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ON BERNSTEIN TYPE THEOREMS IN FINSLER SPACES WITH THE VOLUME FORM INDUCED FROM THE PROJECTIVE SPHERE BUNDLE

QUN HE AND YI-BING SHEN

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Abstract. By using the volume form induced from the projective sphere bundle of the Finsler manifold, we study the Finsler minimal submanifolds. It is proved that such a volume form for the Randers metric $F = \alpha + \beta$ in a Randers space is just that for the Riemannian metric α, and therefore the Bernstein type theorem in the special Randers space of dimension ≤ 8 is true. Moreover, a Bernstein type theorem in the 3-dimensional Minkowski space is established by considering the volume form induced from the projective sphere bundle.

In classical differential geometry there is a well-known Bernstein theorem which says that any complete minimal graphs in the Euclidean 3-space are planes. There are various generalizations of the Bernstein theorem to higher dimensions (see [12], [6], [16], etc. for details).

Recently, by using the Busemann-Hausdorff volume form, Z. Shen ([9]) investigated the geometry of Finsler submanifolds from a new point of view. By avoiding any connections to Finsler geometry, he introduced the notions of the mean curvature and the normal curvature for Finsler submanifolds. Being based on it, minimal surfaces and a Bernstein type theorem on a special Randers space were considered in [14] and [13]. As is well known, there is another volume form induced from the projective sphere bundle of the Finsler manifold ([2]), which appeared once in [5] and [1]. By using this volume form, analogues such as the mean curvature and the second fundamental form for Finsler submanifolds were introduced in [7] and coincide with the usual notions for the Riemannian case.

In this paper we shall continue the work of [7]. By using the volume form induced from the projective sphere bundle of the Finsler manifold, we study properties of Finsler minimal submanifolds and establish the Bernstein type theorems for Finsler minimal graphs in the Minkowski space and the Randers space.

The contents of the paper are arranged as follows. First, in §1 we describe the volume form induced from the projective sphere bundle SM of an oriented Finsler

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manifold (M, F) and the mean curvature form μ of Finsler submanifolds, which is similar to that introduced in [9]. In §2, we prove that the volume form induced from the projective sphere bundle for the Randers metric $F = \alpha+\beta$ in a Randers space is just the volume form for the Riemannian metric α (Theorem 2.1). If α is Euclidean, $F = \alpha + \beta$ is called a special Randers metric. Hence, by the Bernstein theorem for Euclidean minimal graphs [12], it follows that, under considering the volume form induced from the projective sphere bundle, any complete minimal graphs in the special Randers m-space with $m \leq 8$ are affine hyperplanes (Theorem 2.2). Moreover, it is proved that complete stable minimal surfaces in a special Randers 3-space are planes (Theorem 2.3). In §3, we consider hypersurfaces in the Minkowski space $(\widetilde{V}, \widetilde{F})$. It is proved that a constant mean curvature graph with respect to the volume form induced from the projective sphere bundle in $(\widetilde{V}, \widetilde{F})$ satisfies the so-called equation of mean curvature type (Theorem 3.1). Therefore, by [11], any complete minimal graphs in a 3-dimensional Minkowski space with the volume form induced from the projective sphere bundle are planes (Theorem 3.2).

It should be remarked that the mean curvature form of Finsler submanifolds used here by us is different from that considered in [9, 13], so that the critical points (minimal surfaces) of the volume functionals are not the same. By considering the Busemann-Hausdorff volume form, a Bernstein type theorem on a special Randers 3-space has been shown in [13].

1. Finsler volume forms and minimal immersions

Let M be an n-dimensional smooth manifold and let $\pi : TM \to M$ be the natural projection. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ satisfying the following properties: (i) F is smooth on $TM \setminus \{0\}$; (ii) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$; (iii) the induced quadratic form g is positive definite, where

$$g := g_{ij}(x, y)dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}. \tag{1.1}$$

Here and from now on, $[F]_{y^i}$, $[F]_{y^iy^j}$ mean $\frac{\partial F}{\partial y^i}$, $\frac{\partial^2 F}{\partial y^i \partial y^j}$, etc., and we shall use the following convention of index ranges unless otherwise stated:

$$1 \leq i, j, \cdots \leq n; \quad 1 \leq a, b, \cdots \leq n-1; \quad \bar{a} = n + a; \quad 1 \leq \alpha, \beta, \cdots \leq m \ (> n).$$

The simplest Finsler manifolds are Minkowski spaces, on which the metric function F is independent of x.

