FUNCTIONAL CALCULUS AND *-REGULARITY OF A CLASS OF BANACH ALGEBRAS

CHI-WAI LEUNG AND CHI-KEUNG NG

(Communicated by David R. Larson)

Abstract. Suppose that \((A, G, \alpha)\) is a \(C^*\)-dynamical system such that \(G\) is of polynomial growth. If \(A\) is finite dimensional, we show that any element in \(K(G; A)\) has slow growth and that \(L^1(G, A)\) is *-regular. Furthermore, if \(G\) is discrete and \(\pi\) is a “nice representation” of \(A\), we define a new Banach *-algebra \(l^1_\pi(G, A)\) which coincides with \(l^1(G, A)\) when \(A\) is finite dimensional. We also show that any element in \(K(G; A)\) has slow growth and \(l^1_\pi(G, A)\) is *-regular.

1. Introduction and preliminaries

For a Banach *-algebra \(B\), we denote by \(B^*\) the unitalization of \(B\) together with the *-algebra norm defined by \(\|b + \lambda 1\| = \|b\| + |\lambda|\). We also denote by \(C^*(B)\) the enveloping \(C^*\)-algebra of \(B\) and \(\Phi : B \to C^*(B)\) the canonical embedding (not necessarily injective).

In the following, we assume that \(C^*(B) \neq (0)\). Moreover, throughout this paper, all *-representations of Banach *-algebras are assumed to be non-degenerate and all ideals are closed. For any *-representation \(\pi\) of \(B\), there is a unique *-representation \(\pi_*\) of \(C^*(B)\) such that \(\pi = \pi_* \circ \Phi\). Let \(\text{Prim} C^*(B)\) be the space of primitive ideals of \(C^*(B)\) and let \(\text{Prim}* B\) be the space of kernels of topological irreducible *-representations of \(B\), both equipped with the Jacobson topology. Moreover, \(\Phi\) induces a continuous surjection \(\Psi : \text{Prim} C^*(B) \to \text{Prim}* B\) (see e.g. [7, Corollary 10.5.7]).

Definition 1.1 ([7, 10.5.8]). A Banach *-algebra \(B\) is said to be *-regular if the canonical map \(\Psi : \text{Prim} C^*(B) \to \text{Prim}* B\) is a homeomorphism.

Remark 1.2. \(B\) is *-regular if and only if \(B/^*\text{rad}(B)\) is *-regular (where \(^*\text{rad}(B)\) is the *-radical of \(B\)).

We now recall the following result from [2]. As we are in a more general setting, we repeat their argument here for clarity.

Received by the editors June 23, 2004 and, in revised form, August 19, 2004 and October 13, 2004.

2000 Mathematics Subject Classification. Primary 47A60, 32A65.
Key words and phrases. Banach algebras, functional calculus, *-regular.
This work was jointly supported by Hong Kong RGC Direct Grant and the National Natural Science Foundation of China (10371058).
Proposition 1.3 ([2 Satz 1]). A Banach $*$-algebra $B$ is $*$-regular if and only if for any $*$-representations $\pi$ and $\rho$ of $B$, the inclusion $\ker \pi \subseteq \ker \rho$ will imply that $\|\rho(b)\| \leq \|\pi(b)\|$ for all $b \in B$.

Proof. Suppose that $B$ is $*$-regular. Let $E := \{ P \in \text{Prim } C^*(B) : \ker \pi_* \subseteq P \}$.

As $\Phi(\ker \pi) = \ker \pi_* \cap \Phi(B)$, we see that $\cap_{P \in E} \Phi(\Psi(P)) = \cap_{P \in E} P \cap \Phi(B) = \Phi(\ker \pi)$. Let $I \in \text{Prim } C^*(B)$ such that $\ker \rho_* \subseteq I$. Then

$$\bigcap_{P \in E} \Phi(\Psi(P)) \subseteq \Phi(\ker \rho) \subseteq I \cap \Phi(B) = \Phi(\Psi(I)),$$

which implies that $\bigcap_{Q \in \Psi(E)} Q \subseteq \Psi(I)$ (as $\ker \Phi \subseteq J$ for any $J \in \text{Prim}_* B$). Since $\Psi$ is a homeomorphism and $E$ is closed, $\Psi(I) \in \Phi(E)$ and $I \in E$. This shows that $\ker \pi_* \subseteq \ker \rho_*$ and so $\|\rho_*(x)\| \leq \|\pi_*(x)\|$ for any $x \in C^*(B)$. Conversely, the hypothesis clearly implies that $\Psi$ is injective (as $\ker \pi = \ker \rho$ will then imply $\|\rho_*(x)\| = \|\pi_*(x)\|$ for any $x \in C^*(B)$). Let $E \subseteq \text{Prim } C^*(B)$ be a closed subset. For any $\ker \tau \in \text{hull}(\ker \Psi(E))$, we have

$$\ker \tau \supseteq \bigcap_{\ker \sigma_* \in E} \ker \sigma = \ker(\bigoplus_{\ker \sigma_* \in E} \sigma).$$

