the image of L_n^1 in $M_n(\tilde{W})$ is totally bounded for $\|\cdot\|_n^\sim$. Consequently, the image of $\mathcal{L}^1$ in $\tilde{W}$ is totally bounded for $\|\cdot\|^\sim$.

(2) Suppose that the image of L_1^1 in $\tilde{W}$ is totally bounded for $\|\cdot\|_1^\sim$. From the proof of (1), we see that the image of $\mathcal{L}^1$ in $\tilde{W}$ is totally bounded for $\|\cdot\|^\sim$. For any $n\in\mathbb{N}$, let $\varphi\in GCS_n(V)$ and $\epsilon>0$ be given, and let $B(\varphi,\epsilon)$ be the D_{L_n}-ball of radius ϵ at φ. By the total boundedness of $\tilde{L}_n^1$, we can find elements $a_1,a_2,\cdots,a_k\in L_n^1$ such that the $\|\cdot\|_n^\sim$-balls of radius $\frac{\epsilon}{6}$ at the $\tilde{a}_j$'s cover $\tilde{L}_n^1$.

Assume that $\varphi=[\varphi_{pq}]$ and $a_j=[a_{st}^{(j)}]$ for $j\in\{1,2,\cdots,k\}$. Let $Q=\big\{\psi=[\psi_{ij}]\in GCS_n(V):|\psi_{pq}(a_{st}^{(j)})-\varphi_{pq}(a_{st}^{(j)})|<\frac{\epsilon}{6n^4},s,t,p,q\in\{1,2,\cdots,n\},j\in\{1,2,\cdots,k\}\big\}$. Then Q is BW-open and $\varphi\in Q$. Set $N=\{\psi\in GCS_n(V):\|\langle\!\langle\varphi-\psi,a_j\rangle\!\rangle\|<\frac{\epsilon}{6},1\le j\le k\}$. Then $Q\subseteq N$ and so N is a BW-neighborhood of φ. For any $a\in L_n^1$, there are a $j\in\{1,2,\cdots,k\}$ and a $b\in K_n$ such that $\|a-(a_j+b)\|_n<\frac{\epsilon}{6}$. Hence for any $\psi\in N$, we have that $\|\langle\!\langle\varphi-\psi,a\rangle\!\rangle\|\le\|\varphi\|_{cb}\|a-(a_j+b)\|_n+\|\langle\!\langle\varphi-\psi,a_j\rangle\!\rangle\|+\|\psi\|_{cb}\|(a_j+b)-a\|_n<\frac{\epsilon}{2}$. By Lemma 5.2 in [10], we obtain $\|\langle\!\langle\varphi-\psi,a\rangle\!\rangle\|\le\frac{\epsilon}{2}$ for all $a\in L_r^1$ with $r\in\mathbb{N}$ arbitrary. Thus $D_{L_n}(\varphi,\psi)\le\frac{\epsilon}{2}$. Consequently $N\subseteq B(\varphi,\epsilon)$. Since W separates the points of $\mathcal{GCS}(V)$, by Proposition 3.3 the D_{L_n}-topology on $GCS_n(V)$ is finer than the BW-topology. Therefore, the D_{L_n}-topology on $GCS_n(V)$ agrees with the BW-topology. We conclude that the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{GCS}(V)$ agrees with the BW-topology. □

For a sequence $\{r_n\}$ of positive constants, we set $CB_n^{r_n}(\tilde{V})=\{f\in M_n((\tilde{V})^*):\|f\|_{cb}\le r_n\}$, and this is just the subset of $M_n(V^*)$ consisting of those $f\in M_n(V^*)$ such that $\|f\|_{cb}\le r_n$ and $f(k)=0_n$ for $k\in K_1$. If $CB_n^{r_n}(\tilde{V})\subseteq(GCS_n(V)-GCS_n(V))+i(GCS_n(V)-GCS_n(V))$ for each $n\in\mathbb{N}$, we say that V has the *matrix* $\{r_n\}$*-decomposable property* about $\mathcal{K}$; if $r_n=r$ for all $n\in\mathbb{N}$, we say that V has the *matrix* r*-decomposable property* about $\mathcal{K}$.

Proposition 3.8. *If* V *has the matrix* $\{r_n\}$*-decomposable property about* $\mathcal{K}$ *and* $\mathcal{D}_{\mathcal{L}}$ *is bounded, that is, each* D_{L_n} *is bounded, then there is a sequence* $\{d_n\}$ *of positive constants such that* $\|\tilde{a}\|_n^\sim\le d_n$ *for all* $a\in L_n^1$ *and* $n\in\mathbb{N}$.