The projection $\pi : TM \to M$ gives rise to the pull-back bundle π^*TM and its dual π^*T^*M, which sit over $TM \setminus \{0\}$. We shall work on $TM \setminus \{0\}$ and rigidly use only objects that are invariant under positive rescaling in y, so that one may view them as objects on the projective sphere bundle SM using homogeneous coordinates.

In π^*T^*M there is a global section $\omega = [F]_{y^i}dx^i$, called the *Hilbert form*, whose dual is $l = l^i\frac{\partial}{\partial x^i}$, $l^i = y^i/F$, called the *distinguished* field. Each fibre of π^*T^*M has a positively oriented orthonormal coframe $\{\omega^i\}$ with $\omega^n = \omega$. Expand ω^i as $v^i_j dx^j$, whereby the stipulated orientation implies that $v := \det(v^i_j) = \sqrt{\det(g_{ij})}$. Set

$$\omega^{n+i} = v^i_j \delta y^j, \qquad \delta y^i = \frac{1}{F}(dy^i + N^i_j dx^j). \tag{1.2}$$

The collection $\{\omega^i, \omega^{n+i}\}$ forms an orthonormal basis on $T^*(TM\setminus\{0\})$ with respect to the Sasaki-type metric $g_{ij}dx^i\otimes dx^j + g_{ij}\delta y^i\otimes\delta y^j$ [2]. The pull-back of the Sasaki-type metric from $TM\setminus\{0\}$ to SM is a Riemannian metric

$$\hat{g} = g_{ij}dx^i\otimes dx^j + \delta_{ab}\omega^{\bar{a}}\otimes\omega^{\bar{b}}. \tag{1.3}$$

Thus, the volume element dV_{SM} of SM with the metric $\hat{g}$ is

$$dV_{SM} = \omega^1\wedge\cdots\wedge\omega^{2n-1} = \sqrt{\det(g_{ij})}dx\wedge\omega^{n+1}\wedge\cdots\wedge\omega^{2n-1}, \tag{1.4}$$

where $dx = dx^1\wedge\cdots\wedge dx^n$. It is easy to see that (1.4) can be rewritten as [7]

$$dV_{SM} = \Omega d\tau\wedge dx, \tag{1.5}$$

where

$$\Omega := \det\left(\frac{g_{ij}}{F}\right), \qquad d\tau := \sum_{i=1}^{n}(-1)^{i-1}y^i dy^1\wedge\cdots\wedge\widehat{dy^i}\wedge\cdots\wedge dy^n. \tag{1.6}$$

The *volume form* of a Finsler n-manifold (M,F) is defined by [7]

$$dV_M := \sigma(x)dx, \qquad \sigma(x) := \frac{1}{c_{n-1}}\int_{S_xM}\Omega d\tau, \tag{1.7}$$

where c_{n-1} denotes the volume of the unit Euclidean $(n-1)$-sphere S^{n-1}, $S_xM = \{y\in T_xM | F(y) = ||y|| = 1\}$. This definition is essentially the same as in [5], and dV_M is also called the *Holmes-Thompson volume form* when F is the Minkowski metric (cf. [10], §2.2, or [17]).

Let (M,F) and $(\widetilde{M},\widetilde{F})$ be Finsler manifolds, and let $f: M\to\widetilde{M}$ be an immersion. If $F(x,y) = \widetilde{F}(f(x), df(y))$ for all $(x,y)\in TM\setminus\{0\}$, then f is said to be an *isometric immersion*. It is clear that

$$g_{ij}(x,y) = \tilde{g}_{\alpha\beta}(\tilde{x},\tilde{y})f_i^\alpha f_j^\beta \tag{1.8}$$

for the isometric immersion $f:(M,F)\to(\widetilde{M},\widetilde{F})$, where

$$\tilde{x}^\alpha = f^\alpha(x), \quad \tilde{y}^\alpha = f_i^\alpha y^i, \quad f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}. \tag{1.9}$$