Therefore, by the hypothesis, $\|\tau_*(x)\| \leq \sup\{\|\sigma_*(x)\| : \ker \sigma_* \in E\} ~ (x \in C^*(B))$. Hence $\ker \tau_* \in E$ (as $E$ is closed) and $\ker \tau \in \Psi(E)$.

Corollary 1.4. Let $B$ be a Banach $*$-algebra.

(a) Suppose that there is a dense subset $B_0$ of $B_{sa}$ such that for any $b \in B_0$ and any smooth function $\varphi: \mathbb{R}_+ \to \mathbb{C}$ with compact support and $\varphi(0) = 0$, there exists $c \in B$ with $\mu(c) = \varphi(\mu(b^2))$ for any $*$-representation $\mu$ of $B$. Then $B$ is $*$-regular.

(b) Suppose that $\{ B_i \}_{i \in I}$ is a directed family of Banach $*$-subalgebras of $B$ (i.e. $B_i \subseteq B_j$ if $i \leq j$) such that $\bigcup_{i \in I} B_i$ is dense in $B$. If all $B_i$ are $*$-regular, then so is $B$.

Proof. (a) Let $(\pi, H_\pi)$ and $(\rho, H_\rho)$ be two $*$-representations of $B$ such that $\ker \pi \subseteq \ker \rho$. Assume that there exists $a \in B$ such that $\|\pi(a)\| < \|\rho(a)\|$. By the density of $B_0$ and the $C^*$-identity, there exists $b \in B_0$ such that $\|\pi(b^2)\| < \|\rho(b^2)\|$. Consider a smooth function $\varphi: \mathbb{R}_+ \to \mathbb{C}$ with compact support such that $\varphi(\mathbb{R}_+) \subseteq [0, 1]$,

$$\varphi([0, \|\pi(b^2)\|]) = \{0\} \quad \text{and} \quad \varphi(\|\rho(b^2)\|) = 1.$$

Let $c \in B$ be the element given by the hypothesis. Then $\pi(c) = \varphi(\pi(b^2)) \in \mathcal{L}(H_\pi)_+$. From the equalities:

$$\sigma(\pi(c)) = \varphi(\sigma(\pi(b^2))) = \{0\},$$

we see that $c \in \ker \pi$, where $\sigma(x)$ denotes the spectrum of $x$. However, $\sigma(\rho(c)) = \varphi(\sigma(\rho(b^2))) \neq \{0\}$, which gives the contradiction that $c \notin \ker \rho$.

(b) Suppose that $\pi$ and $\rho$ are two $*$-representations of $B$ such that $\ker \pi \subseteq \ker \rho$. It is clear that for any $i \in I$, one has

$$\ker (\pi|_{B_i}) \subseteq \ker (\rho|_{B_i})$$

(where $\pi|_{B_i}$ is the non-degenerated part of the restriction of $\pi$ on $B_i$ and so is $\rho|_{B_i}$). Therefore, by Proposition 1.3 $\|\rho(b)\| \leq \|\pi(b)\|$ ($b \in B_i$). Now, the result follows from the density of $\bigcup_{i \in I} B_i$ as well as Proposition 1.3. 

\[ \square \]
Remark 1.5. (a) Corollary [14, a] is the argument in [2, Satz 2]. Note that we do not assume that the spectrum of $b^2$ is in $\mathbb{R}_+$.

(b) Recall that $b \in B$ is said to have slow growth if there exists $k \in \mathbb{N}$ such that $\|e^{ibt}\| = O(|t|^k)$ for $t \in \mathbb{R}$ (see [1]).

It was proved in [2] that $L^1(G)$ is a $*$-regular Banach $*$-algebra if $G$ is a polynomial growth group. The main step in their proof is the observation that if $G$ is of polynomial growth, then by the argument of a main result in [3], any self-adjoint element in $K(G)$ (the space of continuous functions with compact supports) has slow growth. It is natural to ask if a similar thing holds for a $C^*$-dynamical system $(A, G, \alpha)$. In this paper, we will consider two particular cases: the case when $A$ is finite dimensional and the case when $G$ is discrete.

