

PHANTOM DEPTH AND FLAT BASE CHANGE

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ABSTRACT. We prove that if $f : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism, $S/\mathfrak{m}S$ is Cohen-Macaulay and F -injective, and R and S share a weak test element, then a tight closure analogue of the (standard) formula for depth and regular sequences across flat base change holds. As a corollary, it follows that phantom depth commutes with completion for excellent local rings. We give examples to show that the analogue does not hold for surjective base change.

All rings considered in this paper are Noetherian, local, and of positive prime characteristic $p > 0$. For such rings R (among others), Hochster and Huneke [HH90] developed a theory of “tight closure” for finitely-generated R -modules. In [Abe94], Ian Aberbach defined a tight closure analogue of depth, called *phantom depth*, and showed that it satisfies (analogues of) many properties we expect depth to satisfy. One such property is a “phantom Auslander-Buchsbaum theorem”, which is like the classical Auslander-Buchsbaum theorem but with both depth and projective dimension¹ replaced by their “phantom” analogues. In [Eps] the present author showed that under mild conditions on R and M , the phantom depth of a finitely generated R -module M is the length of *any* maximal phantom regular sequence on M , as Aberbach [Abe94] had proved in the special case that M has finite phantom projective dimension.

Consider the following standard, extremely useful facts:

Base change formulas for depth (see e.g. [BH97], section 1.2). *Let $\phi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings, let M be a finitely generated R -module, let $\mathbf{x} = x_1, \dots, x_a \in \mathfrak{m}$ be an M -regular sequence and let $\mathbf{y} = y_1, \dots, y_b \in \mathfrak{n}$ be an $(S/\mathfrak{m}S)$ -regular sequence. Then $\phi(\mathbf{x}), \mathbf{y}$ is an $(S \otimes_R M)$ -regular sequence. Furthermore, if \mathbf{x} and \mathbf{y} are maximal regular sequences on M , $S/\mathfrak{m}S$ respectively, then the sequence $\phi(\mathbf{x}), \mathbf{y}$ is a maximal $(S \otimes_R M)$ -regular sequence. In particular, we have*

$$\text{depth}_R M + \text{depth } S/\mathfrak{m}S = \text{depth}_S(S \otimes_R M).$$

If instead of being flat, ϕ is surjective, then for any finitely-generated S -module N ,

$$\text{depth}_R N = \text{depth}_S N.$$

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¹“Phantom projective dimension”, a tight closure analogue to the classical notion of projective dimension, was introduced in [HH90] and further developed in [HH93], [AHH93], and [Eps].

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It seems natural to ask: what parts of these base change formulas hold when we replace “depth” and “regular sequence” with their phantom analogues?

Naïvely, one would hope that all the resulting statements held verbatim. However, base change in tight closure theory is a messy business. One often has to impose conditions on the closed fiber and the residue fields, and the rings sometimes need to share a weak test element for the proofs to work. See [HH94], [Ene00], [Has01], [Abe01], [AE02], [HH00], and [BS02] for work along these lines, and see [Sin99] for an interesting counterexample. These authors investigate preservation of such properties as F -rationality, weak and strong F -regularity, whether tight closure commutes with localization, and extension of the test ideal.

In the spirit of the some of the aforementioned papers, we prove the following “phantom analogue” of the flat base change formula for depth in Section 2:

Main Theorem. *Let $(R, \mathfrak{m}) \xrightarrow{\phi} (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings of prime characteristic $p > 0$. Let M be a finitely generated R -module satisfying avoidance, and suppose that R and S share a q_0 -weak test element c and that the closed fiber $S/\mathfrak{m}S$ is Cohen-Macaulay and F -injective. Then²*

$$\text{ph.depth}_R M + \text{depth } S/\mathfrak{m}S = \text{ph.depth}_S(S \otimes_R M).$$

Unfortunately, no corresponding analogue of the surjective base change formula holds, as we show in two counterexamples in Section 3.

