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EVENTUAL ARM AND LEG WIDTHS IN COCHARACTERS OF P. I. ALGEBRAS

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ABSTRACT. Given a p.i. algebra A, we study which partitions λ correspond to characters with non-zero multiplicities in the cocharacter sequence of A. We define the $\omega_0(A)$, the eventual arm width to be the maximal d so that such λ can have d parts arbitrarily large, and $\omega_1(A)$ to be the maximum h so that the conjugate λ' could have h arbitrarily large parts. Our main result is that for any A, $\omega_0(A) \geq \omega_1(A)$.

INTRODUCTION

Let A be a p.i. algebra over a field F of characteristic zero. Let A have cocharacter sequence $\{\chi_n(A)\}_{n=0}^{\infty}$, where each $\chi_n(A)$ decomposes into a sum of irreducible characters with multiplicities

$$\chi_n(A) = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda \chi^\lambda.$$

In [1] Amitsur and Regev studied which χ^λ occur with non-zero multiplicity, $m_\lambda \neq 0$. Identifying a partition with its Young diagram, they proved that there exist a k and ℓ such that $m_\lambda = 0$ outside of the $k \times \ell$ hook, i.e., $m_\lambda = 0$ unless λ has at most k parts of length greater than or equal to ℓ. See Figure 1 (next page). The set of such partitions of n is denoted $H(k, \ell; n)$ and the union $\bigcup_n H(k, \ell; n)$ is denoted $H(k, \ell)$.

Pictorially, we think of the first k rows of λ as the arm and the remaining part as the leg. Define the eventual arm width of A to be $\omega_0(A) =$ the maximum d such that there exists λ with λ_d arbitrarily large and $m_\lambda \neq 0$. The definition of the eventual leg width $\omega_1(A)$ is similar, using the conjugate partition λ'; namely, $\omega_1(A) =$ the maximum h such that there exists λ with λ'_h arbitrarily large and $m_\lambda \neq 0$. So, $\omega_0(A)$ and $\omega_1(A)$ are minimal such that the cocharacter sequence is supported in diagrams as in Figure 2, for some s.

Kemer defined the verbally prime p.i. algebras to be $M_k(F)$, $M_k(E)$ and $M_{k,\ell}$. In these cases the eventual arm and leg widths are known: $\omega_0(M_k(F)) = k^2$ and $\omega_1(M_k(F)) = 0$; $\omega_0(M_k(E)) = k^2$ and $\omega_1(M_k(E)) = k^2$; and $\omega_0(M_{k,\ell}) = k^2+\ell^2$ and $\omega_1(M_{k,\ell}) = 2k\ell$. Observing this, Regev conjectured (privately) that $\omega_0(A) \geq \omega_1(A)$ for any A. The main goal of this paper is to prove Regev's conjecture.

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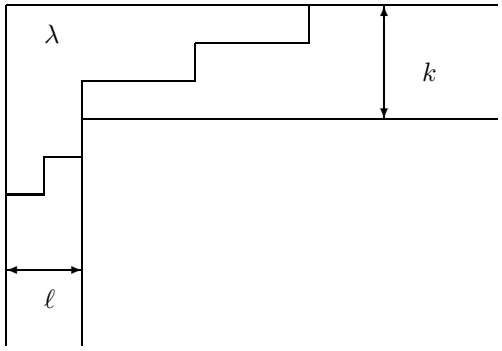

FIGURE 1. The $k \times \ell$ hook

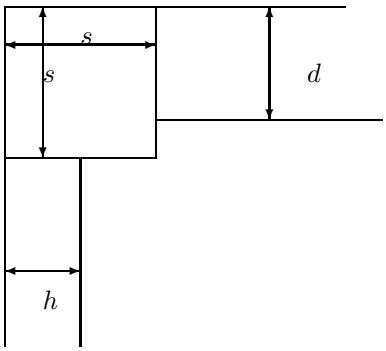

FIGURE 2. Definition of h and d

Here are two examples to show the limitations of this theorem. First, the theorem does not imply that any individual λ will have both λ_d and λ'_h large. For an example of this, consider $A = M_k(F) \oplus E$. Then for each λ,

$$\max\{m_\lambda(M_k(F)), m_\lambda(E)\} \leq m_\lambda(A) \leq m_\lambda(M_k(F)) + m_\lambda(E).$$

