MULTIPLE POSITIVE SOLUTIONS OF SINGULAR PROBLEMS
BY VARIATIONAL METHODS

RAVI P. AGARWAL, KANISHKA PERERA, AND DONAL O'REGAN

(Communicated by Carmen C. Chicone)

ABSTRACT. The purpose of this paper is to use an appropriate variational framework to obtain positive solutions of some singular boundary value problems.

1. Introduction

Consider the boundary value problem

\begin{equation}
\begin{cases}
- y'' = f(t, y), & 0 < t < 1, \\
y(0) = y(1) = 0
\end{cases}
\end{equation}

where we only assume that \( f \in C((0, 1) \times (0, \infty), [0, \infty)) \) satisfies

\begin{equation}
2\varepsilon \leq f(t, y) \leq Cy^{-\gamma}, \quad (t, y) \in (0, 1) \times (0, \varepsilon),
\end{equation}

for some \( \varepsilon, C > 0 \) and \( \gamma \in (0, 1) \), so that it may be singular at \( y = 0 \) (here of course \( C \) could depend on \( \varepsilon \)). A typical example is

\begin{equation}
f(t, y) = y^{-\gamma} + g(t, y)
\end{equation}

with \( g \in C((0, 1) \times [0, \infty), [0, \infty)) \).

Define \( f_{\varepsilon} \in C((0, 1) \times \mathbb{R}, [0, \infty)) \) by

\begin{equation}
f_{\varepsilon}(t, y) = f(t, y - \varphi_{\varepsilon}(t)) + \varphi_{\varepsilon}(t)
\end{equation}

where \( \varphi_{\varepsilon}(t) = \varepsilon t(1 - t) \) is the solution of

\begin{equation}
\begin{cases}
- y'' = 2\varepsilon, & 0 < t < 1, \\
y(0) = y(1) = 0
\end{cases}
\end{equation}

and \( y^\pm = \max \{\pm y, 0\} \), and consider

\begin{equation}
\begin{cases}
- y'' = f_{\varepsilon}(t, y), & 0 < t < 1, \\
y(0) = y(1) = 0
\end{cases}
\end{equation}

By (1.2),

\begin{equation}
2\varepsilon \leq f_{\varepsilon}(t, y) \leq C\varphi_{\varepsilon}(t)^{-\gamma}, \quad (t, y) \in (0, 1) \times (-\infty, \varepsilon).
\end{equation}
We observe that if $y$ is a solution of (1.6), then $y \geq \varphi_\varepsilon$ and hence also a solution of (1.1). To see this suppose (1.8) $y(t) < \varphi_\varepsilon(t)$ for some $t$.

By Lemma 2.8.1 of Agarwal and O’Regan [1],

(1.9) $y(t) \geq t(1-t) |y_0|, \quad t \in [0,1],$

where $|y_0| = \max_{t \in [0,1]} |y(t)|$, so (1.8) implies $|y_0| < \varepsilon$. But then $-y'' \geq 2\varepsilon = -\varphi_\varepsilon''$ by (1.7), so $y \geq \varphi_\varepsilon$, contradicting (1.8). Conversely, every solution of (1.1) is a solution of (1.6).

Since $\varphi_\varepsilon^{-\gamma} \in L^1(0,1)$, we see from (1.7) that solutions of (1.6) are the critical points of the $C^1$ functional

(1.10) $\Phi(y) = \int_0^1 \left( \frac{1}{2} |y'(t)|^2 - F_\varepsilon(t, y(t)) \right) dt, \quad y \in H = H^1_0(0,1),$

where $F_\varepsilon(t, y) = \int_\varepsilon^y f_\varepsilon(t, x) \, dx$ and $H^1_0(0,1)$ is the usual Sobolev space, normed by

(1.11) $\|y\| = \left( \int_0^1 |y'(t)|^2 \, dt \right)^{1/2}.$

The purpose of this paper is to use this variational framework to obtain positive solutions of (1.1).

We will show that, under additional assumptions on the behavior of $f$ at infinity, $\Phi$ satisfies the compactness condition of Cerami [2]:

**C**: every sequence $\{y_m\} \subset H$ such that

(1.12) $\Phi(y_m)$ is bounded, \quad $(1 + \|y_m\|)\|\Phi'(y_m)\| \to 0$

has a convergent subsequence.