1. BACKGROUND

There are many excellent accounts of tight closure theory, including the seminal paper [HH90] and the monograph [Hun96], so in this note we will only cover the points most salient to our work here. If M is a finitely generated R -module, where (R, \mathfrak{m}) is a Noetherian local ring of prime characteristic $p > 0$, let $\varphi : X \rightarrow Y$ be a minimal free presentation of M . If we fix bases for the free modules X and Y , then φ can be thought of as a matrix (φ_{ij}) of elements of \mathfrak{m} . For a power $q = p^e$ of p , let $\varphi^q : X \rightarrow Y$ be the homomorphism defined by the matrix (φ_{ij}^q) . Then we set $F_R^e(M) := F^e(M) := \text{coker } \varphi^q$. This module is called the q th *Frobenius power* of M . For an element $z \in M$, let y be its preimage in Y . Then z^q is the image of the element y in $F^e(M)$. It is standard that z^q and $F^e(M)$ are independent of the choice of free modules and bases in the minimal free presentation, and indeed $F^e(-) := F_R^e(-)$ can be made into a right-exact functor from the category of finitely-generated R -modules to itself. In particular, if $f : L \rightarrow M$ is a map of finitely-generated R -modules, we get a corresponding map $F^e(f) : F^e(L) \rightarrow F^e(M)$. If $i : N \hookrightarrow M$ is a submodule, then $N_M^{[q]}$ will denote the image of $F^e(N)$ in $F^e(M)$ under the map $F^e(i)$. Note that if e and e' are positive integers and S is an R -algebra, we have that $F^{e+e'} = F^e \circ F^{e'}$ and $F_S^e(S \otimes_R -) = S \otimes_R F_R^e(-)$ as functors on the category of finitely-generated R -modules.

We now have enough to define tight closure of a submodule. Denote by R° the complement of the union of the minimal primes of R . For a submodule $N \subseteq M$, we say that an element $z \in M$ is in the *tight closure of N in M* (in symbols, $z \in N_M^*$)

²Since $S/\mathfrak{m}S$ is Cohen-Macaulay, we could have written $\dim S/\mathfrak{m}S$, $\text{ph.depth}_{S/\mathfrak{m}S} S/\mathfrak{m}S$, or $\text{ph.depth}_S S/\mathfrak{m}S$ instead of $\text{depth } S/\mathfrak{m}S$. However, if the closed fiber were not Cohen-Macaulay, these four numbers could differ. One of the problems in trying to extend the analogy to a situation where the closed fiber is not Cohen-Macaulay would be to choose which of the above four invariants (if any) provides the correct middle term in the displayed formula in the theorem.

if there is some $c \in R^\circ$ and some integer e_0 such that for all $e \geq e_0$, $cz^{p^e} \in N_M^{[p^e]}$. If c and $q_0 = p^{e_0}$ can be chosen uniformly for all such triples (z, M, N) , then we say that c is a *weak test element* (or a q_0 -*weak test element*, if we want to emphasize the power q_0) for R . We say that c is a *completely stable q_0 -weak test element* for R if its image is a q_0 -weak test element for $\hat{R}_{\mathfrak{p}}$ whenever $c \in \mathfrak{p} \in \text{Spec } \hat{R}$. In intricate work, Hochster and Huneke [HH94] showed that whenever R is essentially of finite type over an excellent local ring, it has a completely stable weak test element. In order to simplify our definitions and proofs, we often assume that R has a weak test element.

Let $G^e(M) := F^e(M)/0_{F^e(M)}^*$, the q th *reduced Frobenius power* of M . We say that M satisfies *avoidance* if for any quotient module N of M and any ideal $I \subseteq R$ such that

$$I \subseteq \bigcup_{e \geq 0} \text{Ass } G^e(N),$$

there is some $e \geq 0$ and some $\mathfrak{p} \in \text{Ass } G^e(N)$ such that $I \subseteq \mathfrak{p}$. In particular, if $\mathfrak{m} \subseteq \bigcup_{e \geq 0} \text{Ass } G^e(N)$, then $\mathfrak{m} \in \bigcup_{e \geq 0} \text{Ass } G^e(N)$.

Avoidance is a weak condition. For example, it holds whenever R satisfies countable prime avoidance, which is the case if R is complete [Bur72, Lemma 3] or contains an uncountable field [HH00, Remark 2.17]. It also occurs whenever the union $\bigcup_{e \geq 0} \text{Ass } G^e(M)$ has only finitely many maximal elements, a condition for which no counterexamples were known to exist until recently [SS04].