Assume that M is complete and $\mathcal{D}\subset M$ is any compact domain. Let $f_t: M\to\widetilde{M}$, $t\in(-\varepsilon,\varepsilon)$, be a smooth variation of f with $f_0 = f$ and $f_t|_{M\setminus\mathcal{D}} = f|_{M\setminus\mathcal{D}}$. Then $\{f_t\}$ induces a variation vector field $\tilde{V}$ along f defined by

$$\tilde{V} := \frac{\partial f_t}{\partial t}|_{t=0} = \tilde{V}^\alpha\frac{\partial}{\partial\tilde{x}^\alpha}, \qquad \tilde{V}|_{M\setminus\mathcal{D}} = 0. \tag{1.10}$$

f_t induces a family of Finsler metrics $F_t = (f_t)^*\widetilde{F}$, i.e., $F_t(x,y) = \widetilde{F}(f_t(x), df_t(y))$. By (1.6) and (1.7), the volume of $(\mathcal{D}, F_t)$ is

$$V_t(\mathcal{D}) = \int_{\mathcal{D}} dV_t = \int_{\mathcal{D}}\left(\frac{1}{c_{n-1}}\int_{S_xM_t}\Omega_t d\tau\right)dx,$$

where

$$\Omega_t = \det\left(\frac{1}{F_t}g_{t|ij}\right) = \det\left(\frac{1}{F_t}\tilde{g}_{\alpha\beta}f_{t|i}^\alpha f_{t|j}^\beta\right), \quad \Omega_0 = \Omega. \tag{1.11}$$

It is easy to see that

$$\frac{d}{dt}V_t(\mathcal{D})|_{t=0} = \int_{\mathcal{D}}\left(\frac{1}{c_{n-1}}\int_{S_xM}(\frac{\partial}{\partial t}\Omega_t)|_{t=0}d\tau\right)dx. \tag{1.12}$$

Set

$$f^\alpha_{ij} = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}, \tag{1.13}$$

$$h := \frac{h^\alpha}{F^2}\frac{\partial}{\partial \tilde{x}^\alpha}, \qquad h^\alpha = f^\alpha_{ij} y^i y^j - f^\alpha_k G^k + \widetilde{G}^\alpha, \tag{1.14}$$

where G^k and $\widetilde{G}^\alpha$ are the geodesic coefficients for (M, F) and $(\widetilde{M}, \widetilde{F})$, respectively.

Let $\pi^*\mathcal{N} = (\pi^*TM)^\perp$ be the orthogonal complement of π^*TM in $\pi^*(f^{-1}T\widetilde{M})$ with respect to $\tilde{g}$, and let

$$\mathcal{N}^* = \{\xi \in \mathcal{C}(f^{-1}T^*\widetilde{M}) \mid \xi(df(X)) = 0, \forall X \in \mathcal{C}(TM)\},$$

which is called the *normal bundle* of f [9]. Clearly, $\pi^*\mathcal{N}^*$ is the dual bundle of $\pi^*\mathcal{N}$. By (1.14), we can see that $h \in \pi^*\mathcal{N}$ [7]. For some $\tilde{N} \in \pi^*\mathcal{N}$, we define $\mu_{\tilde{N}} \in \mathcal{N}^*$ by

$$\mu_{\tilde{N}}(\tilde{X}) := \frac{\int_{S_xM} \tilde{g}(\tilde{N}, \tilde{X})\Omega d\tau}{\int_{S_xM} \Omega d\tau} = \frac{\int_{S_xM} \tilde{g}(\tilde{N}, \tilde{X})\Omega d\tau}{c_{n-1}\sigma(x)} \tag{1.15}$$

for any $\tilde{X} \in \mathcal{C}(f^{-1}T\widetilde{M})$. Then $\mu_{\tilde{N}}$ is a global section of $f^{-1}T^*\widetilde{M}$. Set

$$\mu = \mu_h = \frac{1}{c_{n-1}\sigma}\sum_\alpha \left(\int_{S_xM} \frac{h_\alpha}{F^2}\Omega d\tau\right) d\tilde{x}^\alpha, \tag{1.16}$$

where h is defined by (1.14). By (1.11)$\sim$(1.16), a straightforward calculation (see [7] for details) gives the following (cf. [7])

Theorem 1.1. *Let $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ be an isometric immersion, let f_t be a smooth variation with $f_0 = f$ and let the variation field $\tilde{V}$ satisfy* (1.10)*. Then the first variation formula of the volume for $\mathcal{D} \subset M$ is*

$$\frac{d}{dt}V_t(\mathcal{D})|_{t=0} = -n\int_{\mathcal{D}} \mu(\tilde{V}) dV_M, \tag{1.17}$$

where μ is defined by (1.16) *and dV_M is defined by* (1.7)*.*

Definition 1.2. An isometric immersion $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ is called a *minimal immersion* if any compact domain of M is the critical point of its volume functional with respect to any variation vector field (1.10).