Proof. Assume that each D_{L_n} is bounded by c_n. Then for $a \in M_n(W)$ with $L_n(a) \leq 1$, we have that

$$\begin{aligned}
\|\tilde{a}\|_n^{\sim} &= \sup\{\|\langle\langle f, \tilde{a}\rangle\rangle\| : f \in CB_n(\tilde{V}), \|f\|_{cb} \leq 1\} \\
&\leq r_n^{-1} \sup\{\|\langle\langle(\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4), \tilde{a}\rangle\rangle\| : \varphi_i \in GCS_n(V), i = 1, 2, 3, 4\} \\
&\leq 2r_n^{-1} \sup\{\|\langle\langle\varphi_1 - \varphi_2, \tilde{a}\rangle\rangle\| : \varphi_1, \varphi_2 \in GCS_n(V)\} \\
&= 2r_n^{-1} \sup\{\|\langle\langle\varphi_1, a\rangle\rangle - \langle\langle\varphi_2, a\rangle\rangle\| : \varphi_1, \varphi_2 \in GCS_n(V)\} \\
&\leq 2r_n^{-1} \sup\{D_{L_n}(\varphi_1, \varphi_2) : \varphi_1, \varphi_2 \in GCS_n(V)\} \leq 2c_n r_n^{-1}
\end{aligned}$$

by Theorem 4.2 in [18] and Lemma 2.1 in [8]. Letting $d_n = 2c_n r_n^{-1}$, we have that $\|\tilde{a}\|_n^{\sim} \leq d_n$ for all $a \in L_n^1$. □

Here, we give another additional premise such that for the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{GCS}(V)$ to agree with the BW-topology it is necessary that the image of $\mathcal{L}^1$ in $\tilde{W}$ is totally bounded for $\|\cdot\|^{\sim}$.

Theorem 3.9. *With the notation of Theorem* 3.7*, suppose that V has the matrix $\{r_n\}$-decomposable property about $\mathcal{K} = (K_n)$. If the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{GCS}(V)$ agrees with the BW-topology, then the image of $\mathcal{L}^1$ in $\tilde{W}$ is totally bounded for $\|\cdot\|^{\sim}$.*

Proof. If each D_{L_n}-topology agrees with the BW-topology on $GCS_n(V)$, then each D_{L_n} must be bounded since $GCS_n(V)$ is BW-compact. By Proposition 3.8, there exists a sequence $\{d_n\}$ of positive constants such that $\|\tilde{a}\|_n^{\sim} \leq d_n$ for $a \in L_n^1$ and $n \in \mathbb{N}$. Choose $k_n > d_n$. Then $\|\tilde{a}\|_n^{\sim} < k_n$ if $a \in L_n^1$. So the image of $B_n^{k_n}$ in $M_n(\tilde{W})$ is the same as the image of L_n^1. For $r, n \in \mathbb{N}$, $a \in L_n^1$ and $\varphi, \psi \in GCS_r(V)$, we have that $\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| \leq D_{L_r}(\varphi, \psi)$. Let $\epsilon > 0$. For each $\varphi \in GCS_n(V)$, let U_φ be an open neighborhood of φ such that $\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\| < \frac{\epsilon}{3}$ for $a \in L_n^1 (n \in \mathbb{N})$, and $\psi \in U_\varphi$. Now $\{U_\varphi : \varphi \in GCS_n(V)\}$ is an open covering of $GCS_n(V)$. Since $GCS_n(V)$ is BW-compact, there are points $\varphi_1, \cdots, \varphi_k$ in $GCS_n(V)$ such that $GCS_n(V) = \bigcup_{j=1}^k U_{\varphi_j}$. Assume that $\varphi_j = [\varphi_{st}^{(j)}], j = 1, \cdots, k$, and denote $D = \{\ddot{a} : \ddot{a} = (\varphi_{11}^{(1)}(a_{11}), \varphi_{11}^{(1)}(a_{12}), \cdots, \varphi_{11}^{(1)}(a_{nn}), \varphi_{12}^{(1)}(a_{11}), \varphi_{12}^{(1)}(a_{12}), \cdots, \varphi_{12}^{(1)}(a_{nn}), \cdots, \varphi_{nn}^{(1)}(a_{11}), \varphi_{nn}^{(1)}(a_{12}), \cdots, \varphi_{nn}^{(1)}(a_{nn}), \varphi_{11}^{(2)}(a_{11}), \cdots, \varphi_{nn}^{(2)}(a_{nn}), \cdots, \varphi_{11}^{(k)}(a_{11}), \cdots, \varphi_{nn}^{(k)}(a_{nn})), a = [a_{ij}] \in B_n^{k_n}\}$. For $a \in B_n^{k_n}$, we have