Next we provide the following definition of phantom M -regular sequences and phantom depth. It is *a priori* different from the original one given in [Abe94], but as shown in [Eps], they are equivalent when R has a weak test element.

Definition. Let R be a Noetherian ring of prime characteristic $p > 0$ containing a weak test element, and let M be a finitely generated R -module. Then we say an element $x \in R$ is *phantom M -regular* if $xM \neq M$ and $0 :_{F^e(M)} x^{p^e} \subseteq 0_{F^e(M)}^*$ for all $e \geq 0$.

A *phantom zerodivisor* of M is an element $x \in R$ which is not phantom M -regular.

A sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of R is a *phantom M -regular sequence* if $\mathbf{x}M \neq M$ and x_i is phantom $(M/(x_1, \dots, x_{i-1})M)$ -regular for $1 \leq i \leq n$.

The *phantom depth* of M is the length of the longest phantom M -regular sequence in \mathfrak{m} . It is denoted by $\text{ph.depth}_{\mathfrak{m}} M$ or $\text{ph.depth}_R M$.

Clearly, any M -regular sequence is a phantom M -regular sequence. Note also that the phantom depth of R as a module over itself can be determined in a different way. Namely, the *minheight* [HH93] of \mathfrak{m} (denoted $\text{mnht } \mathfrak{m}$) is defined to be $\max\{\text{ht } \mathfrak{m}/\mathfrak{p}_j \mid 1 \leq j \leq t\}$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ are the minimal primes of R . As Aberbach notes in the first paragraph of the proof of [Abe94, Theorem 3.2.7], if R is the homomorphic image of a Cohen-Macaulay ring, then $\text{mnht } \mathfrak{m} = \text{ph.depth}_R R$.

If (C, d) is a complex of finitely generated R -modules, i is an integer, and $Z_i = \ker d_i$ and $B_i = \text{im } d_{i+1}$ are the cycle and boundary submodules of C_i , then we say that $H_i(C)$ is *phantom* if $Z_i \subseteq (B_i)_{C_i}^*$ (following [HH93]). The following characterization of phantom M -regular sequences in terms of phantomness of Koszul homology will be crucial. It is an analogue to the classical characterization of M -regular sequences in terms of the vanishing of Koszul homology.

Theorem 1.1 ([Eps]). *Let (R, \mathfrak{m}) be a Noetherian local ring with a weak test element c , and let M be a finitely generated R -module which satisfies avoidance. Let $\mathbf{x} = x_1, \dots, x_n$ be any sequence of elements of \mathfrak{m} . Then the following conditions are equivalent:*

- (1) $\mathbf{x} = x_1, \dots, x_n$ is a phantom M -regular sequence,
- (2) $H_1(\mathbf{x}^{[p^e]}; F^e(M))$ is phantom for all $e \geq 0$,
- (3) $H_j(\mathbf{x}^{[p^e]}; F^e(M))$ is phantom for all $e \geq 0$ and all $j \geq 1$.

In the special case where the phantom depth of a module is zero, we have the following useful lemma:

Lemma 1.2. *Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic $p > 0$ containing a q_0 -weak test element c , and let M be a finitely generated R -module. Then the set of phantom zerodivisors for M in \mathfrak{m} is the union $\bigcup_{e \geq 0} \text{Ass } G^e(M)$. Hence if M satisfies avoidance and $\text{ph.depth}_R(M) = 0$, then $\mathfrak{m} \in \text{Ass } G^e(M)$ for some e .*

Proof. For the first containment, suppose that x is a phantom zerodivisor for M . Then there is some $e \geq 0$ such that $0 :_{F^e(M)} x^q \not\subseteq 0_{F^e(M)}^*$. That is, there is some $z \in F^e(M) \setminus 0_{F^e(M)}^*$ with $x^q z = 0$. Then $x^q \bar{z} = \bar{0}$ in $G^e(M)$, where $\bar{z} \neq \bar{0}$, so there is some $\mathfrak{p} \in \text{Ass } G^e(M)$ with $x^q \in \mathfrak{p}$. Since \mathfrak{p} is prime and thus radical, $x \in \mathfrak{p}$.