Hence,

$$\chi_n(A) = \sum_{i=0}^{n} \chi^{(n-i,1^i)} + \sum_{\lambda \in H(k^2,0;n)} m_\lambda \chi^\lambda$$

with $m_\lambda \neq 0$ for λ equal to any $a \times k^2$ rectangle. So, $\omega_0(A) = k^2$ and $\omega_1(A) = 1$, but $m_\lambda = 0$ for any λ with $\lambda_{k^2} \geq 2$ and $\lambda'_1 \geq k^2$.

The second example shows why we use $\omega_0(A)$ and $\omega_1(A)$ instead of just k and ℓ. Let A be the set of 2×2 matrices $\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$ in which $a, b \in E$. Then $m_\lambda \neq 0$ if and only if λ is of the form $(a, 1^b)$ or $(a, 2, 1^b)$. This implies that the cocharacter sequence is supported in the hook $H(1, 2)$ and in no strictly smaller hook, even though $1 < 2$. On the other hand, $\omega_0(A) = \omega_1(A) = 1$ in this case.

THE PROOF

Our proof is an extension of the work of Giambruno and Zaicev in [4], which in turn is based on the work of Kemer. Given any p.i. algebra, Kemer showed that

there exists a finite-dimensional, $\mathbb{Z}_2$-graded algebra B such that A is p.i. equivalent to the Grassmann envelope of B,

$$A \sim G(B) =_{DEF} B^{(0)} \otimes E^{(0)} + B^{(1)} \otimes E^{(1)},$$

where E is the infinite-dimensional Grassmann algebra with its natural $\mathbb{Z}_2$-grading. Without affecting the cocharacter, we may assume that $A = G(B)$ and that F is algebraically closed. Next, by Wedderburn's principle theorem, B can be decomposed as $B = B' \times J$, where B' is a semisimple graded algebra and J is the Jacobson radical. We further note that J will be nilpotent, say of index q, $J^q = 0$; and B' can be written as a sum of graded simple algebras $B' = B_1 \times \cdots \times B_K$.

Up to isomorphism, there are three types of graded simple algebras over an algebraically closed field: (1) $M_n(F)$, concentrated in degree 0, (2) $M_n(F)$ with a grading in which the degree zero part has dimension $k^2 + \ell^2$ and the degree one part has dimension $2k\ell$ for some $k + \ell = n$, and (3) $M_n(F) + M_n(F)t$, where t has degree one and $t^2 = 1$. In all three cases the dimension of the degree zero part is greater than or equal to the dimension of the degree one part. This property is inherited by direct sums, and we record this fact as a lemma.

Lemma 1. *If B is a $\mathbb{Z}_2$-graded semisimple algebra, then* $\dim B^{(0)} \geq \dim B^{(1)}$.

In the decomposition $B = B_1 \times \cdots \times B_K \times J$, we consider subsets $C_1, \ldots, C_k$ of distinct elements of the list of simple algebras $B_1, \ldots, B_K$ such that $C_1 J C_2 J \cdots J C_k \neq 0$. One reason for considering such subsets is this lemma of Giambruno and Zaicev.