Notation 1.6. Throughout this paper, $(A, G, \alpha)$ is a $C^*$-dynamical system. For any $f, g \in K(G; A)$ (continuous maps from $G$ to $A$ with compact supports) and $s \in G$, we define

$$(f \circ g)(s) := \int_G \alpha_t(f(st))g(t^{-1}) \, dt \quad \text{and} \quad f^\#(s) := \Delta(s^{-1})\alpha_s^{-1}(f(s^{-1})^*)$$

(see [3, 2.3]).

If $\Theta$ is a map from $K(G; A)$ to itself defined by $\Theta(f)(t) = \alpha_t^{-1}(f(t))$, then we have $\Theta(f \circ g) = \Theta(f) \circ \Theta(g)$ and $\Theta(f^\#) = \Theta(f)^\#$ (where $f \circ g(s) := \int_G f(r)\alpha_r(g(r^{-1}s)) \, dr$ and $f^\#(t) := \Delta(t)^{-1}\alpha_t(f(t^{-1})^*)$). Moreover, any covariant representation $(\pi, u)$ of $(A, G, \alpha)$, i.e.

$$\pi(\alpha_t(a)) = u_t \pi(a) u_{t^{-1}},$$

induces a $*$-representation $u \star \pi$ of $(K(G, A), \star, ^\#)$ which is defined by

$$u \star \pi(f)(\xi) = \int_G u_t \pi(f(t))\xi \, dt.$$

2. The finite dimensional $C^*$-algebra case

Throughout this section, we assume that in the $C^*$-dynamical system $(A, G, \alpha)$, $A$ is a finite dimensional $C^*$-algebra with the $C^*$-norm $\| \cdot \|_A$. Suppose that $A = \bigoplus_{k=1}^n M_{m_k}$. Then $\text{Tr} := \frac{1}{n} \sum_{k=1}^n \text{Tr}_{m_k}$ is a normalised trace on $A$ (where $\text{Tr}_{m_k}$ is the normalised trace on the component $M_{m_k}$). Since any automorphism of $A$ is the composition of an inner automorphism with a swapping of components of $A$ that have the same dimensions (i.e. $m_k$), $\text{Tr}$ is $\alpha$-invariant (i.e. $\text{Tr}(\alpha_s(a)) = \text{Tr}(a)$ for any $a \in A$ and $s \in G$).

Remark 2.1. (a) Suppose that $(H_T, \rho)$ is the GNS representation corresponding to $\text{Tr}$. Let $\mathcal{H}$ be the Hilbert space $L^2(G) \otimes H_T$. If we regard $A \subseteq H_T$ and $K(G; A) \subseteq L^2(G) \otimes H_T$, then for any $f, g \in K(G; A)$, we have

$$(f, g)_{\mathcal{H}} = \int_G \text{Tr}(f(t)^* g(t)) \, dt$$

(note that we use the convention that the inner product is anti-linear in the first variable).

(b) Let $f$ be a measurable map from $G$ to $A$ (i.e., there exists a sequence of measurable simple maps that converges to $f$ almost everywhere). As usual, we
define, for $1 \leq p < \infty$,
\[
\|f\|_p := \left( \int_G \|f(t)\|_A^p \, dt \right)^{1/p}
\]
and
\[
\|f\|_\infty := \inf \left\{ \sup_{t \in \Delta} \|f(t)\|_A : \Delta \subseteq G; \mu_G(G \setminus \Delta) = 0 \right\}
\]
where $\mu_G$ is the Haar measure on $G$. Let $L^p(G; A) = \{ f : G \to A \mid f$ is measurable and $\|f\|_p < \infty \}$ (strictly speaking, we identify two such maps if they coincide almost everywhere). It is well known that $(L^1(G; A), \ast, \#)$ is a Banach $*$-algebra under this norm. Moreover, since $A$ is finite dimensional, $\| \cdot \|_2$ is equivalent to the norm $\| \cdot \|_2$ on $K(G; A)$ and so, $L^2(G; A) \cong \mathcal{H}$ (as Banach spaces).

(c) Consider the Hilbert $A$-module $L^2(G) \otimes A$ with the $A$-inner product
\[
(\phi \otimes a, \psi \otimes b)_A := \left( \int_G \phi(t)^* \psi(t) \, dt \right) a^* b
\]
for $\phi, \psi \in L^2(G); a, b \in A$. Then the canonical $*$-homomorphism $\mu : L^1(G; A) \to L_A(L^2(G) \otimes A)$ induces an injective $*$-representation $T : L^1(G; A) \to \mathcal{L}(\mathcal{H})$ (note that $\mathcal{H} = (L^2(G) \otimes A) \otimes_\rho H_T$).