Conversely, let $x \in \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass } G^e(M)$ for some e . Then there is some $z \in F^e(M)$, $z \notin 0_{F^e(M)}^*$, with $\mathfrak{p} = \bar{0} :_{G^e(M)} \bar{z}$, which means that $xz \in 0_{F^e(M)}^*$. Then for all large powers $q' \gg 0$ of p ,

$$x^{qq'} cz^{q'} = x^{qq'-q'} \cdot (c(xz)^{q'}) = x^{qq'-q'} \cdot 0 = 0.$$

If x is phantom M -regular, the displayed equation shows that $cz^{q'} \in 0_{F^{e+e'}(M)}^*$. Hence, $c^{q_0+1} z^{q'q_0} = c(cz^{q'})^{q_0} = 0$, so that since q' is any large enough power of p , we conclude that $z \in 0_{F^e(M)}^*$, contrary to assumption. Thus, x is a phantom zerodivisor for M .

The last statement now follows directly from the definitions. \square

The final preliminary result that we need is the fact that the sets of associated primes of the “reduced Frobenius powers” of a module are increasing:

Lemma 1.3. *Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic $p > 0$ and let M be a finitely generated R -module. Then for any $e \geq 0$, $\text{Ass } G^e(M) \subseteq \text{Ass } G^{e+1}(M)$.*

Proof. Without loss of generality, assume that $e = 0$, and let $\mathfrak{q} \in \text{Ass } G^0(M)$. Then there is some $z \in M \setminus 0_M^*$ such that $\mathfrak{q} = 0_M^* : z$. Let $I = 0_{F^1(M)}^* : z^p$. We have

$$\mathfrak{q}^{[p]} z^p = (\mathfrak{q}z)_M^{[p]} \subseteq (0_M^*)_M^{[p]} \subseteq 0_{F^1(M)}^*.$$

Hence $\mathfrak{q}^{[p]} \subseteq 0_{F^1(M)}^* : z^p = I$.

On the other hand, let $a \in I$. Then $a^p z^p = a^{p-1} (az^p) \in 0_{F^1(M)}^*$. So for $q' \gg 0$,

$$c(az)^{pq'} = c(a^p z^p)^{q'} = 0,$$

which means that $az \in 0_M^*$, so $a \in 0_M^* : z = \mathfrak{q}$.

We have shown that $\mathfrak{q}^{[p]} \subseteq I \subseteq \mathfrak{q}$, which means that \mathfrak{q} is minimal over I , so that $\mathfrak{q} \in \text{Ass } R/I$. Therefore there is some $b \in R$ such that

$$\mathfrak{q} = I : b = (0_{F^1(M)}^* : z^p) : b = 0_{F^1(M)}^* : bz^p,$$

which proves that $\mathfrak{q} \in \text{Ass } G^1(M)$. □

2. FLAT BASE CHANGE: PROOF OF THE MAIN THEOREM

Recall that a Noetherian Cohen-Macaulay local ring (R, \mathfrak{m}) of prime characteristic $p > 0$ is said to be *F-injective* if for any proper ideal I of R and any $x \in R$ such that $x^p \in I^{[p]}$, it follows that $x \in I$.

Proof of the main theorem. First we will prove the “ \leq ” direction. Even more, it turns out that if $\mathbf{a} = a_1, \dots, a_r$ is a phantom M -regular sequence in \mathfrak{m} and $\mathbf{z} = z_1, \dots, z_s$ is an $S/\mathfrak{m}S$ regular sequence in \mathfrak{n} , then $\phi(\mathbf{a}), \mathbf{z}$ is a phantom ${}_S(S \otimes_R M)$ -regular sequence, just as one would expect from the classical case. For brevity, let $M' = S \otimes_R M$.