Lemma 2 (Lemma 15 of [4])**.** *Let $A = G(B)$ be as above, and assume that there is a non-zero product $C_1 J C_2 J \cdots J C_k \neq 0$. Let $d = \dim(C_1^{(0)} \oplus \cdots \oplus C_K^{(0)})$ and $h = \dim(C_1^{(1)} \oplus \cdots \oplus C_K^{(1)})$. Then $\chi(A)$ contains irreducible components χ^λ with non-zero multiplicities in which λ_d and λ'_h are arbitrarily large.*

The rest of this section will consist of a proof of a converse to this lemma. Let

$$d = \max\{\dim(C_1^{(0)} \oplus \cdots \oplus C_k^{(0)}) \mid C_1 J \cdots J C_k \neq 0\},$$
$$h = \max\{\dim(D_1^{(1)} \oplus \cdots \oplus D_l^{(1)}) \mid D_1 J \cdots J D_l \neq 0\},$$

where $C_1, \cdots, C_k$ and $D_1, \cdots, D_l$ run over distinct elements of $\{B_1, \ldots, B_K\}$. It follows from Lemma 1 that

$$\begin{aligned} d &= \dim(C_1^{(0)} \oplus \cdots \oplus C_k^{(0)}) \geq \dim(D_1^{(0)} \oplus \cdots \oplus D_l^{(0)}) \\ &\geq \dim(D_1^{(1)} \oplus \cdots \oplus D_l^{(1)}) = h. \end{aligned}$$

So, we need to prove that $d = \omega_0(A)$ and $h = \omega_1(A)$. It follows from Lemma 2 that $d \leq \omega_0(A)$ and $h \leq \omega_1(A)$. In order to finish the proof, we need to show that if λ is a partition in which λ_{d+1} or λ'_{h+1} is bigger than some constant, then for every λ tableau T_λ the polynomial identified with the corresponding idempotent e_{T_λ} is an identity for A. In light of the Amitsur-Regev theorem, if suffices to prove this lemma.

Lemma 3. *There exists a constant t such that if λ is a partition with either t columns of height equal to $s > d$ or with t rows of length equal to $s > h$, then every e_{T_λ} is an identity for A.*

Before proving Lemma 3, we show how it implies our main theorem.

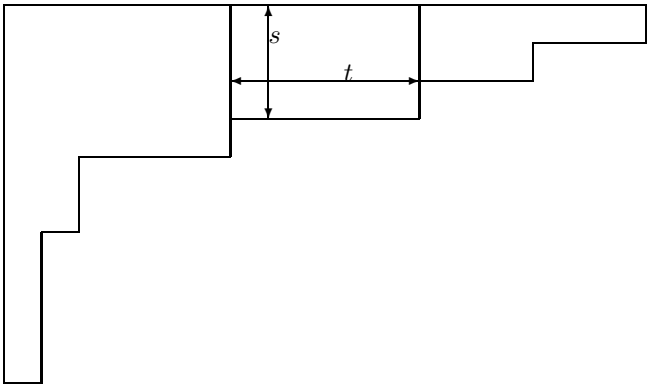

FIGURE 3. Proof of Lemma 3

Theorem 4. *If A is a p.i. algebra in characteristic zero, then $\omega_0(A) = d \geq \omega_1(A) = h$.*

Proof. By Amitsur-Regev, the partitions λ in $\chi(A)$ may have up to ℓ columns of length $\geq k$, for some ℓ and k. By Lemma 2, $k \geq d$. If $k = d$, we are done. If not, let λ' have parts $(\lambda'_1, \lambda'_2, \ldots)$. Note that λ'_i is the length of the i^{th} column of the tableau. Each of $\lambda'_{\ell+1}, \lambda'_{\ell+2}, \ldots$ is less than or equal to k, and by Lemma 3, for each s between $d+1$ and k, there are at most t columns of length s. Hence, $\lambda'_i \leq d$ for all $i > \ell + t(k-d)$, and so $\omega_0(A) \leq d$. Combining with Lemma 2 shows that $\omega_0(A) = d$. The proof of $\omega_1(A) = h$ is similar. □

Proof of Lemma 3. We focus on the case in which λ has t columns of height equal to $s > d$. Let T be a tableau of shape λ and with row symmetrizer R and column symmetrizer C. Let $r = \sum_{\rho \in R} \rho$ and $c = \sum_{\kappa \in C} (-1)^\kappa \kappa$ and let $e = cr$. We identify e with a multilinear polynomial $e(x_1, \ldots, x_n)$ and show that it is an identity for A.