**Lemma 2.2.** (a) If $f \in K(G; A)$, then $\sum_{k=1}^\infty \frac{(if)^k}{k!}$ converges to $u(f)$ in $L^1(G; A) \cap C_0(G; A)$ (and so we can regard $u(f)$ as an element in $L^2(G; A) \cong \mathcal{H}$).

(b) Suppose that $G$ is unimodular. If $g, h \in L^1(G; A) \cap L^\infty(G; A)$ such that $T(g)^* T(g) \leq T(h)^* T(h)$, then $\|g\|_\mathcal{H} \leq \|h\|_\mathcal{H}$.

**Proof.** (a) For any $k, l \in K(G; A)$, we have $\|k \ast l\|_\infty \leq \|k\|_1 \|l\|_\infty$. Therefore,
\[
\sum_{n=1}^\infty \frac{\|(if)^n\|_\infty}{n!} \leq \sum_{n=0}^\infty \frac{\|f\|^n}{(n+1)!} \leq e\|f\|_1 \|f\|_\infty < \infty
\]
and $u(f) \in C_0(G; A)$.

(b) By the assumption, for all $\xi \in K(G; A) \subseteq \mathcal{H}$,
\[
(2.1) \quad \|g \ast \xi\|_\mathcal{H}^2 = (T(g)^* T(g)\xi, \xi)_\mathcal{H} \leq (T(h)^* T(h)\xi, \xi)_\mathcal{H} = \|h \ast \xi\|_\mathcal{H}^2.
\]
If $(f_j)_{j \in I}$ is a net in $L^1(G; A) \cap L^\infty(G; A)$ such that $\|f_j\|_1 \to 0$ and there exists $\kappa \in \mathbb{R}_+$ with $\|f_j\|_\infty < \kappa$ (for all $j \in I$), then
\[
(2.2) \quad \|f_j\|_\mathcal{H}^2 = \int_G \text{Tr} [f_j(t)^* f_j(t)] \, dt \leq \kappa \|\text{Tr}\| \int_G \|f_j(t)\|_A \, dt.
\]
Now suppose that $(\xi_i) \subseteq K(G; A)$ is a contractive approximate identity for $L^1(G; A)$. Notice that as $G$ is unimodular, $\|k \ast l\|_\infty \leq \|k\|_\infty \|l\|_1$ for any $k, l \in K(G; A)$. Thus $\|g \ast \xi_i - g\|_\infty \leq 2\|g\|_\infty$ and $\|g \ast \xi_i - g\|_1 \to 0$. Therefore, inequality (2.2) implies that $\|g \ast \xi_i - g\|_\mathcal{H} \to 0$ and the same is true for $h$. Now, the required inequality follows from (2.1). \qed

**Proposition 2.3.** Suppose that $G$ is of polynomial growth and $f \in K(G; A)$ with $f^\# = f$. Then $f$ has slow growth.

**Proof.** By Lemma 2.2 [3 Lemme 4] and Remark 2.1 (b), we see that $\|u(f)\|_2 \leq C_0 \|f\|_2$ for some constant $C_0 > 0$. Now, the same argument as that in [3] Lemme 6] will imply the result. \qed
Using the above proposition and the argument in [3] Lemme 7, the “smooth functional calculus” can be defined for any \( f \in K(G; A) \) in such a way that the hypothesis of Corollary \([4]\) holds. Since \( K(G; A) \) is dense in \( L^1(G; A) \), we see that \( L^1(G; A) \) is \( * \)-regular. This gives the following generalisations of [2] Satz 2. Note that part (c) is also a partial generalisation of [3] Remark 1 (i.e. \( L^1(G; A) \) is \( * \)-regular if \( G \) is abelian).

**Theorem 2.4.** Suppose that \( G \) is a polynomial growth group, \( A \) is a finite dimensional \( C^* \)-algebra and \( \alpha \) is an action of \( G \) on \( A \). Let \( f \in K(G; A) \) with \( f = f^\# \) and let \( \varphi \) be a smooth and integrable complex function on \( \mathbb{R} \).

(a) \( \varphi\{f\} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(r)e_{ir}f \, dr \) exists in the unitalisation, \( L^1(G; A)_\mathbb{C} \), of \( L^1(G; A) \) (where \( \hat{\varphi} \) is the Fourier transform of \( \varphi \)) and \( \varphi\{f\} \in L^1(G; A) \) if \( \varphi(0) = 0 \).

(b) For any covariant representation \((\nu, \nu) \) of \((A, G, \alpha)\), we have \( \nu \ast \nu(\varphi\{f\}) = \varphi(\nu \ast \nu(f)) \).