First note that $\phi(\mathbf{a})$ is a phantom M' -regular sequence. By induction we need only show this for the one-element sequence $a = a_1$. For integers $e \gg 0$, we have

$$\begin{aligned} 0 :_{F_S^e(M')} \phi(a_1)^q &= S \otimes_R (0 :_{F_R^e(M)} a_1^q) \\ &\subseteq S \otimes_R 0_{F_R^e(M)}^* \\ &\subseteq 0_{F_S^e(M')}^*, \end{aligned}$$

where $q = p^e$. The equality follows from flatness on colons. To see the first inclusion, note first that $0 :_{F_R^e(M)} a_1^q \subseteq 0_{F_R^e(M)}^*$ by definition of phantom M -regularity, and then apply flatness of S to this inclusion. For the final inclusion, it is easy to see that the image of $S \otimes_R 0_{F_R^e(M)}^*$ under the map $S \otimes_R (0_{F_R^e(M)}^* \hookrightarrow F_R^e(M))$ is contained in $0_{F_S^e(M')}^*$, and apply flatness one more time to see that the map in question is injective. Since the displayed containment holds for all $e \gg 0$, it follows that $\phi(a_1)$ is a phantom M' -regular element.

Since $F_S^e(S \otimes_R M)/\phi(\mathbf{a})^{[q]}F_S^e(S \otimes_R M) \cong S \otimes_R F_R^e(M/\mathbf{a}M)$ for any $e \geq 0$ and $\text{ph.depth}_R(M/\mathbf{a}M) = 0$, we can replace M by $M/\mathbf{a}M$ in order to assume without loss of generality that $\text{ph.depth}_R M = 0$. Lemma 1.2 then guarantees that there is some $e' \geq 0$ with $\mathfrak{m} \in \text{Ass}_R G_R^{e'}(M)$. Clearly we have that for any $e \geq 0$, $\mathbf{z}^{[q]}$ is an $(S/\mathfrak{m}S)$ -regular sequence. Then by a standard result (e.g. [BH97, Lemma 1.2.17(b)]), $\mathbf{z}^{[q]}$ is an $(S \otimes_R N)$ -regular sequence for any finitely generated R -module N . Since $F_S^e(M') = S \otimes_R F_R^e(M)$, applying this standard result to $F_R^e(M)$ together with the Koszul homology criterion for regular sequences shows that $H_1(z_1^q, \dots, z_s^q; F_S^e(M')) = 0$, which certainly implies phantomness, for all $e \geq 0$. Thus, by Theorem 1.1, \mathbf{z} is a phantom M' -regular sequence.

This completes one direction. Next, assume that $\text{ph.depth}_R M = 0$, and we will prove that $\text{depth } S/\mathfrak{m}S = \text{ph.depth}_S M'$. Let $\mathbf{z} = z_1, \dots, z_t$ be a maximal $(S/\mathfrak{m}S)$ -regular sequence in \mathfrak{n} . As above, \mathbf{z} is an M' -regular (and hence phantom M' -regular) sequence, so for maximality we need to show that $\text{ph.depth}_S (M'/\mathbf{z}M') = 0$.

Since the phantom depth of M is 0, we have by Lemma 1.2 that there is some $e \geq 0$ with $\mathfrak{m} \in \text{Ass}_R G_R^e(M)$. This means that there is some $u \in F_R^e(M)$ with

$$\mathfrak{m} = 0_{F_R^e(M)}^* :_R u.$$

Note that since the sets of associated primes of the $G_R^{e'}(M)$'s are increasing (by Lemma 1.3), we may assume that $e \geq e_0$. Then by flatness of $R \rightarrow S/(\mathbf{z}^{[q]})$ [BH97, Lemma 1.2.17(b)] the map

$$S/(\mathfrak{m}, \mathbf{z}^{[q]})S = (S/\mathbf{z}^{[q]}) \otimes_R R/\mathfrak{m} \xrightarrow{\beta} (S/\mathbf{z}^{[q]}) \otimes_R G_R^e(M) = \frac{(S/\mathbf{z}^{[q]}) \otimes F_R^e(M)}{(S/\mathbf{z}^{[q]}) \otimes 0_{F_R^e(M)}^*},$$

which sends 1 to u , is an injection.