$$\begin{aligned}
\|\ddot{a}\| &= \sqrt{\textstyle\sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n |\varphi_{st}^{(j)}(a_{rl})|^2} \\
&= \sqrt{\textstyle\sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n |e_{sr}^* \langle\langle\varphi_j, a\rangle\rangle e_{tl}|^2} \\
&\leq \sqrt{\textstyle\sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n \|\langle\langle\varphi_j, a\rangle\rangle\|^2} \\
&\leq \sqrt{\textstyle\sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n \|\varphi_j\|_{cb}^2 \|a\|_n^2} \leq n^2 k_n \sqrt{k},
\end{aligned}$$

where e_{ij} is the $nr \times 1$ column matrix $[0, \cdots, 0, 1_{(i-1)n+j}, 0, \cdots, 0]^*$. So D is totally bounded, and hence there are elements $a_1, a_2, \cdots, a_m \in B_n^{k_n}$ such that $D \subseteq \bigcup_{j=1}^m \{\ddot{a} : \|\ddot{a} - \ddot{a}_j\| < \frac{\epsilon}{3n^2}, a \in B_n^{k_n}\}$. Now for $a \in B_n^{k_n}$, there exists a $p \in \{1, 2, \cdots, m\}$ such that $\|\ddot{a} - \ddot{a}_p\| < \frac{\epsilon}{3}$. For any $\varphi \in GCS_n(V)$, there is a $j \in \{1, 2, \cdots, k\}$ such that $\varphi \in U_{\varphi_j}$. So by Cauchy's inequality, we have

$$\begin{aligned}
\|\langle\langle a - a_p, \varphi\rangle\rangle\| &\leq \|\langle\langle a - a_p, \varphi_j\rangle\rangle\| + \|\langle\langle a - a_p, \varphi - \varphi_j\rangle\rangle\| \\
&\leq n^2 \|\ddot{a} - \ddot{a}_p\| + \|\langle\langle a, \varphi - \varphi_j\rangle\rangle\| + \|\langle\langle a_p, \varphi - \varphi_j\rangle\rangle\| < \epsilon.
\end{aligned}$$

By Lemma 2.1 in [8], we have that

$$\begin{aligned}
\|\tilde{a}-\tilde{a}_p\|_n^{\sim} &= \sup\{\|\langle\langle \tilde{a}-\tilde{a}_p, f\rangle\rangle\| : f \in M_n(\tilde{V}), \|f\|_{cb} \le 1\} \\
&= r_n^{-1}\sup\{\|\langle\langle \tilde{a}-\tilde{a}_p, (\varphi_1-\varphi_2)+i(\varphi_3-\varphi_4)\rangle\rangle\| : \varphi_i \in GCS_n(V), i=1,2,3,4\} \\
&\le 2r_n^{-1}\sup\{\|\langle\langle \tilde{a}-\tilde{a}_p, \varphi_1-\varphi_2\rangle\rangle\| : \varphi_1,\varphi_2 \in GCS_n(V)\} \\
&= 2r_n^{-1}\sup\{\|\langle\langle a-a_p, \varphi_1-\varphi_2\rangle\rangle\| : \varphi_1,\varphi_2 \in GCS_n(V)\} \\
&\le 4r_n^{-1}\sup\{\|\langle\langle a-a_p, \varphi\rangle\rangle\| : \varphi \in GCS_n(V)\} \\
&\le 4r_n^{-1}\epsilon.
\end{aligned}$$

Therefore, the image of $B_n^{k_n}$ in $M_n(\tilde{W})$ is totally bounded for $\|\cdot\|_n^{\sim}$; then so is L_n^1. From the arbitrariness of n, the image of $\mathcal{L}^1$ in $\tilde{W}$ is totally bounded for $\|\cdot\|^{\sim}$. □