This condition is weaker than the usual Palais-Smale condition, but can be used in place of it when constructing deformations of sublevel sets via negative pseudo-gradient flows, and therefore also in minimax theorems such as the mountain pass lemma.

By a standard argument it suffices to show that $\{y_m\}$ is bounded when verifying (C). Moreover,

(1.13) \begin{align*}
\|y_m\|^2 &= - \left( \int_0^1 f_\varepsilon(t, y_m(t))y_m(t) \, dt + \langle \Phi'(y_m), y_m \rangle \right) \\
&\leq |y_m|_0 \int_{y_m < 0} f_\varepsilon(t, y_m(t)) \, dt + \|\Phi'(y_m)\| \|y_m\|
\end{align*}

and hence

(1.14) $\|y_m\| \leq C \int_0^1 \varphi_\varepsilon(t)^{-\gamma} \, dt + o(1)$

since $|\cdot|_0 \leq \|\cdot\|$ and by (1.7), so we will only need to check that $\{y_m^+\}$ is bounded.

We refer the reader to Agarwal and O’Regan [1] for a broad introduction to singular problems and to Rabinowitz [3] for variational methods.
2. An existence principle

Assume

(f): there is a constant \( M > 0 \), independent of \( \lambda \), such that \( \| y \| \neq M \) for every solution \( y > 0 \) to

\[
\begin{aligned}
- y'' &= \lambda f(t, y), \quad 0 < t < 1, \\
y(0) &= y(1) = 0
\end{aligned}
\]

(2.1)

for each \( \lambda \in (0, 1] \).

Note that (f) holds if there exists an a priori bound of the norm of the solutions of the problem.

Proposition 2.1. If (1.2) and (f) hold, then (1.1) has a positive solution.

Proof. We will show that \( \Phi \) assumes its infimum on

\[
B = \{ y \in H : \| y \| \leq M \}
\]

at some point \( y_0 \in \partial B \), which is then a local minimizer, if \( \varepsilon \) is chosen small enough.

Clearly, \( \inf \Phi(B) > -\infty \). Let \( \{ y_m \} \) be a minimizing sequence. Passing to a subsequence we may assume that \( y_m \) converges to some \( y_0 \in B \) weakly in \( H \), strongly in \( L^2(0, 1) \), and a.e. in \( (0, 1) \). Then

\[
\Phi(y_0) = \frac{1}{2} \| y_0 \|^2 - \int_0^1 F_\varepsilon(t, y_0(t)) \, dt
\]

(2.3)

\[
\leq \liminf \frac{1}{2} \| y_m \|^2 - \lim \int_0^1 F_\varepsilon(t, y_m(t)) \, dt
\]

\[
= \lim \Phi(y_m) = \inf_{y \in B} \Phi(y),
\]

so \( \Phi(y_0) = \inf \Phi(B) \).

Suppose that \( y_0 \in \partial B \). Then it is also a minimizer of \( \Phi|_{\partial B} \), so the gradient of \( \Phi \) at \( y_0 \) points in the direction of the inward normal to \( \partial B \), i.e.,

\[
\Phi'(y_0) = -\nu y_0
\]

(2.4)

or

\[
-y''_0 = \frac{1}{1 + \nu} f_\varepsilon(t, y_0),
\]

(2.5)

for some \( \nu \geq 0 \). If \( y_0 \geq \varphi_\varepsilon \), (2.5) reduces to (2.1) with \( \lambda = \frac{1}{1 + \nu} \in (0, 1] \), so, as in the introduction, it follows that \( y_0 < \varepsilon \). But then multiplying (2.5) by \( y_0 \) and integrating by parts gives

\[
M^2 = \frac{1}{1 + \nu} \int_0^1 y_0(t)f_\varepsilon(t, y_0(t)) \, dt \leq C\varepsilon \int_0^1 \varphi_\varepsilon(t)^{-\gamma} \, dt = C\varepsilon^{1-\gamma}
\]

by (1.7), where \( C \) is a generic positive constant, which is impossible if \( \varepsilon \) is sufficiently small. \( \Box \)

For example, consider

\[
\begin{aligned}
- y'' &= \mu f(t, y), \quad 0 < t < 1, \\
y(0) &= y(1) = 0
\end{aligned}
\]