Also, setting $q = p^e$, there is some $b \in S$ such that

$$\mathfrak{n} = (\mathfrak{m}, \mathbf{z}^{[q]})S :_S b,$$

since $\mathbf{z}^{[q]}$ is a maximal $(S/\mathfrak{m}S)$ -regular sequence in \mathfrak{n} . We then have that the map

$$S/\mathfrak{n} \xrightarrow{\alpha} S/(\mathfrak{m}, \mathbf{z}^{[q]})S$$

that sends 1 to b is an injection as well.

Now, consider the following

Lemma 2.1. *Let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings, both of prime characteristic $p > 0$, with Cohen-Macaulay F -injective closed fiber and a shared q_0 -weak test element c . Suppose \mathbf{z} is a system of parameters for the S -module $(S/\mathfrak{m}S)$ and the image of b is nonzero in $S/(\mathfrak{m}, \mathbf{z})S$. If N is a finitely generated R -module and u is not in 0_N^* , then bu is not in $0_{(S/\mathbf{z}) \otimes_R N}^*$, where the tight closure is taken over S .*

This lemma has the same conclusion as [AE02, Lemma 3.1], and it has exactly the same proof (except that “for all q ” must be replaced by “for all $q \gg 0$ ”), although the hypotheses differ. For completeness, we reproduce a version of the proof here:

Proof of Lemma 2.1. We prove the contrapositive. That is, assuming that $bu \in 0_{(S/\mathbf{z}) \otimes_R N}^*$ (with the tight closure taken over S), we will show that $u \in 0_N^*$.

We have that for all powers $q \geq q_0$ of p ,

$$\overline{c(bu)^q} = \bar{0} \in S/\mathbf{z}^{[q]} \otimes_R F_R^e(N).$$

Then by flatness of the map $R \rightarrow S/\mathbf{z}^{[q]}$,

$$(1) \quad \overline{b^q} \in 0 :_{S/\mathbf{z}^{[q]}} \overline{cu^q} = S/\mathbf{z}^{[q]} \otimes_R (0 :_R cu^q).$$

If $cu^q \neq 0$, then $0 :_R cu^q \subseteq \mathfrak{m}$, from which we conclude that

$$(2) \quad S/\mathbf{z}^{[q]} \otimes_R (0 :_R cu^q) \subseteq S/\mathbf{z}^{[q]} \otimes_R \mathfrak{m} = \frac{(\mathfrak{m}, \mathbf{z}^{[q]})S}{(\mathbf{z}^{[q]})}.$$

Combining (1) with (2), we have that $b^q \in (\mathfrak{m}, \mathbf{z}^{[q]})S$, from which we conclude, by F -injectivity of $S/\mathfrak{m}S$, that $b \in (\mathfrak{m}, \mathbf{z})S$, which contradicts our assumption on b . Hence $cu^q = 0$. Since q was allowed to be any power of p larger than q_0 , it follows that $u \in 0_N^*$. \square

It follows from Lemma 2.1 that, in our case,

$$bu \notin 0_{(S/\mathbf{z}^{[q]}) \otimes_R F_R^e(M)}^* = 0_{F_S^e((S/\mathbf{z}) \otimes_R M)}^* = 0_{F_S^e(M'/\mathbf{z}M')}^*,$$

where the tight closures are computed over S . Thus, S/\mathfrak{n} injects into $G_S^e(M'/\mathbf{z}M')$, so that $\text{ph.depth}_S(M'/\mathbf{z}M') = 0$, which means that \mathbf{z} is indeed maximal as a phantom M' -regular sequence.

Finally, consider the case where $\text{ph.depth}_R M > 0$. Then if \mathbf{a} is a maximal phantom M -regular sequence and \mathbf{z} is a maximal $(S/\mathfrak{m}S)$ -regular sequence, then $M/\mathbf{a}M$ has phantom depth 0, so we can apply the above to show that $\phi(\mathbf{a}), \mathbf{z}$ is a maximal phantom M' -regular sequence. \square

Corollary (Phantom depth is unaffected by completion). *Suppose R is a Noetherian local ring of prime characteristic $p > 0$ containing a completely stable weak test element. For any finitely generated R -module M that satisfies avoidance,*

$$\text{ph.depth}_R M = \text{ph.depth}_{\hat{R}} \hat{M}.$$

3. SURJECTIVE BASE CHANGE: COUNTEREXAMPLES

In general, if (R, \mathfrak{m}) is a Noetherian local ring, $S = R/I$ is a quotient of it, and M is a finitely generated S -module, we have that $\text{depth}_{\mathfrak{m}} M = \text{depth}_{\mathfrak{m}/I} M$. However, unlike depth, phantom depth can depend on the ring over which it is calculated. In particular, $\text{ph.depth}_R M$ may differ from $\text{ph.depth}_{R/I} M$.