4. Matrix metrics from Dirac operators

Let $\mathcal{A}$ be a unital C^*-algebra. An *unbounded Fredholm module* $(\mathcal{H}, D)$ over $\mathcal{A}$ is [5, 15]: a Hilbert space $\mathcal{H}$ which is a left $\mathcal{A}$-module, that is, a Hilbert space $\mathcal{H}$ and a *-representation of $\mathcal{A}$ on $\mathcal{H}$; an unbounded, self-adjoint operator D (the generalized Dirac operator) on $\mathcal{H}$ such that $\mathcal{B} = \{a \in \mathcal{A}$: $[D, a]$ is densely defined and extends to a bounded operator on $\mathcal{H}\}$ is norm dense in $\mathcal{A}$; $(1 + D^2)^{-1}$ is a compact operator. Given a triple $(\mathcal{A}, \mathcal{H}, D)$, where $(\mathcal{H}, D)$ is an unbounded Fredholm module over a unital C^*-algebra $\mathcal{A}$, then $\mathcal{B}(\mathcal{H})$ is a Banach $\mathcal{A}$-module, and the mapping $T : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H})$, defined by $Ta = [D, a]$, is a densely defined derivation from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. For $n \in \mathbb{N}$, we define $L_n(a) = \|T_n(a)\|, a \in M_n(\mathcal{B})$, and denote $\mathcal{L} = (L_n)$. For the unital C^*-algebra $\mathcal{A}$, the *matrix state space* of $\mathcal{A}$ is the collection $\mathcal{CS}(\mathcal{A}) = (CS_n(\mathcal{A}))$ of matrix states $CS_n(\mathcal{A}) = \{\varphi : \varphi$ is a completely positive linear mapping from $\mathcal{A}$ to M_n, $\varphi(1) = 1_n\}$, where 1 is the identity of $\mathcal{A}$. This space is important in the study of C^*-algebras [19].

Proposition 4.1. *$\mathcal{L}$ is a matrix seminorm on $\mathcal{B}$, and $M_n(\mathbb{C}1) \subseteq K_n = \{a \in M_n(\mathcal{B}) : L_n(a) = 0\}$.*

Proof. It is clear that L_n is a seminorm on $M_n(\mathcal{B})$ for each $n \in \mathbb{N}$. For $a = [a_{ij}] \in M_m(\mathcal{B})$, $b = [b_{st}] \in M_n(\mathcal{B})$, $\alpha = [\alpha_{kl}] \in M_{n,m}$ and $\beta = [\beta_{uv}] \in M_{m,n}$, we have that

$$\begin{aligned}
L_{m+n}(a \oplus b) =& \|T_{m+n}(a \oplus b)\| = \|T_m(a) \oplus T_n(b)\| \\
=& \|[[D, a_{ij}]] \oplus [[D, b_{st}]]\| = \max\{\|[[D, a_{ij}]]\|, \|[[D, b_{st}]]\|\} \\
=& \max\{\|T_m(a)\|, \|T_n(b)\|\} = \max\{L_m(a), L_n(b)\}
\end{aligned}$$

and

$$\begin{aligned}
L_n(\alpha a\beta) =& \|T_n(\alpha a\beta)\| = \|T_n[\sum_{i,j=1}^{m} \alpha_{si}a_{ij}\beta_{jt}]\| = \|[\sum_{ij=1}^{m} \alpha_{si}T(a_{ij})\beta_{jt}]\| \\
=& \|\alpha T_m(a)\beta\| = \|\alpha[[D, a_{ij}]]\beta\| \le \|\alpha\|\|[[D, a_{ij}]]\|\|\beta\| \\
=& \|\alpha\|\|T_m(a)\|\|\beta\| = \|\alpha\|L_m(a)\|\beta\|.
\end{aligned}$$

So $\mathcal{L}$ is a matrix seminorm on $\mathcal{B}$. For any $[a_{ij}1] \in M_n(\mathbb{C}1)$, we have that $L_n([a_{ij}1]) = \|T_n([a_{ij}1])\| = \|[T(a_{ij}1)]\| = \|[0]\| = 0$, that is, $M_n(\mathbb{C}1) \subseteq K_n$. □

Proposition 4.2. *Let $P_n = M_n(\mathbb{C}1)$ and $\mathcal{P} = (P_n)$. Then $\mathcal{A}$ has the matrix 2-decomposable property about $\mathcal{P}$.*