(2.7)
where \( \mu > 0 \) is a parameter and \( f \in C((0,1) \times (0,\infty), [0,\infty)) \) satisfies (1.2). If \( y > 0 \) is a solution of
\[
\begin{cases}
- y'' = \lambda \mu f(t, y), & 0 < t < 1, \\
y(0) = y(1) = 0
\end{cases}
\]
with \( \|y\| = M \),
\[
M^2 = \lambda \mu \int_0^1 y(t) f(t, y(t)) \, dt \leq \mu \sup_{(t,y) \in (0,1) \times (0,M)} yf(t,y),
\]
so (2.7) has a positive solution for
\[
\mu \leq \sup_{M > 0} \frac{M^2}{\sup_{(t,y) \in (0,1) \times (0,M)} yf(t,y)}.
\]

Similarly,
\[
\begin{cases}
- y'' = y^{-\gamma} + \mu g(t, y), & 0 < t < 1, \\
y(0) = y(1) = 0
\end{cases}
\]
where \( \gamma \in (0,1) \), \( \mu > 0 \) is a parameter, and \( g \in C((0,1) \times [0,\infty), [0,\infty)) \) has a positive solution for
\[
\mu \leq \sup_{M > 0} \frac{M^{1-\gamma} (M^{1+\gamma} - 1)}{\sup_{(t,y) \in (0,1) \times (0,M)} yg(t,y)}.
\]

3. ASYMPTOTICALLY LINEAR CASE

Assume
\[
f(t, y) \leq Cy, \quad (t, y) \in (0,1) \times [\varepsilon, \infty),
\]
for some \( C > 0 \). We say that (1.1) is resonant if
\[
\frac{f(t,y)}{y} \to \lambda_1 \quad \text{as} \quad y \to \infty
\]
where \( \lambda_1 = \pi^2 \) is the first eigenvalue of
\[
\begin{cases}
- y'' = \lambda y, & 0 < t < 1, \\
y(0) = y(1) = 0.
\end{cases}
\]
Denote by
\[
H(t,y) = F_\varepsilon(t,y) - \frac{1}{2} f_\varepsilon(t,y)
\]
the nonquadratic part of \( F_\varepsilon \).

**Theorem 3.1.** If (1.2) and (3.1) hold, then (1.1) has a positive solution in the following cases:

(i) **Nonresonance below \( \lambda_1 \):**
\[
f(t,y) \leq ay + C, \quad (t,y) \in (0,1) \times [\varepsilon, \infty),
\]
for some \( a < \lambda_1 \) and \( C > 0 \),
Solutions of singular problems by variational methods

(ii) Resonance: \[3.2\] holds, \[
H(t, y) \leq C, \quad (t, y) \in (0, 1) \times [\varepsilon, \infty),
\]
for some \(C > 0\), and \[
H(t, y) \to -\infty \quad \text{as } y \to \infty.
\]

Proof. (i) By \(1.7\) and \(3.5\), \[
\begin{aligned}
F_\varepsilon(t, y) &= \begin{cases} 
0, & y < \varepsilon, \\
\frac{a}{2} y^2 + Cy, & y \geq \varepsilon,
\end{cases}
\end{aligned}
\]
and, since \(a < \lambda_1\), it follows from Wirtinger’s inequality that \(\Phi\) is bounded from below and coercive, and hence satisfies (C) and admits a global minimizer.

(ii) For \(y \geq \varepsilon\), \[
\begin{aligned}
\frac{\partial}{\partial y} \left( F_\varepsilon(t, y) y^2 \right) &= -2 H(t, y) y^3, \\
F_\varepsilon(t, y) y^2 &\to \frac{\lambda_1}{2} \quad \text{as } y \to \infty
\end{aligned}
\]
by \(3.2\), so \[
\begin{aligned}
F_\varepsilon(t, y) &= \left( \frac{\lambda_1}{2} + 2 \int_\varepsilon^\infty \frac{H(t, x)}{x^3} dx \right) y^2 \\
&\leq \frac{\lambda_1}{2} y^2 + C
\end{aligned}
\]
by \(3.6\), and hence Wirtinger’s inequality implies \(\Phi\) is bounded from below.