If e is not an identity for A, then we may assume that it has a non-zero evaluation in which each x_i is substituted by either an element of B' or J. Moreover, we may assume that each x_i in B' is taken from one of $C_1, \ldots, C_K$ with $C_1 J \cdots J C_K \neq 0$. Let

$$\dim(C_1^{(0)} \oplus \cdots \oplus C_k^{(0)}) = d' \leq d \text{ and } \dim(C_1^{(1)} \oplus \cdots \oplus C_k^{(1)}) = h' \leq h.$$

Let $v_1, \ldots, v_{d'}$ and $w_1, \ldots, w_{h'}$ be bases for the degree zero and degree one parts of B', respectively. We may assume that each x_i that takes a value in B' is either of the form $v_j \otimes \epsilon_j$ or $w_j \otimes \epsilon_j$, where $\epsilon_j \in E$ is degree zero in the first case and degree one in the second. It follows that if $e(x_1, \ldots, x_n)$ is symmetric in a set of variables, then at most d' of them can have degree zero substitutions in B'; and if it is antisymmetric in a set of variables, then at most h' of them can have degree one substitutions in B'.

Let Y be the set of variables x_i corresponding to letters i in the $t \times s$ rectangle in the tableau T; see Figure 3.

For each permutation $\kappa \in C$, $\kappa e = (-1)^\kappa e$, and so, in particular, e is alternating in the elements in each column of Y. Hence, each column contains at most h' degree one elements, and so the total number of degree one substitutions in Y is at most th'. On the other hand, let $e = \sum_{\kappa \in C} e_\kappa$, where $e_\kappa = (-1)^\kappa \kappa r$. If a certain

substitution is non-zero on e, it must be non-zero on at least one of the e_κ. Now, for each $\rho \in R$,

$$\kappa\rho\kappa^{-1}e_\kappa = e_\kappa.$$

Note that $\kappa\rho\kappa^{-1}$ is a row permutation of the tableau κT, and that in both T and κT the $t \times s$ rectangle has the same entries, Y. Hence, in each of the rows of this rectangle in κT, there can be at most d' degree zero substitutions. Keeping in mind that the number of radical substitutions is at most $q-1$, the number of degree one substitutions is at least $st - sd' - q + 1$. Combining, we get

$$th' \geq st - sd' - q + 1.$$

This implies the bound $t \leq (sd' + q - 1)/(s - h')$, which is false for large t.

The case of $\omega_1(A) = h$ is similar with the technical difference that we use the basis of idempotents constructed as rc rather than cr. □

Examples

In this section we compute ω_0 and ω_1 for various generic p.i. algebras. We first will use the results of the previous section to develop an effective way to compute these numbers. The main idea is inspired by [3], and we now recall some of the machinery from that work.

Definition 5. Let each of the algebras $B_1, \ldots, B_n$ be isomorphic to one of the verbally prime algebras $M_{k_i}(F)$, M_{k_i,ℓ_i} or $M_{n_i}(E)$. So, the elements of each B_i are matrices with entries from E. Define $B_1 \circ \cdots \circ B_n$ to be the algebra of all matrices of the form

$$\begin{pmatrix} B_1 & * & \cdots & * \\ 0 & B_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

with blocks of elements from $B_1, \ldots, B_n$ on the diagonal, zeros below, and elements from E above.

Algebras of this form are called prime product algebras. We note that the T-ideal of identities $Id(B_1 \circ \cdots \circ B_n)$ is the product $Id(B_1) \cdots Id(B_n)$. It follows from [2] that

$$\omega_0(B_1 \circ \cdots \circ B_n) = \omega_0(B_1) + \cdots + \omega_0(B_n),$$
$$\omega_1(B_1 \circ \cdots \circ B_n) = \omega_1(B_1) + \cdots + \omega_1(B_n).$$

We also need Lemma 2.3 from [3]:

Lemma 6. *Let $A \sim G(B)$, $B = B_1 \times \cdots \times B_K \times J$, as above. If $B_{i_1} J \cdots J B_{i_k} \neq 0$, then*

$$G(B_{i_1}) \circ \cdots \circ G(B_{i_k}) \in \mathrm{var}(A)$$

i.e., $G(B_{i_1}) \circ \cdots \circ G(B_{i_k})$ satisfies all of the identities of A.