We call μ defined by (1.16) the *mean curvature form* of f, of which the *norm* is defined by

$$||\mu|| := \sup_{\tilde{X} \in \mathcal{C}(f^{-1}T\widetilde{M})} \frac{|\mu(\tilde{X})|}{||\tilde{X}||}.$$

It is obvious that $||\mu|| = 0$ if and only if $\mu = 0$. Thus, we have (cf. [7])

Theorem 1.3. *An isometric immersion $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ is minimal if and only if the mean curvature form μ defined by* (1.16) *vanishes identically.*

Recall that for an isometric immersion $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ we have (see formulas (2.14) and (3.14) of Chapter V in [8])

$$G^k = f^\alpha_l g^{lk} \tilde{g}_{\alpha\beta}(f^\beta_{ij} y^i y^j + \widetilde{G}^\beta),$$

from which together with (1.14) it follows that

$$h_\beta = \tilde{g}_{\beta\gamma} h^\gamma = T_{\alpha\beta}(f^\alpha_{ij} y^i y^j + \widetilde{G}^\alpha), \tag{1.18}$$

where

$$T_{\alpha\beta} := \tilde{g}_{\alpha\beta} - \tilde{g}_{\alpha\gamma} f^\gamma_i f^\sigma_j g^{ij} \tilde{g}_{\sigma\beta}. \tag{1.19}$$

Hence, by Theorem 1.3, we see that f is minimal if and only if

$$\int_{S_x M} \frac{1}{F^2} T_{\alpha\beta}(f^\alpha_{ij} y^i y^j + \widetilde{G}^\alpha)\Omega d\tau = 0 \tag{1.20}$$

for all β. Let $p^\perp : \pi^*(f^{-1}T\widetilde{M}) \to \pi^*\mathcal{N} = (\pi^*TM)^\perp$ be the orthogonal projection with respect to $\tilde{g}$, and let $X^\perp = p^\perp X$ for $X \in \mathcal{C}(\pi^*(f^{-1}T\widetilde{M}))$. Thus,

$$T(X, Y) = \tilde{g}(p^\perp X, Y) = \tilde{g}(X^\perp, Y), \tag{1.21}$$

where $T = T_{\alpha\beta} d\tilde{x}^\alpha \otimes d\tilde{x}^\beta$. Then (1.20) can be rewritten as

$$\int_{S_x M} \frac{1}{F^2} \tilde{g}(v^\perp, \frac{\partial}{\partial \tilde{x}^\alpha})(f^\alpha_{ij} y^i y^j + \widetilde{G}^\alpha)\Omega d\tau = 0 \tag{1.22}$$

for any vector field $v \in \mathcal{C}(f^{-1}T\widetilde{M}))$.

2. Submanifolds in Randers spaces

Let (M, F) be a Randers space, where

$$F = \alpha + \beta = \sqrt{a_{ij} y^i y^j} + b_i y^i, \quad ||\beta|| = \sqrt{a^{ij} b_i b_j} = b \ (0 \le b < 1). \tag{2.1}$$

By [2], we know that

$$\det(g_{ij}) = a \left(\frac{F}{\alpha}\right)^{n+1}, \qquad a = \det(a_{ij}). \tag{2.2}$$

Thus, we have

$$dV_M = \sigma(x) dx = \frac{dx}{c_{n-1}} \int_{S_x M} \Omega d\tau,$$

$$\int_{S_x M} \Omega d\tau = \int_{S_x M} \frac{\det(g_{ij})}{F^n} d\tau = a \int_{S_x M} \frac{F}{\alpha^{n+1}} d\tau = \sqrt{a} \int_{S_x} (1 + b_i y^i) dV_{S_x}, \tag{2.3}$$

where

$$S_x = \left\{ y \in R^n | a_{ij} y^i y^j = 1 \right\}, \quad dV_{S_x} = \sqrt{a} d\tau. \tag{2.4}$$

Let $\{\lambda_i\}$ be the eigenvalues of the matrix (a_{ij}), of which the corresponding unit eigenvectors are $\{v_i\}$ with respect to the Euclidean metric $\langle\ ,\ \rangle$ in R^n. Set

$$y^i = \sum_k v^i_k \frac{z^k}{\sqrt{\lambda_k}} \qquad \text{with} \qquad |z|^2 = \langle z, z \rangle = 1, \tag{2.5}$$

so that $y \in S_x$. Thus, we have

$$\begin{aligned} dV_{S_x} &= \sqrt{a} \sum_{i=1}^n (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge d\widehat{y}^i \wedge \cdots \wedge dy^n \\ &= \sum_{i=1}^n (-1)^{i-1} z^i dz^1 \wedge \cdots \wedge d\widehat{z}^i \wedge \cdots \wedge dz^n = dV_{S^{n-1}}. \end{aligned} \tag{2.6}$$

Because $S^{n-1}$is symmetric with respect to every z_k and the z_k's are odd functions, we get

$$\int_{S_x} y^i dV_{S_x} = \sum_k v_k^i \frac{1}{\sqrt{\lambda_k}} \int_{S^{n-1}} z^k dV_{S^{n-1}} = 0, \tag{2.7}$$

from which it follows that

$$dV_M = \frac{\sqrt{a}dx}{c_{n-1}} \int_{S_x} (1 + b_i y^i) dV_{S_x} = \sqrt{a}dx. \tag{2.8}$$

Hence, from (2.8) we have the following.