To verify (C), let \(\{y_m\}\) satisfy \(1.12\) and suppose \(\rho_m := \|y_m\| \to \infty\). Since \(\{y_m\}\) is bounded, for a subsequence, \(\tilde{y}_m := \frac{y_m}{\rho_m}\) converges to some \(\tilde{y} \geq 0\) weakly in \(H\), strongly in \(L^2(0, 1)\), and a.e. in \((0, 1)\). Then

\[
\int_0^1 \tilde{y}_m'(t) (\tilde{y}_m'(t) - \tilde{y}'(t)) dt = \int_0^1 g_m(t) dt + \frac{\langle \Phi'(y_m), \tilde{y}_m - \tilde{y} \rangle}{\rho_m}
\]
where \(g_m(t) = \frac{f(t, y_m(t))}{\rho_m} (\tilde{y}_m(t) - \tilde{y}(t))\). By \(1.7\) and \(3.1\),

\[
|g_m(t)| \leq C (\varphi(x)^{\gamma} + 1) |\tilde{y}_m(t) - \tilde{y}(t)|
\]
and hence \(g_m \to 0\) a.e. and \(|g_m| \leq C (\varphi^{\gamma} + 1) \in L^1(0, 1)\), so passing to the limit in \(1.11\) gives \(\|\tilde{y}\| = 1\); in particular, \(\tilde{y} \neq 0\). By \(1.7\) and \(3.6\),

\[
H(t, y) \leq \begin{cases} 
C \varphi(x)^{-\gamma} |y|, & y < 0, \\
0, & 0 \leq y < \varepsilon, \\
C, & y \geq \varepsilon,
\end{cases}
\]
and \(|y_m^{-}\) is bounded, so

\[
\int_{\tilde{y} > 0} H(t, y_m(t)) dt \to -\infty
\]
by (3.7) and ∫ y>0 H(t, y_m(t)) dt is bounded from above. Hence

\[ \frac{1}{2} \langle \Phi'(y_m), y_m \rangle - \Phi(y_m) = \int_0^1 H(t, y_m(t)) dt \]

\[ = \int_{y>0} H(t, y_m(t)) dt + \int_{y=0} H(t, y_m(t)) dt \to -\infty, \]

contrary to assumption. □

**Theorem 3.2.** If (1.2), (3.1), and (f) hold, then (1.1) has two positive solutions in the following cases:

(i) Resonance: (3.2) holds,

\[ H(t, y) \geq -C, \quad (t, y) \in (0, 1) \times [\varepsilon, \infty), \]

for some C > 0, and

(3.17)

\[ H(t, y) \to +\infty \quad \text{as} \quad y \to \infty, \]

(ii) Nonresonance above \( \lambda_1 \):

\[ f(t, y) \geq by - C, \quad (t, y) \in (0, 1) \times [\varepsilon, \infty), \]

for some \( b > \lambda_1 \) and \( C > 0 \).

**Proof.** By (see the proof of) Proposition 2.1, \( \Phi \) has a local minimizer \( y_0 \in B \) and \( \inf \Phi(\partial B) \geq \Phi(y_0) \). We will show that \( \Phi(R\varphi_1) \leq \inf \Phi(\partial B) \) if \( R > M \) is sufficiently large, where \( \varphi_1 > 0 \) is the normalized eigenfunction associated with \( \lambda_1 \), and we will verify (C). Then the mountain pass lemma will give a second critical point at the level

\[ c := \inf_{\gamma \in \Gamma} \max_{y \in \gamma([0,1])} \Phi(y) \]

where

(3.20)

\[ \Gamma = \{ \gamma \in C([0,1], H) : \gamma(0) = y_0, \gamma(1) = R\varphi_1 \} \]

is the class of paths joining \( y_0 \) and \( R\varphi_1 \).

(i) By (3.10), (3.16), and (3.17),

\[ \frac{\lambda_1}{2} y^2 - F_\varepsilon(t, y) = -2y^2 \int_y^\infty \frac{H(t, x)}{x^3} dx \leq C, \quad y \geq \varepsilon \]

and \( \to -\infty \) as \( y \to \infty \),

(3.21)

and by (1.7),

\[ \frac{\lambda_1}{2} y^2 - F_\varepsilon(t, y) \leq C (\varphi_\varepsilon(t)^{-\gamma} + 1), \quad 0 \leq y < \varepsilon, \]

so

\[ \Phi(R\varphi_1) = \int_0^1 \left( \frac{\lambda_1}{2} R^2 \varphi_1(t)^2 - F_\varepsilon(t, R\varphi_1(t)) \right) dt \to -\infty \quad \text{as} \quad R \to \infty. \]

The verification of (C) is similar to that in the proof of Theorem 3.1.
(ii) By (1.7) and (3.18),

\[
F_\varepsilon(t, y) \geq \begin{cases} 
-C \varphi_\varepsilon(t)^{-\gamma}, & 0 \leq y < \varepsilon, \\
\frac{b}{2} y^2 - C y, & y \geq \varepsilon,
\end{cases}
\]

and, since \(b > \lambda_1\), it follows that \(\Phi(R\varphi_1) \to -\infty\) as \(R \to \infty\).