For instance, consider the following situation from [HH93, Remark 2.7], which was also considered by Aberbach in [Abe94]. Let k be a field of prime characteristic $p > 0$, let $T = k[[Y_1, \dots, Y_n, Z]]$ where $n > 1$, let J be the ideal $(Y_1 Z, \dots, Y_n Z)$ of T , set $R = T/J$ and $\mathfrak{m} = \mathfrak{m}_R$, and let the images of each Y_j be denoted by y_j , and the image of Z by z . Then put $x = z - y_1$ and $I = (x)$. We have that x is a non-zerodivisor of R , so it is certainly a phantom R -regular element. Hence, $\text{ppd}_R(R/xR) = 1$, so by Aberbach's phantom Auslander-Buchsbaum theorem [Abe94, Theorem 3.2.7],

$$\text{ph.depth}_R R/I = \text{mnht } \mathfrak{m} - \text{ppd}_R(R/xR) = 1 - 1 = 0.$$

On the other hand,

$$\text{ph.depth}_{R/I} R/I = \text{mnht } \mathfrak{m}/I = n - 1 > 0 = \text{ph.depth}_R R/I.$$

In some sense, surjective base change for phantom depth fails in the above example because R is not equidimensional. However, surjective base change may fail even with equidimensional rings. Let (R, \mathfrak{m}) be a reduced Noetherian local ring of characteristic $p > 0$ which contains a test element c and an ideal I such that \mathfrak{m} is not minimal over I , but such that \mathfrak{m} is an associated prime of I^* , and assume further that the ring R/I is equidimensional.

Then

$$\text{ph.depth}_{R/I} R/I = \text{mnht } \mathfrak{m}/I = \text{ht } \mathfrak{m}/I > 0.$$

On the other hand, since \mathfrak{m} is associated to I^* , there is some $z \in R \setminus I^*$ such that $\mathfrak{m} = I^* :_R z$. Now suppose that there is some $a \in \mathfrak{m}$ which is phantom $R(R/I)$ -regular. Since $a \in \mathfrak{m} = I^* : z$, $az \in I^*$, so that for any power q of p ,

$$ca^q z^q \in I^{[q]}.$$

Since a is phantom $R(R/I)$ -regular, this implies that $cz^q \in (I^{[q]})^*$, so that $c^2 z^q \in I^{[q]}$. Since this holds for all q , it follows that $z \in I^*$, which is a contradiction.

Thus, \mathfrak{m} has no phantom $R(R/I)$ -regular elements. That is,

$$\text{ph.depth}_R R/I = 0.$$

For concreteness, note that such a ring R and ideal I is given (and proved to be such, apart from the equidimensionality and test element hypotheses) in [HH00, Example 2.13]. They let

$$R = K[X, Y, U, V]/(X^3Y^3 + U^3 + V^3) = K[x, y, u, v],$$

where K is a field of prime characteristic p , where $p \neq 0, 3$. They let

$$I = (u, v, x^3) \subseteq R.$$

The only parts of our criteria not explicitly stated by Hochster and Huneke in [HH00] are the existence of a test element c and the equidimensionality of R/I . But as they do state, $R/I \cong K[x, y]/(x^3)$, which is clearly Cohen-Macaulay, hence equidimensional. Note also that R itself is equidimensional, since it is an integral domain. Furthermore, R is a finitely generated algebra over a field, which implies that it has a completely stable test element.

One might protest at this point that R is not a local ring. However, we may replace R by $R_{\mathfrak{m}}$ and I by $IR_{\mathfrak{m}}$, where $\mathfrak{m} = (x, y, u, v)$ is the homogeneous graded maximal ideal, without affecting any of the criteria we needed.

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