The next corollary gives an effective way to compute the eventual arm and leg width of a variety.

Corollary 7. *For $i = 0, 1$, $\omega_i(A)$ is the maximum value of $\omega_i(A_1 \circ \cdots \circ A_n)$, where $A_1 \circ \cdots \circ A_n$ runs over all prime product algebras in* $\mathrm{var}(A)$.

Proof. For any $A' \in \mathrm{var}(A)$, $\omega_i(A') \leq \omega_i(A)$, for $i = 1, 2$. For the opposite inequality, the proof that $d = \omega_0(A)$ and $h = \omega_1(A)$ In Theorem 4 implies that there exist graded simple algebras $C_1, \ldots, C_k$ and $D_1, \ldots, D_\ell$ such that

$$G(C_1) \circ \cdots \circ G(C_k),\ G(D_1) \circ \cdots \circ G(D_\ell) \in \mathrm{var}(A)$$

and

$$\omega_0(G(C_1) \circ \cdots \circ G(C_k)) = \omega_0(A) \text{ and } \omega_1(G(D_1) \circ \cdots \circ G(D_\ell)) = \omega_1(A).$$

□

In Theorem 2.4 of [3] Berele and Regev, based on [4], show that the exponential rate of growth $\exp(A)$ is the maximum value of $\exp(A_1 \circ \cdots \circ A_n)$ where $A_1 \circ \cdots \circ A_n$ again runs over prime product algebras in $\mathrm{var}(A)$. Moreover, for a prime product algebra A, $\exp(A) = \omega_0(A) + \omega_1(A)$. Together this gives another corollary.

Corollary 8. *For any p.i. algebra A, $\omega_0(A) \leq \exp(A) \leq \omega_0(A) + \omega_1(A)$.*

We now turn to the application of Corollary 7 to the computation of $\omega_i(A)$ for various generic p.i. algebras. If f is a polynomial, we write $\omega_i(f)$ for ω_i of the generic algebra satisfying f. By Corollary 7, $\omega_i(f)$ will be the maximum value of $\omega_i(A_1 \circ \cdots \circ A_n)$, where $A_1 \circ \cdots \circ A_n$ runs over prime product algebras satisfying f. In [3] Regev and the author did some computations on the subject of which prime product algebras satisfy certain identities f in order to compute $\exp(f)$. By Lemma 7, once we know which prime product varieties satisfy f, we can compute $\omega_0(f)$ and $\omega_1(f)$ as well as $\exp(f)$. We now record the $\omega_i(f)$ in the cases which follow easily from the computations already done in [3].

Theorem 9. *Let the positive integer k be written as $k = 3q + r$ where $r = 0, 1, 2$. If f is the k^{th} power of the commutator $f = [x, y]^k$, then $\omega_0(f) = k$ and $\omega_1(f) = 2q + \lfloor r/2 \rfloor$; and if f is the k^{th} power of a higher commutator, then $\omega_0(f) = \omega_1(f) = k$.*

Proof. In section 5 of [3] it is shown that if a prime product algebra satisfies a power of a commutator, then each factor must be F, E or $M_{1,1}$. Let a, b and c be the number of factors of each. In the case of a single commutator $f = [x, y]^k$, there is a product $A_1 \circ \cdots \circ A_n$ satisfying f if and only if $a + 2b + 3c \leq k$. Note that each

$$\omega_i(A_1 \circ \cdots \circ A_n) = \omega_i(A_1) + \cdots + \omega_i(A_n) = a\omega_i(F) + b\omega_i(E) + c\omega_i(M_{1,1}).$$

Hence, the arm width $w_0(A_1 \circ \cdots \circ A_n)$ equals $a + b + 2c$. It takes its maximum value of k when $a = k$, $b = c = 0$. The leg width $w_1(A_1 \circ \cdots \circ A_n)$ equals $b + 2c$ and is maximized by making c as large as possible. In this case $c = q$, and if $r = 2$, then $b = 1$, and otherwise $b = 0$.