If (1.12) holds and \(\rho_m := \|y_m\| \to \infty\), passing to a subsequence \(\tilde{y}_m := \frac{y_m}{\rho_m}\) converges to a nontrivial \(\tilde{y} \geq 0\) weakly in \(H\), strongly in \(L^2(0, 1)\), and a.e. in \((0, 1)\) as in the proof of Theorem 3.1. Then

\[
(b - \lambda_1) \int_0^1 \tilde{y}_m(t) \varphi_1(t) \, dt = \frac{1}{\rho_m} \int_0^1 \left( (by_m(t) - f_\varepsilon(t, y_m(t))) \varphi_1(t) \right) \, dt \\
\leq \frac{1}{\rho_m} \left( \int_0^1 C \varphi_1(t) \, dt + \int_{y_m \leq \varepsilon} by_m(t) \varphi_1(t) \, dt + \|\Phi'(y_m)\| \right)
\]

by (3.18), and \(|y_m^\prime|_0\) is bounded, so

\[
(b - \lambda_1) \int_0^1 \tilde{y}(t) \varphi_1(t) \, dt \leq 0,
\]

which is impossible. \(\Box\)

4. SUPERLINEAR CASE

Assume

\[
0 < \theta F_\varepsilon(t, y) \leq y f(t, y), \quad (t, y) \in (0, 1) \times [y_0, \infty),
\]

for some \(\theta > 2\) and \(y_0 > \varepsilon\).

**Theorem 4.1.** If (1.2), (4.1), and (f) hold, then (1.1) has two positive solutions.

**Proof.** As in the proof of Theorem 3.2 it suffices to show that \(\Phi(R\varphi_1) \to -\infty\) as \(R \to \infty\) and to verify (C). The former follows since

\[
F_\varepsilon(t, y) \geq \begin{cases} 
-C \varphi_\varepsilon(t)^{-\gamma}, & 0 \leq y < \varepsilon, \\
F_\varepsilon(t, y_0) \left( \frac{y}{y_0} \right)^\theta, & y \geq y_0,
\end{cases}
\]

by (1.7) and (4.1). As for the latter,

\[
\left( \frac{\theta}{2} - 1 \right) \|y_m\|^2 = \int_0^1 \left( \theta F_\varepsilon(t, y_m(t)) - y_m(t) f_\varepsilon(t, y_m(t)) \right) \, dt \\
+ \theta \Phi(y_m) - \langle \Phi'(y_m), y_m \rangle \\
\leq C \left( \int_{y_m \leq \varepsilon} \varphi_\varepsilon(t)^{-\gamma} |y_m(t)| \, dt + 1 \right),
\]

and the last integral is bounded since \(|y_m^\prime|_0\) is bounded. \(\Box\)
For example, Theorem 4.1 guarantees that
\begin{equation}
\begin{cases}
-y'' = \mu(y^{-\gamma} + y^\beta), & 0 < t < 1, \\
y(0) = y(1) = 0
\end{cases}
\end{equation}
where \( \gamma \in (0, 1) \) and \( \beta > 1 \), has two positive solutions for
\begin{equation}
0 < \mu < \frac{(\gamma + 1)^{\gamma+1}(\beta - 1)^{\beta-1}}{\gamma + \beta}
\end{equation}
(see the example after Proposition 2.1 and (2.10)). The problem
\begin{equation}
\begin{cases}
-y'' = y^{-\gamma} + \mu e^y, & 0 < t < 1, \\
y(0) = y(1) = 0
\end{cases}
\end{equation}
where \( \gamma \in (0, 1) \), has two positive solutions for
\begin{equation}
0 < \mu < \sup_{M > 0} \frac{M^{1+\gamma} - 1}{M^{\gamma} e^M}
\end{equation}
(see (2.12)).

References