Proof. Given $f \in CB_n^2(\tilde{\mathcal{A}})$, suppose that f is self-adjoint, that is, $f^* = f$. By Satz 4.5 in [21] or Proposition 1.3, Remark 1.5 and Theorem 1.6 in [11] and the injectivity of M_n, there are completely positive linear mappings ψ_1 and ψ_2 from $\mathcal{A}$ to M_n such that $f = \psi_1 - \psi_2$ and $\|\psi_1 + \psi_2\| = \|\psi_1 - \psi_2\|_{cb} = \|f\|_{cb}$. But $0_n = f(1) = \psi_1(1) - \psi_2(1)$. So $\psi_1(1) = \psi_2(1)$. Therefore, $\|\psi_1\| = \|\psi_1\|_{cb} = \|\psi_1(1)\| = \|\psi_2(1)\| = \|\psi_2\|_{cb} = \|\psi_2\|$ by Proposition 3.5 in [14]. Since $\|f\|_{cb} \leq 2$, we have that $\|\psi_1(1) + \psi_2(1)\| = \|\psi_1 + \psi_2\| \leq 2$. So $0_n \leq \psi_1(1) = \psi_2(1) \leq 1_n$. If $\psi_1(1) \leq 1_n$ but $\psi_1(1) \neq 1_n$, choosing a $\phi \in CS_1(\mathcal{A})$ and setting $\varphi_1(a) = \psi_1(a) + (1_n - \psi_1(1))\phi(a)$ and $\varphi_2(a) = \psi_2(a) + (1_n - \psi_2(1))\phi(a)$ for $a \in \mathcal{A}$, then we have that $\varphi_1(1) = \varphi_2(1) = 1_n$ and $\varphi_1 - \varphi_2 = \psi_1 - \psi_2$. For $k \in \mathbb{N}$ and $[a_{ij}] \in M_k(\mathcal{A})$, we have that

$$\begin{aligned}(\varphi_1)_k([a_{ij}]) &= [\varphi_1(a_{ij})] = [\psi_1(a_{ij}) + (1_n - \psi_1(1))\phi(a_{ij})] \\ &= [\psi_1(a_{ij})] + [(1_n - \psi_1(1))\phi(a_{ij})] \\ &= (\psi_1)_k([a_{ij}]) + [\phi(a_{ij})] \otimes (1_n - \psi_1(1)) \\ &= (\psi_1)_k([a_{ij}]) + \phi_k([a_{ij}]) \otimes (1_n - \psi_1(1)).\end{aligned}$$

From this observation and the complete positivity of ψ_1 and ϕ, we see that $\varphi_1 \in CS_n(\mathcal{A})$. Similarly, we have that $\varphi_2 \in CS_n(\mathcal{A})$. If f is not self-adjoint, we set $f_1 = \frac{1}{2}(f+f^*)$, $f_2 = \frac{1}{2i}(f-f^*)$. Then f_1 and f_2 are self-adjoint completely bounded linear mappings. If $\|f\|_{cb} \leq 2$, then $\|f_1\|_{cb} \leq 2$ and $\|f_2\|_{cb} \leq 2$. So the result above implies that there exist $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in CS_n(\mathcal{A})$ such that $f = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4)$. Accordingly, $\mathcal{A}$ has the matrix 2-decomposable property about $\mathcal{P}$. □

It is apparent that $\mathbb{C}1$ is a closed subspace of $\mathcal{A}$ and $\mathcal{B}$, so $\mathcal{B}/(\mathbb{C}1)$ is an operator space with the quotient matrix norm $\|\cdot\|^{\sim} = (\|\cdot\|_n^{\sim})$ from $\|\cdot\| = (\|\cdot\|_n)$ on $\mathcal{B}$ [18].

Theorem 4.3. *Let $(\mathcal{H}, D)$ be an unbounded Fredholm module over a unital C^*-algebra, and let the matrix metric $\mathcal{D}_{\mathcal{L}}$ on $\mathcal{CS}(\mathcal{A})$ be defined by*

$$\begin{aligned} D_{L_n}(\varphi, \psi) \quad = \quad & \sup\big\{\|\langle\!\langle \varphi, a\rangle\!\rangle - \langle\!\langle \psi, a\rangle\!\rangle\| : a = [a_{ij}] \in M_r(\mathcal{B}), \\ & L_r(a) = \|[[D, a_{ij}]]\| \leq 1, r \in \mathbb{N}\big\}\end{aligned}$$

for $\varphi, \psi \in CS_n(\mathcal{A})$. Then the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{A})$ agrees with the BW-topology if and only if the image of L_1^1 in $\mathcal{B}/(\mathbb{C}1)$ is totally bounded for $\|\cdot\|_1^{\sim}$.

Proof. First we show that $K_n = M_n(\mathbb{C}1)$ under the conditions of the theorem.