Theorem 2.1. *The volume element of the Randers space $(M, \alpha + \beta)$ is just that of the Riemannian manifold (M, α).*

Let $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ be an isometric immersion into a Randers space $(\widetilde{M}, \widetilde{F})$ with

$$\widetilde{F} = \tilde{\alpha} + \tilde{\beta} = \sqrt{\tilde{a}_{\alpha\beta}(\tilde{x})\tilde{y}^\alpha \tilde{y}^\beta} + \tilde{b}_\alpha(\tilde{x})\tilde{y}^\alpha, \quad ||\tilde{\beta}|| = \sqrt{\tilde{a}^{\alpha\beta}\tilde{b}_\alpha \tilde{b}_\beta} = \tilde{b} \ (0 \le \tilde{b} < 1).$$

Clearly, we have

$$F = f^*\widetilde{F} = \alpha + \beta = \sqrt{a_{ij}y^i y^j} + b_i y^i, \tag{2.9}$$

where

$$a_{ij} = \tilde{a}_{\alpha\beta} f_i^\alpha f_j^\beta, \qquad b_i = \tilde{b}_\alpha f_i^\alpha. \tag{2.10}$$

This means that (M, F) is also a Randers n-space. By Theorem 2.1, the volume element of $(M, \alpha + \beta)$ is just that of the Riemannian manifold (M, α). Therefore, we have

Proposition 2.2. *The mean curvature form μ of the submanifold $(M, \alpha + \beta)$ isometrically immersed in the Randers space $(\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ is just that of the submanifold (M, α) isometrically immersed in the Riemannian manifold $(\widetilde{M}, \tilde{\alpha})$.*

By Theorem 1.2 and Proposition 2.2, we have immediately

Proposition 2.3. *The minimal submanifolds in the Randers space $(\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ are just the minimal submanifolds in the Riemannian manifold $(\widetilde{M}, \tilde{\alpha})$, and vice-versa.*

Recall that a Randers space $(\widetilde{M}, \widetilde{F})$ is said to be *special* if $\widetilde{M}$ is a real vector space $\widetilde{V}$ and $\tilde{\alpha}$ is Euclidean [13]. So, by Bernstein's theorem on minimal graphs in the Euclidean space ([12]), we have immediately

Theorem 2.4. *Any complete minimal graph in a special Randers $(n + 1)$-space $(\widetilde{V}^{n+1}, \widetilde{F})$ with $n \le 7$ is an affine n-subspaces.*

Remark 2.5. By means of the Busemann-Hausdoff measure, a Bernstein type theorem in a special Randers 3-space $(\widetilde{M}, \widetilde{F})$, which is also a Minkowski space and satisfies $0 < ||\tilde{\beta}|| < \frac{1}{\sqrt{3}}$, was given in [13].

Let $f : (M, \alpha+\beta) \to (\widetilde{M}, \tilde{\alpha}+\tilde{\beta})$ be a minimal isometric immersion, and let f_t be a smooth variation with $f_0 = f$. Then the second variation formula of the volume functional for any compact domain $\mathcal{D} \subset M$ is the same as that of (M, α) viewed as a minimal submanifold in the Riemannian manifold $(\widetilde{M}, \tilde{\alpha})$. Recall that a minimal

submanifold is said to be *stable* if its second variation is always nonnegative for any deformation with compact support.