(c) \( L^1(G; A) \) is \( * \)-regular.

### 3. The discrete group case

In this section, we will consider the case when \( G \) is a discrete group (but \( A \) is a general \( C^* \)-algebra). The absence of a bounded trace that gives an equivalent norm on \( A \) makes the situation much more complicated.

Let us start with the easy case when \( G \) is a “locally finite group”. We recall the well-known fact that if \( G \) is finite, then \( K(G; A) = l^1(G; A) = A \times \alpha G \) (note that if \( \{f_n\} \) is a sequence in \( K(G; A) \) converging to an element in \( A \times \alpha G \), then \( \{f_n(t)\} \) is Cauchy for any \( t \in G \) and so \( \{f_n\} \) converges in \( K(G; A) \)).

**Proposition 3.1.** Let \( G \) be the inductive limit of a system of finite groups \( \{G_i\}_{i \in I} \) and let \( \alpha \) be an action of \( G \) on a \( C^* \)-algebra \( A \). Then for any \( f \in K(G; A) \) with \( f = f^\# \) and any smooth and integrable complex function \( \varphi \) on \( \mathbb{R} \), \( \varphi\{f\} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(r)e_{ir}f \, dr \) exists in the \( K(G; A)_\mathbb{C} \) and \( l^1(G; A) \) is \( * \)-regular.

**Proof.** Note that \( K(G_i; A) = A \times \alpha_i G_i \) is \( * \)-regular (\( \alpha_i \) being the restriction of \( \alpha \) on \( G_i \)). Moreover, it is easy to see that \( \bigcup_{i \in I} l^1(G_i) \) is dense in \( l^1(G) \). Since \( l^1(G; A) = l^1(G) \otimes^\pi A \) (where \( \otimes^\pi \) is the projective tensor product), it is easy to see that \( \bigcup_{i \in I} l^1(G_i; A) \) is dense in \( l^1(G; A) \). Now this result follows from Corollary \([1, 4]\) b). The existence of the functional calculus follows from the fact that for any \( f \in K(G; A) \) there exists \( i \in I \) such that \( f \in K(G_i; A) = A \times \alpha_i G_i \). \( \square \)

Next, we consider the case when \( G \) is discrete and has polynomial growth. As in the previous section, we want to use a similar argument to that of [3]. In order to do this, we need to replace \( \{\| \cdot \|_p\}_{p \in [1, \infty]} \) by another series of norms \( \{n_{\pi, p}\}_{p \in [1, \infty]} \) such that \( n_{\pi, 2} \) is the one given by the Hilbert \( C^* \)-module \( l^2(G) \otimes A \).

**Remark 3.2.** Suppose that \( \pi \) is a representation of \( A \) on \( H \). For any \( f : G \to A \) and any \( 1 \leq p < \infty \), we define

\[
n_{\pi, p}(f) = \sup \left\{ \left( \sum_{t \in G} \|\pi(f(t))\xi\|^p \right)^{1/p} : \xi \in H \text{ and } \|\xi\| \leq 1 \right\}.
\]

(a) It is clear that \( n_{\pi, p} \) is a semi-norm on \( K(G; A) \) and if \( \pi \) is faithful, then \( n_{\pi, p} \) is a norm on \( K(G; A) \). Moreover, \( n_{\pi, p}(f) \leq \|f\|_p \).
(b) Suppose that \( f \in K(G; A) \). Then
\[
\sum_{t \in G} \|\pi(f(t))\xi\|^2 = \langle \pi(\sum_{t \in G} f(t)^* f(t)) \xi, \xi \rangle.
\]
Hence if \( \pi \) is faithful, then \( n_{\pi,2}^2(f) = \|\sum_{t \in G} f(t)^* f(t)\| \) and \( n_{\pi,2} \) is the norm on \( K(G; A) \) induced from the Hilbert \( A \)-module \( l^2(G) \otimes A \).

(c) Suppose that \( \xi \in H \) and \( f, g \in K(G; A) \). Then
\[
(3.1) \quad \sum_{s \in G} \|\pi(f * g(s))\xi\| \leq \sum_{v \in G} \sum_{u \in G} \|\pi(\alpha_{v-1}(u)g(v))\xi\| \leq \sum_{v \in G} \sum_{u \in G} \|f(u)\| \|\pi(g(v))\xi\| \leq \|f\|_1 \ n_{\pi,1}(g) \ \|\xi\|
\]
and so, \( n_{\pi,1}(f * g) \leq \|f\|_1 \ n_{\pi,1}(g) \).