Suppose that the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{A})$ agrees with the BW-topology. For each $n \in \mathbb{N}$, $CS_n(\mathcal{A})$ is BW-compact implies that D_{L_n} is bounded. If $a = [a_{ij}] \in M_n(\mathcal{B}) \setminus M_n(\mathbb{C}1)$ and $L_n(a) = 0$, then there exists an $a_{ij} \in \mathcal{B} \setminus (\mathbb{C}1)$ such that $L_1(a_{ij}) = 0$ since $\mathcal{L}$ is a matrix seminorm by Proposition 4.1. From the definition of $\mathcal{L}$, we see that $L_n(a^*) = L_n(a)$ for each $a \in M_n(\mathcal{B})$. There is a self-adjoint $b \in \mathcal{B} \setminus (\mathbb{C}1)$ with $L_1(b) = 0$. So there is a bounded linear functional f on $\mathcal{A}$ such that $\|f\| = 1$, $f(y) = 0$ for each y in $\mathbb{C}1$, and $f(b) \neq 0$. Since b is self-adjoint, there is a bounded hermitian linear functional h on $\mathcal{A}$ such that $h(1) = 0$ but $h(b) \neq 0$. By the Hahn-Jordan decomposition theorem [12], there are positive linear functionals φ_1 and φ_2 on $\mathcal{A}$ such that $h = \varphi_1 - \varphi_2$ and $\|h\| = \|\varphi_1\| + \|\varphi_2\|$. From the condition $h(1) = 0$, we get $\varphi_1(1) = \varphi_2(1)$. Thus there are $\psi_1, \psi_2 \in CS_1(\mathcal{A})$ and $c > 0$ such that $h = c(\psi_1 - \psi_2)$. Now, for any $r \in \mathbb{R}^+$ we have that $L_1(rb) = 0 < 1$. Therefore, $D_{L_1}(\psi_1, \psi_2) \geq |\psi_1(rb) - \psi_2(rb)| = \frac{r}{c}|h(b)|$ for all $r \in \mathbb{R}^+$. This contradicts the boundedness of D_{L_1}. So $K_n = M_n(\mathbb{C}1)$ by Proposition 4.1.

Assume that the image of $\mathcal{L}^1$ in $\mathcal{B}/(\mathbb{C}1)$ is totally bounded for $\|\cdot\|^{\sim}$. If there is an $a = [a_{ij}] \in K_n$ and $a \notin M_n(\mathbb{C}1)$, then $\|\tilde{a}\|_n^{\sim} \neq 0$. Since $ra \in K_n$ for any $r \in \mathbb{R}$, we get that $\|\widetilde{ra}\|_n^{\sim} = |r|\|\tilde{a}\|_n^{\sim} \to +\infty$ $(r \to \infty)$. This contradicts the boundedness of the image of $\mathcal{L}^1$ in $\mathcal{B}/(\mathbb{C}1)$. Thus we also have that $K_n = M_n(\mathbb{C}1)$ by Proposition 4.1.

For $n \in \mathbb{N}$, let $\varphi = [\varphi_{ij}], \psi = [\psi_{ij}] \in CS_n(\mathcal{A})$ and $\varphi \neq \psi$. Then there exist $i_0, j_0 \in \{1, 2, \cdots, n\}$ such that $\varphi_{i_0,j_0} \neq \psi_{i_0,j_0}$. Since $\mathcal{B}$ is norm dense in $\mathcal{A}$ and $\varphi_{i_0,j_0}, \psi_{i_0,j_0} \in \mathcal{A}^*$, there is a $b \in \mathcal{B}$ such that $\varphi_{i_0,j_0}(b) \neq \psi_{i_0,j_0}(b)$. Therefore, $\varphi(b) \neq \psi(b)$. So $\mathcal{B}$ separates the points of $\mathcal{CS}(\mathcal{A})$.

Since a unital and completely contractive linear mapping from $\mathcal{A}$ into M_n is completely positive (Proposition 3.4 in [14]), the conditions in the theorem imply that the matrix state space $\mathcal{CS}(\mathcal{A})$ of $\mathcal{A}$ is the graded set $\mathcal{GCS}(\mathcal{A})$ that was defined in section 3 with $\varphi_n^{(0)}(1) = 1_n$. Now the theorem follows from Proposition 4.2, Theorem 3.7 and Theorem 3.9. □

Remark 4.4. Though D_{L_n} does not always define a bounded metric (see Proposition 4 in [4]), the conditions in the theorem above imply that D_{L_n} is bounded from the proof of it.