In the case of a power of a higher commutator, [3] does not settle whether the restraint is $a + b + 3c \leq k$ or $a + b + 2c \leq k$. In either event, we maximize both $w_0 = a + b + 2c$ and $\omega_1 = b + 2c$ by setting $b = k$ and $a = c = 0$. □

Theorem 10. *Let f be a power of a standard identity $f = s_n(x_1, \ldots, x_n)^k$ with $n \geq 4$. Then $\omega_0(f) = k\lfloor n/2 \rfloor^2$.*

Proof. In the proof of Theorem 6.10 in [3], we consider

$$A = M_{\lfloor n/2 \rfloor}(F) \circ \cdots \circ M_{\lfloor n/2 \rfloor}(F),$$

which is an algebra satisfying s_n^k. Then $\omega_0(A) = k\lfloor n/2\rfloor^2$ and $\omega_1(A) = 0$. Theorem 6.10 there shows that $\exp(A) = \exp(f)$. Hence,

$$k\lfloor n/2\rfloor^2 = \omega_0(A) \leq \omega_0(f) \leq \exp(f) = k\lfloor n/2\rfloor^2,$$

and the theorem follows. □

We remark that we are not able to compute $\omega_1(f)$ where f is a power of a standard identity at this time. Such a computation would require knowing which prime product algebras satisfy which powers of standard identities. It is not even known which verbally prime algebras satisfy which powers of standard identities, although an investigation of this question would certainly be worthwhile.

The last case we consider are Amitsur's Capelli-type polynomials. For each k, ℓ there is a polynomial $E_{k,\ell}$ with the property that an algebra A satisfies $E_{k,\ell}$ if and only if the cocharacters are contained in the $k \times \ell$ hook. It is immediate that $\omega_0(E_{k,\ell}) \leq k$ and $\omega_1(E_{k,\ell}) \leq \ell$, and we would want to know how much of a gap there could be. In light of Theorem 4 it is reasonable to consider the cases $k \geq \ell$ and $k < \ell$ separately. Here is a partial answer.

Theorem 11. *If $k \geq \ell$, then $k-1 \leq \omega_0(E_{k,\ell}) \leq k$ and $\ell - 2 \leq \omega_1(E_{k,\ell}) \leq \ell$, and if $k < \ell$, then $k-1 \leq \omega_1(E_{k,\ell}) \leq \omega_0(E_{k,\ell}) \leq k$. Moreover, if $\ell \geq k+3$, then $\omega_0(E_{k,\ell}) = \omega_1(E_{k,\ell}) = k$.*

Proof. In the case of $k \geq \ell$ see the proof of Theorem 4.5 in [3]. In the language of that proof, u is the gap $k - \omega_0(E_{k,\ell})$ and v is the gap $\ell - \omega_1(E_{k,\ell})$, and the proof constructs a prime product algebra A satisfying $E_{k,\ell}$ with $u \leq 1$ and $v \leq 2$.

In the case of $k < \ell$, we write $k-1$ as a sum of four squares $n = n_1^2 + \cdots + n_4^2$. By Remark 4.2 of [3] $A = M_{n_1}(E) \circ \cdots \circ M_{n_4}(E)$ has cocharacter in the hook $H(k-1+u, k-1+v)$ for any u, v with $(u+1)(v+1) \geq 4$. Taking $u = v = 1$ shows that A satisfies $E_{k,\ell}$, and $\omega_i(A) = k-1$ for $i = 1, 2$.

In the case of $\ell \geq k+3$ we modify the above and write k itself as a sum of four squares and take $u = 0$ and $v = 3$. □

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