(d) In general, it may not be true that \( n_{\pi,p}(f) = n_{\pi,p}(f^\#) \).

**Proposition 3.3.** Suppose that \( A \) is finite dimensional and \( \pi \) is any faithful representation of \( A \). Then \( n_{\pi,1} \sim \| \cdot \|_1 \).

**Proof.** Let \( A = \bigoplus_{k=1}^N M_{m_k}(\mathbb{C}) \). For any \( a = ((a_{ij}^{(1)}), \ldots, (a_{ij}^{(N)})) \in A \), we define
\[
\|a\|_s = \sum_{k=1}^N \sum_{i,j=1}^{m_k} |a_{ij}^{(k)}|.
\]
Since \( \| \cdot \|_s \) is equivalent to the \( C^* \)-norm \( \| \cdot \|_A \), on \( A \), there exists a \( \kappa > 0 \) such that \( \|a\|_A \leq \kappa \|a\|_s \) (\( a \in A \)). Therefore, if \( f(t) = ((f(t)_{ij}^{(1)}), \ldots, (f(t)_{ij}^{(N)})) \in A \), then
\[
\|f\|_1 \leq \kappa \sum_{t \in G} \sum_{i=1}^N \sum_{j=1}^{m_k} |f(t)_{ij}^{(k)}|.
\]
Let \( \pi^{(k)} \) be the representation defined by \( \pi^{(k)}(a_{ij}^{(1)}, \ldots, a_{ij}^{(N)}) = \pi(a_i^{(1)}, \ldots, a_i^{(N)}, 0, \ldots, 0) \). There exists \( \xi_i \in H \) such that \( \|\xi_i\| \leq 1 \) and \( |f(t)_{ij}^{(k)}| \leq |\langle \pi^{(k)}(f(t)) \xi_j, \xi_i \rangle| \) (\( t \in G \)). Thus, for fixed \( i \) and \( j \),
\[
\sum_{t \in G} |f(t)_{ij}^{(k)}| \leq \sum_{t \in G} \|\pi^{(k)}(f(t))\xi_j\| \leq n_{\pi,1}(f).
\]
Consequently, \( n_{\pi,1}(f) \leq \|f\|_1 \leq \kappa \left( \sum_{k=1}^N m_k^2 \right) n_{\pi,1}(f). \]

**Example 3.4.** Suppose that \( G = \mathbb{Z} \) and \( A = K(l^2(\mathbb{Z})) \). Let \( \{e_k\}_{k \in \mathbb{Z}} \) be the canonical basis for \( l^2(\mathbb{Z}) \) and let \( p^{(k)} \in A \) be defined by \( p^{(k)}(e_l) = \delta_{k,l} e_k \). Fix \( m \in \mathbb{N} \). Define \( f_m \in K(G; A) \) by
\[
f_m(k) = \begin{cases} p^{(k)} & \text{if } |k| \leq m, \\ 0 & \text{otherwise}. \end{cases}
\]
Then clearly $\|f_m\|_1 = 2m + 1$. However, if $\pi$ is the canonical representation of $A$ on $l^2(\mathbb{Z})$, then
\[
n_{\pi,1}(f_m) = \sup \left\{ \sum_{k=-m}^{m} \|\pi(f_m(k))\xi\| : \xi \in l^2(\mathbb{Z}); \|\xi\| \leq 1 \right\}
= \sup \left\{ \sum_{k=-m}^{m} |\xi_k| : (\xi_i) \in l^2(\mathbb{Z}); \sum_{k \in \mathbb{Z}} |\xi_i|^2 \leq 1 \right\} = \sqrt{2m + 1}.
\]
Therefore, $n_{\pi,1}$ is not equivalent to $\| \cdot \|_1$.

**Proposition 3.5.** Suppose that $(\pi, u, H)$ is a covariant representation of $(A, G, \alpha)$. Then $\|f\|_{\pi,1} = \max\{n_{\pi,1}(f), n_{\pi,1}(f^\#)\}$ is an involutive algebra semi-norm on $(K(G,A), \star, \#)$. If $l^1_{\pi}(G; A)$ is the completion of the quotient of $K(G; A)$ under $\| \cdot \|_{\pi,1}$, then there exists a contractive Banach $\star$-algebra homomorphism $\epsilon_{\pi}$ from $l^1(G; A)$ to $l^1_{\pi}(G; A)$. Moreover, there exists a contractive representation $\mu_{\pi,u}$ of $l^1_{\pi}(G; A)$ on $H$ such that $\mu_{\pi,u} \circ \epsilon_{\pi} = u \star \pi$.