5. Matrix metrics from matrix Lipschitz seminorms

A complex vector space V is said to be *matrix ordered* if V is a *-vector space, each $M_n(V)$ is partially ordered and $\gamma^* M_n(V)^+ \gamma \subseteq M_m(V)^+$ when $\gamma \in M_{n,m}$. A *matrix order unit space* $(\mathcal{V}, 1)$ is a matrix ordered space $\mathcal{V}$ together with a distinguished order unit 1 satisfying the conditions: $\mathcal{V}^+$ is a proper cone with the order unit 1 and each of the cones $M_n(\mathcal{V})^+$ is Archimedean. Each matrix order unit space $(\mathcal{V}, 1)$ may be provided with the norm

$$\|v\| = \inf\left\{t \in \mathbb{R} : \begin{bmatrix} t1 & v \\ v^* & t1 \end{bmatrix} \geq 0\right\}.$$

As in section 3, we will not assume that $\mathcal{V}$ is complete for the norm.

A *Lipschitz seminorm* on an order unit space S is a seminorm on S such that its null space is the scalar multiples of the order unit [17]. Here we give a non-commutative version of Lipschitz seminorms.

Definition 5.1. Given a matrix order unit space $(\mathcal{V}, 1)$, a *matrix Lipschitz seminorm* $\mathcal{L}$ on $(\mathcal{V}, 1)$ is a sequence of seminorms $L_n : M_n(\mathcal{V}) \longmapsto [0, +\infty)$ such that the null space of each L_n is $M_n(\mathbb{C}1)$, $L_{m+n}(v \oplus w) = \max\{L_m(v), L_n(w)\}$, $L_n(\alpha v \beta) \leq \|\alpha\| L_m(v) \|\beta\|$ and $L_m(v^*) = L_m(v)$ for any $v \in M_m(\mathcal{V})$, $w \in M_n(\mathcal{V})$, $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$.

By an *operator system* one means a self-adjoint subspace $\mathcal{R}$, which is not necessarily closed, in a unital C^*-algebra $\mathcal{A}$ such that $\mathcal{R}$ contains the identity $1 \in \mathcal{A}$. Parallel to the abstract characterization of Kadison's function systems as order unit space [13], Choi-Effros's representation theorem says that every matrix order unit space is completely order isomorphic to an operator system [3]. For a matrix order unit space $(\mathcal{V}, 1)$, the *matrix state space* is the collection $\mathcal{CS}(\mathcal{V}) = (CS_n(\mathcal{V}))$ of matrix states

$$CS_n(\mathcal{V}) = \{\varphi\colon \varphi \text{ is a unital completely positive linear mapping from } \mathcal{V} \text{ to } M_n\},$$

[20]. We consider $\mathcal{CS}(\mathcal{V})$ the non-commutative version of the state space of an order unit space.

Proposition 5.2. *Let $(\mathcal{V}, 1)$ be a matrix order unit space, and let $P_n = M_n(\mathbb{C}1)$ and $\mathcal{P} = (P_n)$. Then $\mathcal{V}$ has the matrix 2-decomposable property about $\mathcal{P}$.*

Proof. Since every matrix order unit space is completely order isomorphic to an operator system, there is a unital C^*-algebra $(\mathcal{A}, 1_{\mathcal{A}})$ such that $\mathcal{V} \subseteq \mathcal{A}$ and $1_{\mathcal{A}} = 1$. Because $\mathbb{C}1$ is a closed subspace of $\mathcal{V}$, we have the complete isometry $(\mathcal{V}/(\mathbb{C}1))^* \cong (\mathbb{C}1)^{\perp}$, where $(\mathbb{C}1)^{\perp} = \{f \in \mathcal{V} : f(1) = 0\}$[2]. So $(\tilde{\mathcal{V}})^*$ is just the subspace of $\mathcal{V}^*$ consisting of those $f \in \mathcal{V}^*$ such that $f(1) = 0$.

For $f \in CB_n^2(\tilde{\mathcal{V}})$, there exists a completely bounded linear mapping g from $\mathcal{A}$ to M_n, which extends f, with $\|g\|_{cb} = \|f\|_{cb}$ by Theorem 7.2 in [14]. From Proposition 4.2, there are $\psi_1, \psi_2, \psi_3, \psi_4 \in CS_n(\mathcal{A})$ such that $g = (\psi_1 - \psi_2) + i(\psi_3 - \psi_4)$. Let $\varphi_i = \psi_i|_{\mathcal{V}}$ $(i = 1, 2, 3, 4)$, the restriction of ψ_i to $\mathcal{V}$. Clearly each $\varphi_i \in CS_n(\mathcal{V})$. So we obtain that $f = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4)$. Therefore, $\mathcal{V}$ has the matrix 2-decomposable property about $\mathcal{P}$. □

