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ON MIXING AND COMPLETELY MIXING PROPERTIES OF POSITIVE L^1-CONTRACTIONS OF FINITE VON NEUMANN ALGEBRAS

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Abstract. Akcoglu and Suchaston proved the following result: Let $T : L^1(X,\mathcal{F},\mu) \to L^1(X,\mathcal{F},\mu)$ be a positive contraction. Assume that for $z \in L^1(X,\mathcal{F},\mu)$ the sequence $(T^n z)$ converges weakly in $L^1(X,\mathcal{F},\mu)$. Then either $\lim\limits_{n\to\infty} \|T^n z\| = 0$ or there exists a positive function $h \in L^1(X,\mathcal{F},\mu)$, $h \neq 0$ such that $Th = h$. In the paper we prove an extension of this result in a finite von Neumann algebra setting, and as a consequence we obtain that if a positive contraction of a noncommutative L^1-space has no nonzero positive invariant element, then its mixing property implies the completely mixing property.

1. Introduction

It is known (see [17]) that there are several notions of mixing (i.e. weak mixing, mixing, completely mixing, etc.) of measure-preserving transformations on a measure space in ergodic theory. It is important to know how these notions are related with each other. A lot of papers are devoted to this topic. For example, recently, in [5] relations between the notions of weak mixing and weak wandering have been studied.

In this paper we deal with the notions of mixing and completely mixing. Now recall them. Let $(X,\mathcal{F},\mu)$ be a measure space with probability measure μ. Let $L^1(X,\mathcal{F},\mu)$ be the associated L^1-space. A linear operator $T : L^1(X,\mathcal{F},\mu) \to L^1(X,\mathcal{F},\mu)$ is called a *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$. Let $L^1_0 = \{f \in L^1(X,\mathcal{F},\mu) : \int f d\mu = 0\}$. A positive contraction T in $L^1(X,\mathcal{F},\mu)$ is called *mixing* (resp. *completely mixing*) if $T^n f$ tends weakly to 0 for all $f \in L^1(X,\mathcal{F},\mu)$ (resp. $\|T^n f\|$ tends to 0 for all $f \in L^1_0$). A relation between these two notions was given by Krengel and Sucheston (see [18]), which can be formulated as follows (see [17], Ch.8, Thm 1.4):

Theorem 1.1. *Let $T : L^1(X,\mathcal{F},\mu) \to L^1(X,\mathcal{F},\mu)$ be a positive contraction. Assume that there exists no nonzero $y \in L^1(X,\mathcal{F},\mu)$, $y \geq 0$, such that $Ty = y$. If for $z \in L^1(X,\mathcal{F},\mu)$ the sequence $(T^n z)$ converges weakly to some element of*

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$L^1(X, \mathcal{F}, \mu)$, then $\lim_{n\to\infty} \|T^n z\| = 0$. In particular, if T is mixing, then T is completely mixing.

This theorem gives an answer to the problem of whether K-automorphisms of σ-finite measure space are mixing, and shows that, in fact, invertible mixing measure-preserving transformations of σ-finite infinite space do not exist (see [18]) (see for review [17]).

Later in [1] Akcoglu and Sucheston proved an extension of Theorem 1.1, which is formulated as follows.

Theorem 1.2. *Let $T : L^1(X, \mathcal{F}, \mu) \to L^1(X, \mathcal{F}, \mu)$ be a positive contraction. Assume that for $z \in L^1(X, \mathcal{F}, \mu)$ the sequence $(T^n z)$ converges weakly in $L^1(X, \mathcal{F}, \mu)$. Then either $\lim_{n\to\infty} \|T^n z\| = 0$ or there exists a positive function $h \in L^1(X, \mathcal{F}, \mu)$, $h \neq 0$ such that $Th = h$.*

By means of this theorem in [1], [11] an extension of the Blum-Hanson theorem was proved [6], which states that if T is a positive contraction on $L^1(X, \mathcal{F}, \mu)$, then T is mixing if and only if $\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f$ converges strongly in $L^1(X, \mathcal{F}, \mu)$ for every $f \in L^1(X, \mathcal{F}, \mu)$ and an increasing sequence of integers $\{k_n\}$: $0 \leq k_1 < k_2 \cdots$. Other extensions of this result have been given in [4].

The formulated Theorem 1.2 has a lot of applications; here we mention only a few of them. Namely, using it in [8], the existence of an invariant measure for a given positive contraction T on $L^1(X, \mathcal{F}, \mu)$ was proved, and in [25] a criterion of strong asymptotic stability for positive contractions was given by means of Theorem 1.2.

