MAPS INTO COMPLEX SPACE

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Abstract. If the dimension of \( M \) is denoted by \( 2k - 1 \) or \( 2k \), then a generic map \( F : M \to C^k \) satisfies \( dF_1 \wedge \ldots \wedge dF_k \neq 0 \), while in certain cases there is no map \( F : M \to C^{k+1} \) that satisfies \( dF_1 \wedge \ldots \wedge dF_{k+1} \neq 0 \).

1. Statement of results

Let \( F : M \to R^2 \) be any map. The components of \( F \) may be approximated by Morse functions. From this, we see that the rank of the Jacobian of a generic map of \( M \to R^2 \) is everywhere greater than zero. This can be reformulated as: A generic map \( F : M \to C^1 \) has, at all points, \( dF \neq 0 \). This suggests the question: For a given value of \( n \), what is the largest value of \( r \) for which every manifold of dimension \( n \) has a map \( F = (F_1, \ldots, F_r) \) into \( C^r \) with \( dF_1 \wedge \ldots \wedge dF_r \) never zero?

Given \( n \), set \( r = \left\lfloor \frac{n}{2} \right\rfloor \), the smallest integer greater than or equal to \( \frac{n}{2} \).

Theorem 1.1. For every manifold \( M^n \) of dimension \( n \) there is a generic set of maps \( F : M^n \to C^r \) with \( dF_1 \wedge \ldots \wedge dF_r \) never equal to zero.

By a generic set of functions or maps on a compact manifold we mean a set that is open in the \( C^1 \) topology and dense in the \( C^\infty \) topology; on a non-compact manifold we mean the countable intersection of such sets. Note that the theorem states that \( M^n \) can be mapped into \( R^k \), \( k \leq n \) for \( n \) even and \( k \leq n + 1 \) for \( n \) odd, in a special way.

Corollary 1. Every manifold \( M^n \) of dimension \( n \) admits an involutive sub-bundle of rank \( k \) for each \( k \geq \left\lfloor \frac{n}{2} \right\rfloor \).

A bundle \( V \subset CTM \) is involutive if the Lie bracket condition \([V, V] \subset V\) holds for all local sections. This is an interesting restriction, motivated by partial differential equations and generalizing the Frobenius condition for sub-bundles of \( TM \). Here \( \left\lfloor \frac{n}{2} \right\rfloor \) is the largest integer less than or equal to \( \frac{n}{2} \) and so \( n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \). The corollary follows from the theorem by taking the annihilator of \( \{dF_1, \ldots, dF_r\} \).

The value for \( r \) is sharp (for \( n = 4k \)) if we insist that the same \( r \) works for all manifolds of a fixed dimension. The basic idea is that every complex bundle of rank \( N \) over \( M^n \), \( N > \frac{n}{2} \), admits \( N - \left\lfloor \frac{n}{2} \right\rfloor \) independent global sections. The rank of \( CT^*M^n \) is \( n \) and \( dF_1, \ldots, dF_r \) provide \( r \) independent global sections. Thus for \( r > n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \), we have a restriction on the bundle. Here is one simple way to formulate this restriction. Note that we use only the existence of the global

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section and ignore the more refined information that each section is given by a closed one-form.

**Theorem 1.2.** Let \( F : M^{4k} \to C^{2k+1} \) satisfy \( dF_1 \land \ldots \land dF_{2k+1} \) never equal to zero. Then the top Pontryagin class of \( M \) is zero.

Since there exist manifolds \( M^{4k} \) with \( P_k \neq 0 \), \( r = 2k \) is the largest \( r \) that works for all manifolds of dimension \( 4k \).

**Examples.** There exists a map \( F : CP^2 \to C^2 \) with \( dF_1 \land dF_2 \) never equal to zero; any map \( F : CP^2 \to C^3 \) has \( dF_1 \land dF_2 \land dF_3 \) somewhere equal to zero. Whether there exists a map of \( CP^1 \to C^2 \) with \( dF_1 \land dF_2 \neq 0 \) is left open by these theorems. Note that the existence of such a map implies the (true) statement that \( CTS^2 \) is trivial. There does exist such a map into \( C^1 \), and it can be obtained by projecting \( S^2 \) into any plane. Also note that the case \( n = 2 \) is related to symplectic structures. Namely, identify \( R^4 \) with \( C^2 \) by using \( z_1 = x_1 - iy_1 \) and \( z_2 = x_2 + iy_2 \). Then \( dz_1 \land dz_2 = \omega_1 + \omega_2 \) for symplectic forms \( \omega_1 \) and \( \omega_2 \), positive with respect to the orientation given by \((x_1, x_2, y_1, y_2)\). No symplectic structure on \( R^4 \) has a compact symplectic submanifold. So if \( F : S^2 \to C^2 \), then \( F^\ast \omega_1 \) and \( F^\ast \omega_2 \) must have zeroes, and we are asking if \( F \) can be chosen such that they have no common zeroes.