Theorem 5.3. *Let $(\mathcal{V}, 1)$ be a matrix order unit space with operator space norm $\|\cdot\| = (\|\cdot\|_n)$, and let $\mathcal{L} = (L_n)$ be a matrix Lipschitz seminorm on $(\mathcal{V}, 1)$. Then the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{V})$ agrees with the BW-topology if and only if the image of L_1^1 in $\tilde{\mathcal{V}}$ is totally bounded for $\|\cdot\|_1^{\sim}$.*

Proof. If f is a completely bounded linear mapping from $\mathcal{V}$ to M_n and $f(1) = 1_n$ and $\|f\|_{cb} = 1$, then f is completely positive by Proposition 3.4 in [14]. So the matrix state space $\mathcal{CS}(\mathcal{V})$ of $\mathcal{V}$ is the graded set $\mathcal{GCS}(\mathcal{V})$ that was defined in section 3 with $\varphi_n^{(0)}(1) = 1_n$. Now by Proposition 5.2, Theorem 3.7 and Theorem 3.9, the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{V})$ agrees with the BW-topology if and only if the image of L_1^1 in $\tilde{\mathcal{V}}$ is totally bounded for $\|\cdot\|_1^{\sim}$. □

Acknowledgements

I would like to thank the referee for helpful comments and suggestions that prompted a number of clarifications and corrections. This research was partially supported by China Scholarship Council, National Natural Science Foundation of China (Grant No. 10271090), and Shanghai Priority Academic Discipline.

References

[1] W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141-224. MR0253059 (40:6274)

[2] D. P. Blecher, *The standard dual of an operator space*, Pacific J. Math. **153** (1992), 15-30. MR1145913 (93d:47083)

[3] M. -D. Choi, E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209. MR0430809 (55:3814)

[4] A. Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory Dynam. Systems, **9**(1989), 207–220. MR1007407 (90i:46124)

[5] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994. MR1303779 (95j:46063)

[6] E. G. Effros, *Aspects of noncommutative order*, C^*-algebras and Applications to Physics (Proc. Second Japan-USA Sem., Los Angeles, Calif., 1977), pp. 1–40, Lecture Notes in Math., 650, Springer, Berlin, 1978. MR0504750 (80j:46093)

[7] E. G. Effros, *Some quantizations and reflections inspired by the Gelfand-Naimark theorem*, C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 98–113, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994. MR1292011 (95h:46081)

[8] E. G. Effros and Z-J. Ruan, *A new approach to operator spaces*, Canad. Math. Bull. **34** (1991), 329–337. MR1127754 (93a:47045)

[9] E. G. Effros and Z-J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000. MR1793753 (2002a:46082)

[10] E. G. Effros and C. Webster, *Operator analogues of locally convex spaces*, Operator algebras and applications (Samos, 1996), 163–207, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 495, Kluwer Acad. Publ., Dordrecht, 1997. MR1462680 (99b:46081)

[11] U. Haagerup, *Injectivity and decomposition of completely bounded maps*, Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), 170–222, Lecture Notes in Math., 1132, Springer, Berlin, 1985. MR0799569 (87i:46133)

[12] R. V. Kadison, J. R. Ringrose, *Fundamentals of the theory of operator algebras*, Vol. I, Academic Press, New York, 1983. MR0719020 (85j:46099)

[13] R. V. Kadison, *A representation theory for commutative topological algebra*. Mem. Amer. Math. Soc., 7, 1951. MR0044040 (13:360b)

[14] V. I. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics Series, 146. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1986. MR0868472 (88h:46111)

[15] B. Pavlović, *Defining metric spaces via operators from unital C^*-algebras*, Pacific J. Math. **186** (1998), 285–313. MR1663810 (99m:46162)

[16] M. A. Rieffel, *Metrics on states from actions of compact groups*, Doc. Math. **3** (1998), 215–229. MR1647515 (99k:46126)

[17] M. A. Rieffel, *Metrics on state spaces*, Doc. Math. **4** (1999), 559–600. MR1727499 (2001g:46154)

[18] Z. -J. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal. **76** (1988), 217–230. MR0923053 (89h:46082)

[19] R. R. Smith, J. D. Ward, *The geometric structure of generalized state spaces*, J. Funct. Anal. **40** (1981), 170–184. MR0609440 (82i:46094)

[20] C. Webster, S. Winkler, *The Krein-Milman theorem in operator convexity*, Trans. Amer. Math. Soc. **351** (1999), 307–322. MR1615970 (99d:46079)

[21] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal. **40** (1981), 127–150. MR0609438 (83d:46072)

Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China

E-mail address: wwu@math.ecnu.edu.cn