**Proof.** For any $g \in K(G; A)$ and $\xi \in H$, we have
\[
\sum_{s \in G} \|\pi(f \ast g(s))\xi\| \leq \sum_{r \in G} \sum_{t \in G} \|\pi(\alpha_{r^{-1}}(f(t)))\pi(g(r))\xi\| = \sum_{r \in G} \sum_{t \in G} \|\pi(f(t))u_r\pi(g(r))\xi\|
\leq \sum_{r \in G} n_{\pi,1}(f) \|u_r\pi(g(r))\xi\| = n_{\pi,1}(f) n_{\pi,1}(g) \|\xi\|.
\]
Thus, $n_{\pi,1}$ is an algebra semi-norm on $(K(G; A), \star)$ and $\| \cdot \|_{\pi,1}$ is an involutive algebra semi-norm on $(K(G; A), \star, \#)$. The second statement of the proposition follows from Remark 3.6(a). Finally, as $\|u \star \pi(f)\| \leq n_{\pi,1}(f)$, the third statement is easy to obtain. \hfill \Box

**Remark 3.6.** (a) We can regard $l^1_{\pi}(G; A)$ as the completion of the quotient of $l^1(G; A)$ with respect to $\| \cdot \|_{\pi,1}$. In this case, the completion of the quotient of $l^1(G; A)$ \wedge with respect to $\| \cdot \|_{\pi,1}$ coincides with $l^1_{\pi}(G; A)$.

(b) Suppose that $\pi$ is faithful. Then any $x \in l^1_{\pi}(G; A)$ defines a map $f : G \to A$ such that
\[
\sup \left\{ \sum_{t \in G} \|\pi(f(t))\xi\| : \xi \in H \text{ with } \|\xi\| \leq 1 \right\} < \infty
\]
(because any sequence in $K(G; A)$ converging to $x$ will converge pointwisely and the pointwise limits of any two such sequences are the same). Thus, $\epsilon_{\pi} : l^1(G; A) \to l^1_{\pi}(G; A)$ (Proposition 3.5) is injective. Moreover, $\epsilon_{\pi}$ extends to an injection from $l^1(G; A) \wedge$ to $l^1_{\pi}(G; A)$.

(c) Let $G = \mathbb{Z}$ and $A = K(l^2)$, and let $\alpha$ be the trivial action. If $\pi$ is the canonical representation of $A$ and $u$ is the trivial representation of $G$ respectively on $l^2$, then $(\pi, u, l^2)$ is a covariant representation. Suppose that $\epsilon_{\pi}$ is surjective. Then the Open Mapping theorem will imply that $\| \cdot \|_{\pi,1}$ is equivalent to $\| \cdot \|_1$, which contradicts the conclusion of Example 3.4 (note that $\| \cdot \|_{\pi,1} = n_{\pi,1}$ in this case). Therefore, $\epsilon_{\pi}$ is in general not surjective.
Lemma 3.7. Suppose that $G$ is a polynomial growth discrete group and $(\pi, H)$ is a representation of $A$. If $f \in K(G; A)$ and $f = f^\#$, then $\epsilon_\pi(f)$ has slow growth.

Proof. Let $u(f)$ be as in Lemma 2.2(a) and let $\mu : l^1(G; A) \to A \times_\alpha G$ be the canonical $*$-homomorphism. Since $\mu(u(f)) = u(\mu(f))$, we have

$$\mu(u(f)^\# * u(f)) \leq \mu(f * f)$$

(by an analogue of [3, Lemme 4] for $C^*$-algebras). If $E$ is the canonical conditional expectation from $A \times_\alpha G$ to $A$ (see e.g. [4]), then $E(\mu(g)) = g(e)$ for any $g \in l^1(G; A)$. Therefore, using (3.2),

$$\sum_{t \in G} u(f)(t)^* u(f)(t) = (u(f)^\# * u(f))(e)
= E(\mu(u(f)^\# * u(f))) \leq E(\mu(f * f)) = \sum_{t \in G} f(t)^* f(t).$$

Consequently, for any $\xi \in H$ with $\|\xi\| \leq 1$, we have

$$\sum_{t \in G} \|\pi(u(f)(t))\xi\|^2 = \left\langle \pi \left( \sum_{t \in G} u(f)(t)^* u(f)(t) \right) \xi, \xi \right\rangle \leq \left\| \sum_{t \in G} u(f)(t)^* u(f)(t) \right\|
\leq \left\| \sum_{t \in G} f(t)^* f(t) \right\| \leq n_{\pi, 2}(f)^2$$