2. **Proof of Theorem 1.1**

We use notation and results from [1], Chapter 2. Let \( X^{(1)} \) be the space of one jets of maps \( F : M^n \to C^r \) written locally as \( \{(p, F(p), dF_1(p), \ldots, dF_r(p))\} \) and let \( S \) be the subset given by \( \{(p, c, \theta) : \theta_1 \land \ldots \land \theta_r = 0\} \).

**Lemma 2.1.** \( S \) is a stratified subset of \( X^{(1)} \) of codimension \( 2(n + 1 - r) \).

To prove this, we need to show

\[
(1) \quad S = \bigcup_{i=0}^{N} S_i,
\]

where each \( S_i \) is a locally closed sub-manifold of \( X^{(1)} \) satisfying

\[
(2) \quad \text{codim} S_0 = 2(n + 1 - r).
\]

**Proof.** Let \( S_i \) be the subset of \( S \) defined by

\[
\{(p, c, \theta) : \theta_1 \land \ldots \land \theta_{r-i} = 0 \text{ and } \theta_1 \land \ldots \land \theta_{r-i-1} \neq 0\}.
\]

So \( N = r - 1 \) and \( S_N = \{(p, c, \theta : \theta_1 = 0\} \). Then the first condition is obvious. For the second, we count the equations that define \( S_0 \) near one of its points, \( (p, c, \theta^0) \). We have that \( (p, c, \theta) \in S_0 \) provided \( \theta_r \in \text{span}\{\theta_1, \ldots, \theta_{r-1}\} \). Expressing \( \theta_r \) as a linear combination with complex coefficients of basis elements \( \{\theta_1, \ldots, \theta_{r-1}, e_1, \ldots, e_{n-r+1}\} \), we see that there are \( n-r+1 \) independent equations. This establishes the second condition.
We now require $2(n+1-r) > n$. Thus the codimension of $S$ in $X^{(1)}$ is greater than the dimension of $M$. It follows from a basic result of differential topology that any map of $M$ into $X^{(1)}$ may be perturbed so as to not intersect $S$. The (simplest case of) the Thom Transversality Theorem is more precise. Any map of $M \to C^r$ may be perturbed to obtain a map of $M \to C^r$ whose one jet does not intersect $S$. Further, maps of this latter kind are generic. Finally, note that $2(n+1-r) > n$ is equivalent to $r \leq \left\lceil \frac{n}{2} \right\rceil$.

3. Proof of Theorem 1.2

Consider a map $F : M^n \to C^r$ with $dF_1 \wedge \ldots \wedge dF_r$ never equal to zero. Let $V \subset CTM$ be the bundle whose fiber at $p$ is given by

$$V = \{ v : dF_j(v) = 0 \text{ at } p, \ j = 1, \ldots, r \}$$

and let $\Omega \subset CT^*M$ be the span of $\{dF_1, \ldots, dF_r\}$. Use any hermitian metric to get a decomposition

$$CTM = V \oplus Q.$$

**Lemma 3.1.** $\Omega \simeq Q^*.$

**Proof.** Recall that $Q^*$ is the set of linear functionals on the fibers of $Q$. Restricting an element $\theta \in \Omega$ to act on $Q$ produces an element $l_\theta$ of $Q^*$. The map $\theta \to l_\theta$ is injective and, since $\Omega$ and $Q^*$ have the same dimension, it is also surjective.

Each $dF_j$ is a global, non-zero section of $\Omega$. Thus $\Omega$, $Q^*$, and $Q$ are all trivial bundles. Thus, for the Chern class of $CTM$ we have, as in [2], page 164,

$$c(CTM) = c(V \oplus Q) = c(V) \wedge c(Q) = c(V).$$

Now let $P_k(TM) \in H^{4k}(M, \mathbb{Z})$ be the top Pontryagin class of $M$. We have from [2], page 174,

$$P_k(TM) = (-1)^k c_{2k}(CTM) = (-1)^k c_{2k}(V).$$

In the case of Theorem 1.2, $\text{rank} V = 4k-(2k+1) = 2k-1 < 2k$ and so $P_k(TM) = 0$.

**References**


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