By Proposition 2.3 and the generalized Bernstein theorem on minimal surfaces in Euclidean space ([6]), we have immediately

Theorem 2.6. *Any complete stable minimal surface in a special* 3*-dimensional Randers space is a plane.*

3. Hypersurfaces in a Minkowski space

In this section we assume that $(\widetilde{M}, \widetilde{F}) = (\widetilde{V}^{n+1}, \widetilde{F})$ is a Minkowski space of dimension $m = n+1$. Let $\{\tilde{e}_\alpha\}$ be a given orthonormal basis of $\widetilde{V}^{n+1}$ with respect to the Euclidean metric $\langle , \rangle$. Let $f = f^\alpha \tilde{e}_\alpha : (M, F) \to (\widetilde{V}^{n+1}, \widetilde{F})$ be an isometrically immersed hypersurface. Noting that $\widetilde{G}^\alpha = 0$ for $(\widetilde{V}^{n+1}, \widetilde{F})$, it follows from (1.22) that f is minimal if and only if

$$f^\alpha_{ij} \int_{S_xM} \frac{1}{F^2} \tilde{g}(v^\perp, \tilde{e}_\alpha) y^i y^j \Omega d\tau = 0 \tag{3.1}$$

for any $v \in \mathcal{C}(f^{-1}\widetilde{V}^{n+1})$.

Let $\mathbf{n} = n^\alpha \tilde{e}_\alpha$ be the unit normal vector field of $f(M)$ with respect to the Euclidean metric $\langle\ ,\ \rangle$ in $\widetilde{V}^{n+1}$, and let $\tilde{\mathbf{n}} = \tilde{n}^\alpha \tilde{e}_\alpha$ be the unit normal vector field with respect to $\tilde{g}_{\tilde{y}}$ for $\tilde{y} = df(y)$ in $(\widetilde{V}^{n+1}, \widetilde{F})$. These mean that

$$\begin{gathered} \sum_\alpha n^\alpha f^\alpha_i = 0, \qquad \tilde{g}_{\alpha\beta} \tilde{n}^\alpha f^\beta_i = 0, \\ |\mathbf{n}|^2 = \langle \mathbf{n}, \mathbf{n} \rangle = \sum_\alpha (n^\alpha)^2 = 1, \qquad \tilde{g}(\tilde{\mathbf{n}}, \tilde{\mathbf{n}}) = \tilde{g}_{\alpha\beta} \tilde{n}^\alpha \tilde{n}^\beta = 1. \end{gathered} \tag{3.2}$$

It is clear that there is a function $\lambda(x, y)$ on (M, F) such that

$$\lambda n^\alpha = \tilde{g}_{\alpha\beta} \tilde{n}^\beta \qquad \text{with} \qquad \lambda = \tilde{g}(\mathbf{n}, \tilde{\mathbf{n}}) = \langle \mathbf{n}, \tilde{\mathbf{n}} \rangle^{-1}. \tag{3.3}$$

Since $\mathbf{n} \in \mathcal{C}(f^{-1}(\widetilde{V}^{n+1}))$ is linearly independent of $\{\frac{\partial}{\partial x^i}\}$ and $\mu(dfX) = 0$ for $X \in \mathcal{C}(TM)$, we see that $\mu = 0$ if and only if $\mu(\mathbf{n}) = 0$, i.e., by (3.1),

$$f^\alpha_{ij} \int_{S_xM} \frac{1}{F^2} \tilde{g}(\mathbf{n}^\perp, \tilde{e}_\alpha) y^i y^j \Omega d\tau = 0. \tag{3.4}$$

Definition 3.1. The *mean curvature* $\mathcal{H}$ of a hypersurface M in $(\widetilde{V}^{n+1}, \widetilde{F})$ is defined by

$$\mathcal{H}(x) = \frac{1}{n} \mu(\mathbf{n}) = \frac{1}{n c_{n-1} \sigma(x)} f^\alpha_{ij} \int_{S_xM} \frac{1}{F^2} \tilde{g}(\mathbf{n}^\perp, \tilde{e}_\alpha) y^i y^j \Omega d\tau, \tag{3.5}$$

where $\mathbf{n}$ is defined by (3.2).

From (3.2) and (3.3) it follows that

$$\tilde{g}(\mathbf{n}^\perp, \tilde{e}_\alpha) = \tilde{g}(\mathbf{n}, \tilde{\mathbf{n}}) \tilde{g}(\tilde{\mathbf{n}}, \tilde{e}_\alpha) = \lambda \tilde{g}_{\alpha\beta} \tilde{n}^\beta = \lambda^2 n^\alpha. \tag{3.6}$$

Set

$$a_{ij} = a_{ij}(x) = \sum_\alpha f^\alpha_i f^\alpha_j, \qquad a = \det(a_{ij}), \qquad \partial_i = \frac{\partial}{\partial x^i} = f^\alpha_i \tilde{e}_\alpha.$$