In this paper we are going to extend these results for quantum dynamical systems over von Neumann algebras. Here by quantum dynamical systems we mean a linear, positive, weak continuous mapping α of a von Neumann algebra M, with normal faithful trace τ, into itself. It is known (see [7], sec.4.3) that the theory of quantum dynamical systems provides a convenient mathematical description of the irreversible dynamics of an open quantum system. This motivates an interest in the study of conditions for a dynamical system to induce approach to a stationary state, of reflect subjects such as irreducibility (i.e. ergodicity, mixing) and ergodic theorems (see, for example, [2], [9], [13]). By means of a given dynamical system α, a conjugate dynamical system α_* on a predual M_* of the algebra M can be defined as follows: $(\alpha_*\varphi)(x) = \varphi(\alpha(x))$, where $\varphi \in M_*, x \in M$. It is known [20] that a predual M_* of M is isometrically isomorphic to $L^1(M, \tau)$, which is a noncommutative analog of L^1-space. Note that the study of limiting behaviors of α_* is important since it will provide direction towards a proof of some ergodic type theorems for α. Therefore, in the paper we consider only L^1-contractions of $L^1(M, \tau)$. Note that such dynamical systems were considered in [22], [24].

In the paper we prove an analog of Theorem 1.2 for positive contractions of $L^1(M, \tau)$ associated with a finite von Neumann algebra. We think that this theorem will serve to prove the existence of a stationary state for a given dynamical system. Note that the existence of a stationary state is an actual problem in studying ergodic properties of the quantum dynamical system (see [12], [14]). As a consequence of our main result we infer that the mixing property of positive L^1-contraction implies the completely mixing property; i.e. we prove a noncommutative extension of Theorem 1.1. It should be noted that our results are not valid when the von Neumann algebra is semi-finite.

2. Preliminaries

Throughout the paper M would be a von Neumann algebra with the unit $\mathbb{1}$, and let τ be a faithful normal finite trace on M. Therefore we omit this condition from the formulation of the theorems. Recall that an element $x \in M$ is called *self-adjoint* if $x = x^*$. The set of all self-adjoint elements is denoted by M_{sa}. A self-adjoint element $p \in M$ is called a *projection* if $p^2 = p$. The set of all projections in M we will denote by ∇. By M_* we denote a pre-dual space to M (see for definitions [7], [23]).

The map $\|\cdot\|_1 : M \to [0, \infty)$ defined by the formula $\|x\|_1 = \tau(|x|)$ is a norm (see [20]). The completion of M with respect to the norm $\|\cdot\|_1$ is denoted by $L^1(M,\tau)$. It is known [20] that the spaces $L^1(M,\tau)$ and M_* are isometrically isomorphic; therefore they can be identified. Further we will use this fact without noting it again.

Theorem 2.1 ([20])**.** *The space $L^1(M,\tau)$ coincides with the set*

$$L^1 = \{x = \int_{-\infty}^{\infty} \lambda de_\lambda \ : \int_{-\infty}^{\infty} |\lambda| d\tau(e_\lambda) < \infty\}.$$

Moreover,

$$\|x\|_1 = \int_{-\infty}^{\infty} |\lambda| d\tau(e_\lambda).$$

It is known [20] that the equality

$$L^1(M,\tau) = L^1(M_{sa},\tau) + iL^1(M_{sa},\tau)$$

is valid. Note that $L^1(M_{sa},\tau)$ is a pre-dual to M_{sa}.

Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a linear bounded operator. We say that a linear operator T is *positive* if $Tx \geq 0$ whenever $x \geq 0$. A linear operator T is said to be a *contraction* if $\|T(x)\|_1 \leq \|x\|_1$ for all $x \in L^1(M_{sa},\tau)$.

3. Mixing and completely mixing contractions

Let M be a von Neumann algebra with faithful normal finite trace τ. Let $L^1(M,\tau)$ be an L^1-space. In the sequel by $\|\cdot\|$ we mean the norm $\|\cdot\|_1$.

Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a linear contraction. Define

$$\bar{\rho}(T) = \sup\left\{\lim_{n\to\infty} \frac{\|T^n(u-v)\|}{\|u-v\|} : \quad u, v \in L^1(M_{sa},\tau), u, v \geq 0, \|u\| = \|v\|\right\}. \tag{3.1}$$

If $\bar{\rho}(T) = 0$, then T is called *completely mixing*. Note that certain properties of completely mixing quantum dynamical systems have been studied in [3].

Denote

$$X = \{x \in L^1(M_{sa},\tau) : \ \tau(x) = 0\}.$$

Recall that a positive contraction T is *mixing* if for all $x \in X$ and $y \in M$ the following condition holds:

$$\lim_{n\to\infty} \tau(T^n(x)y) = 0.$$

Let T be a positive contraction of $L^1(M,\tau)$, and let $x \in L^1(M,\tau)$ be such that $x \geq 0$, $x \neq 0$. We say that T is *smoothing* with respect to (w.r.t.) x if for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(pT^n x) < \varepsilon$ for every $p \in \nabla$ such that $\tau(p) < \delta$ and for every $n \geq n_0$. A commutative analog of this notion was introduced in [25], [15].