(see Remark 3.2(b)). Thus, $n_{\pi, 2}(u(f)) \leq n_{\pi, 2}(f)$. Let $S$ be the support of $f$. As $G$ has polynomial growth, there exists $N \in \mathbb{N}$ such that $|S^m| = O(m^N)$. Now, a similar argument to that of [3 Lemme 6] will give the result, but since we are in a slightly different setting, we will sketch the proof here for clarity. If $m \in \mathbb{N}$ and $\xi \in H$ with $\|\xi\| \leq 1$, then

$$\sum_{t \in S^{m^2-1}} \|\pi(u(mf)(t))\xi\|^2 \leq \left( \sum_{t \in S^{m^2-1}} \|\pi(u(mf)(t))\xi\|^2 \right)^{1/2} \|S^{m^2-1}\|^{1/2}
\leq n_{\pi, 2}(u(mf)) |S^{m^2-1}|^{1/2} \leq m n_{\pi, 2}(f) |S^{m^2-1}|^{1/2}
\leq C_1 m^{N+1}$$

($C_1$ is independent of $\xi$ and $m$). On the other hand, the same argument as that for [3 Lemme 6] shows that

$$\sum_{t \in G \setminus S^{m^2-1}} \|\pi(u(mf)(t))\xi\| \leq \sum_{t \in G \setminus S^{m^2-1}} \left\| \sum_{k=m^2}^{\infty} \frac{(imf)^k}{k!} (t) \right\|
\leq C_2 m^{-m^2-1} e^{m^2 + m}$$

($C_2$ is independent of $\xi$ and $m$). Equations (3.3) and (3.4) imply that $n_{\pi, 1}(u(mf)) = O(m^{N+1})$. Finally, let $[\lambda]$ be the integral part of $\lambda$. Then we have

$$n_{\pi, 1}(e^{i\lambda f}) = n_{\pi, 1}(e^{i(\lambda-[\lambda]) f} e^{i[\lambda] f}) \leq e^{\|f\|_1 (1 + n_{\pi, 1}(u(i[\lambda] f)))} = O(|\lambda|^{N+1}).$$

Again, using the argument of [3 Lemme 7] and Lemma 1.4(a), we have the following theorem.
Theorem 3.8. Suppose that $G$ is a discrete polynomial growth group. Let $(A, G, \alpha)$ be a $C^*$-dynamical system and let $(\pi, u)$ be a covariant representation of $(A, G, \alpha)$. Let $f \in K(G; A)$ with $f = f^\#$ and let $\varphi$ be a smooth and integrable complex function on $\mathbb{R}$.

(a) The Bochner integral $\varphi_\pi \{f\} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\lambda) e^{i\lambda f} d\lambda$ exists in $\ell_1^\pi(G; A)^*$ and $\varphi_\pi \{f\} \in \ell_1^\pi(G; A)$ if $\varphi(0) = 0$.

(b) Suppose that $u \ast \pi$ extends to a faithful representation of $A \times_\alpha G$. Then one can regard $\mu_{\pi, u}$ as a $*$-homomorphism from $\ell_1^\pi(G; A)^*$ to $A \times_\alpha G$. Under this identification, for any covariant representation $(\nu, v)$ of $(A, G, \alpha)$, we have $(v \ast v)(\mu_{\pi, u}(\varphi_\pi \{f\})) = \varphi((v \ast v)(f))$.

(c) Suppose that $u \ast \pi$ is faithful on $A \times_\alpha G$. Then $\ell_1^\pi(G; A)$ is $*$-regular.

Remark 3.9. (a) Suppose that $u \ast \pi$ extends to a faithful representation of $A \times_\alpha G$. By Theorem 3.8, any non-degenerate $*$-representation of $A \times_\alpha G$ induces (through the map $\mu_{\pi, u}$) a non-degenerate $*$-representation of $\ell_1^\pi(G; A)$. On the other hand, any non-degenerate $*$-representation of $\ell_1^\pi(G; A)$ induces (through the map $\epsilon_\pi$ in Proposition 3.3) a non-degenerate $*$-representation of $\ell_1^G(G; A)$. Since $\mu_{\pi, u} \circ \epsilon_\pi$ is the canonical embedding of $\ell_1^G(G; A)$ in $A \times_\alpha G$, we see that the enveloping $C^*$-algebra of $\ell_1^G(G; A)$ is again $A \times_\alpha G$.

(b) By Proposition 3.3, Theorem 3.8 can be regarded as a partial generalization of Theorem 2.4.

References


Department of Mathematics, The Chinese University of Hong Kong, Hong Kong
E-mail address: cyleung@math.cuhk.edu.hk
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People’s Republic of China
E-mail address: ckng@nankai.edu.cn