Since

$$\begin{pmatrix}\tilde{\mathbf{n}}\\ \partial_i\end{pmatrix}(\tilde{g}_{\alpha\beta})\begin{pmatrix}\tilde{\mathbf{n}}\\ \partial^j\end{pmatrix}^T=\begin{pmatrix}1 & 0\\ 0 & g_{ij}\end{pmatrix},$$

$$\begin{pmatrix}\mathbf{n}\\ \partial_i\end{pmatrix}(\tilde{g}_{\alpha\beta})\begin{pmatrix}\tilde{\mathbf{n}}\\ \partial_j\end{pmatrix}^T=\begin{pmatrix}\lambda & *\\ 0 & g_{ij}\end{pmatrix},\qquad \begin{pmatrix}\mathbf{n}\\ \partial_i\end{pmatrix}\begin{pmatrix}\mathbf{n}\\ \partial_j\end{pmatrix}^T=\begin{pmatrix}1 & 0\\ 0 & a_{ij}\end{pmatrix},$$

we have

$$\det(g_{ij})=\frac{a}{\lambda^2}\det(\tilde{g}_{\alpha\beta}). \tag{3.7}$$

Thus, by Definition 3.1, we have the following.

Proposition 3.2. *Let $f=f^\alpha\tilde{e}_\alpha:(M,F)\to(\widetilde{V}^{n+1},\widetilde{F})$ be an isometrically immersed hypersurface. Then the mean curvature of M can be expressed by*

$$\mathcal{H}=\frac{1}{\int_{S_x}F^2\eta dV_{S_x}}\sum_{\alpha,i,j}n^\alpha f^\alpha_{ij}\int_{S_x}\eta y^iy^j dV_{S_x}, \tag{3.8}$$

where S_x is defined in (2.4)*, and*

$$\eta=\frac{\det(\tilde{g}_{\alpha\beta})}{F^{n+2}}.$$

In particular, f is minimal if and only if

$$\sum_{\alpha,i,j}f^\alpha_{ij}n^\alpha\int_{S_x}\eta y^iy^j dV_{S_x}=0. \tag{3.9}$$

Similarly as in §2, let $\{\lambda_i\}$ be the eigenvalues of the matrix (a_{ij}), of which the corresponding unit eigenvectors are $\{v_i\}$ with respect to the Euclidean metric $\langle\ ,\ \rangle$ in R^n. Then we have

$$\sum_k v^k_iv^k_j=\sum_k v^i_kv^j_k=\delta_{ij},\qquad a_{ij}=\sum_k\lambda_kv^i_kv^j_k,\qquad a^{ij}=\sum\lambda_k^{-1}v^i_kv^j_k, \tag{3.10}$$

where $(a^{ij})=(a_{ij})^{-1}$. From (2.5), (2.6) and (3.9), by reasoning similar to proving (2.7), we can prove that

$$\begin{aligned}\int_{S_x}y^iy^jdV_{S_x}&=\sum_{k,l}v^i_kv^j_l\frac{1}{\sqrt{\lambda_k\lambda_l}}\int_{S^{n-1}}z^kz^ldV_{S^{n-1}}\\ &=\sum_k v^i_kv^j_k\frac{1}{\lambda_k}\int_{S^{n-1}}(z^k)^2dV_{S^{n-1}}.\end{aligned}$$

Because $\int_{S^{n-1}}(z^k)^2dV_{S^{n-1}}=\int_{S^{n-1}}(z^l)^2dV_{S^{n-1}}$ for $k\neq l$, it is obvious that

$$n\int_{S^{n-1}}(z^k)^2dV_{S^{n-1}}=c_{n-1}.$$

From this and (3.10),

$$\int_{S_x}y^iy^jdV_{S_x}=\frac{1}{n}c_{n-1}a^{ij}. \tag{3.11}$$

Define

$$B^{ij}:=\int_{S_xM}\eta y^iy^j\sqrt{a}d\tau=\int_{S_x}\eta y^iy^jdV_{S_x}. \tag{3.12}$$

It is remarkable that $\eta = 1$ and $B^{ij} = \frac{1}{n}c_{n-1}a^{ij}$ when $(\widetilde{V}^{n+1}, \widetilde{F})$ is the Euclidean space. In such a case, the quantity

$$\mathcal{H} = \frac{1}{c_{n-1}} \sum_{\alpha,i,j} n^\alpha f^\alpha_{ij} \int_{S_x} y^i y^j dV_{S_x} = \frac{1}{n} \sum_{\alpha,i,j} n^\alpha f^\alpha_{ij} a^{ij}$$

is just the mean curvature of the Euclidean hypersurface M.

Now assume that M is a graph of $\widetilde{V}^{n+1}$ defined by

$$f(x_1, \cdots, x_n) = (x_1, \cdots, x_n, u(x_1, \cdots, x_n)) \tag{3.13}$$

for $x = (x_1, \cdots, x_n) \in U \subseteq R^n$. Thus, we have

$$a = 1 + |\nabla u|^2, \qquad \mathbf{n} = a^{-1/2}(-u_1, \cdots - u_n, 1),$$

where $\nabla u = (u_1, \cdots, u_n)$ and $u_i = \partial u/\partial x^i$. Since $B^{ij} = B^{ij}(x, u, \nabla u)$ and $a^{ij} = a^{ij}(x, \nabla u)$, we have from (3.8) and (3.12)

Proposition 3.3. *In the Minkowski space $(\widetilde{V}^{n+1}, \widetilde{F})$ the graph* (3.13) *has the constant mean curvature $\mathcal{H}_0$ if and only if*

$$B^{ij}(x, u, \nabla u)u_{ij} = b(x, u, \nabla u), \tag{3.14}$$

where $u_{ij} = \partial^2 u/\partial x^i \partial x^j$, $b(x, u, \nabla u) = \mathcal{H}_0\sqrt{a}\int_{S_x} F^2 \eta dV_{S_x}$.

Definition 3.4. The equation (3.14) is said to be of *mean curvature type* if there are constants C_1 and C_2 such that

$$\begin{gathered} a^{ij}(x, w)\xi_i\xi_j \le B^{ij}(x, u, w)\xi_i\xi_j \le (1 + C_1)a^{ij}(x, w)\xi_i\xi_j, \\ |b(x, u, w)| \le C_2\sqrt{1 + |w|^2}, \end{gathered} \tag{3.15}$$

where $(x, u, w) \in U \times R \times R^n$ and $\xi = (\xi_1, \cdots, \xi_n) \in R^n$.

When $n = 2$, $a^{ij} = \delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}$, the equation of mean curvature type in two variables, has been defined in [11].

Theorem 3.5. *Let $f : U \subseteq R^n \to (\widetilde{V}^{n+1}, \widetilde{F})$ be a graph defined by* (3.13) *so that $M = f(U)$, which has the constant mean curvature $\mathcal{H}_0$. Then there is a constant κ such that the equation $\kappa B^{ij}u_{ij} = \kappa b$ is an elliptic equation of mean curvature type.*

Proof. Since $\eta = (\det(\tilde{g}_{\alpha\beta})/F^{n+2})|_{SM} > 0$, then

$$B^{ij}\xi_i\xi_j = \int_{S_x} \eta y^i y^j \xi_i\xi_j dV_{S_x} = \int_{S_x} \eta(y^i\xi_i)^2 dV_{S_x} \ge 0 \tag{3.16}$$

for $\xi \in R^n$, where the equality holds if and only if $y^i\xi_i = 0$ for $y \in S_x$, i.e., $\xi = 0$. This implies that the equation (3.14) is elliptic.

On the other hand, for $y \in S_x$, we see that $\tilde{y} = df(y) \in S^n \subset \widetilde{V}^{n+1}$. So, we have

$$\min\{\eta(x, y) : y \in S_x\} \ge \min\{\frac{\det(\tilde{g}_{\alpha\beta})}{\widetilde{F}^{n+2}} : \tilde{y} \in S^n\} = \kappa_1 > 0,$$

$$\max\{\eta(x, y) : y \in S_x\} \le \max\{\frac{\det(\tilde{g}_{\alpha\beta})}{\widetilde{F}^{n+2}} : \tilde{y} \in S^n\} = \kappa_2 > 0,$$

where κ_1 and κ_2 are constants. Hence, from (3.11) and (3.16) it follows that

$$\begin{gathered} \kappa_1 a^{ij}\xi_i\xi_j \le B^{ij}\xi_i\xi_j \le \kappa_2 a^{ij}\xi_i\xi_j, \\ \kappa_1|\mathcal{H}_0|\sqrt{a} \le |b| \le \kappa_2|\mathcal{H}_0|\sqrt{a}. \end{gathered}$$

These mean that (3.14) is of mean curvature type. □

By Theorem 3.5 and Theorem 4 of [11], we have immediately

Theorem 3.6. *Any complete minimal graph in the* 3*-dimensional Minkowski space* $(\widetilde{V}^3, \widetilde{F})$ *is a plane.*

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Department of Applied Mathematics, Tongji University, Shanghai 200092, People's Republic of China

E-mail address: hequn@mail.tongji.edu.cn

Department of Mathematics, Zhejiang University, Hangzhou 310028, People's Republic of China

E-mail address: yibingshen@zju.edu.cn