Theorem 3.1. *Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a positive contraction. Assume that there is a positive element $y \in L^1(M,\tau)$ such that T is smoothing w.r.t. y. Then $\lim_{n\to\infty} \|T^n y\| = 0$ or there is a nonzero positive $z \in L^1(M,\tau)$ such that $Tz = z$.*

Proof. The contractivity of T implies that the limit

$$\lim_{\nu\to\infty} \|T^n y\| = \alpha$$

exists. Assume that $\alpha \neq 0$. Define $\lambda : M_{sa} \to \mathbb{R}$ by

$$\lambda(x) = L((\tau(xT^n y)_{n\in\mathbb{N}}))$$

for every $x \in M_{sa}$; here L means a Banach limit (see [17]). We have

$$\lambda(\mathbb{1}) = L((\tau(T^n x)_{n\in\mathbb{N}})) = \lim_{\nu\to\infty} \|T^n x\| = \alpha \neq 0;$$

therefore $\lambda \neq 0$. Besides, λ is a positive functional, since for any positive element $x \in M_{sa}$, $x \geq 0$, we have

$$\tau(xT^n y) = \tau(x^{1/2} T^n y x^{1/2}) \geq 0,$$

for every $n \in \mathbb{N}$.

For arbitrary $x \in M$, we have $x = x_1 + ix_2$ and we define λ by

$$\lambda(x) = \lambda(x_1) + i\lambda(x_2).$$

Let T^{**} be the second dual of T, i.e. $T^{**} : M^{**} \to M^{**}$. The functional λ is T^{**}-invariant. Indeed,

$$\begin{aligned}(T^{**}\lambda)(x) &= \langle x, T^{**}\lambda\rangle = \langle T^* x, \lambda\rangle \\ = L((\tau(T^n y T^* x)_{n\in\mathbb{N}})) &= L((\tau(xT^{n+1} y)_{n\in\mathbb{N}})) \\ &= L((\tau(xT^n y)_{n\in\mathbb{N}})) = \lambda(z).\end{aligned}$$

Let $\lambda = \lambda_n + \lambda_s$ be the Takesaki decomposition (see [23]) of λ on normal and singular components. Since T is normal and $T^{**}\lambda = \lambda$, so using the idea of [14] it can be proved that $T^{**}\lambda_n = \lambda_n$. Now we will show that λ_n is nonzero. Consider a measure $\mu := \lambda \restriction_\nabla$. It is clear that μ is an additive measure on ∇. Let us prove that it is σ-additive. To this end, it is enough to show that $\mu(p_k) \to 0$ whenever $p_{k+1} \leq p_k$ and $p_k \searrow 0$, $p_k \in \nabla$.

Let $\varepsilon > 0$. From $p_n \searrow 0$ we infer that $\tau(p_n) \to 0$ as $n \to \infty$. It follows that there exists $k_\varepsilon \in \mathbb{N}$ such that $\tau(p_k) < \varepsilon$ for all $k \geq k_\varepsilon$. Since T is smoothing w.r.t. y we obtain

$$\tau(p_k T^n y) < \varepsilon, \quad \forall k \geq k_\varepsilon,$$

for every $n \geq n_0$. From a property of the Banach limit we get

$$\lambda(p_k) = L((\tau(p_k T^n y)_{n\in\mathbb{N}}) < \varepsilon \quad \text{for every} \;\; k \geq k_\varepsilon,$$

which implies $\mu(p_k) \to 0$ as $k \to \infty$. This means that the restriction of λ_n on ∇ coincides with μ. Since

$$\tau(p^\perp T^n y) > \tau(T^n y) - \varepsilon \geq \inf \|T^n y\| - \varepsilon = \alpha - \varepsilon$$

as ε has been arbitrary, so $\alpha - \varepsilon > 0$, and hence $\mu(p^\perp) > 0$ for all $p \in \nabla$ such that $\tau(p) < \delta$. Therefore $\mu \neq 0$, and consequently, $\lambda_n \neq 0$.

From this we infer that there exists a positive element $z \in L^1(M,\tau)$ such that

$$\lambda_n(x) = \tau(zx), \quad \forall x \in M.$$

The last equality and $T^{**}\lambda_n = \lambda_n$ yield that

$$\tau(zx) = \langle x, T^{**}\lambda_n\rangle = \langle T^*x, \lambda_n\rangle = \tau(zT^*x) = \tau(Tzx)$$

for every $x \in M$, which implies that $Tz = z$. $\square$

Using the pre-compactness criterion for a subset of $L^1(M, \tau)$ (see [23]) one can prove the following.

Lemma 3.2. *Let $(x_n) \subset L^1(M, \tau)$, $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Assume that $x_n \to x^*$ weakly. Then for an arbitrary $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(p|x_n|) < \varepsilon$ for every $p \in \nabla$ such that $\tau(p) < \delta$ and $\forall n \geq n_0$.*

Corollary 3.3. *Let $x \in L^1(M, \tau)$, $x \geq 0$. Assume that $T^n x \to x^*$ weakly. Then T is smoothing w.r.t. x.*

Remark 3.1. The proved Theorem 3.1 is a noncommutative analog of Akcoglu and Sucheston's result [1]. But they used weak convergence instead of smoothing. Lemma 3.2 shows that the smoothing condition is less restrictive than the one they used.

Before proving the next theorem let us give the following auxiliary lemma.

Lemma 3.4. *Let $x \in L^1(M, \tau)$. If the inequality*

$$\tau(xy) \geq 0 \tag{3.2}$$

is valid for every $y \geq 0$, $y \in M$, then $x \geq 0$.

Proof. Given $x = x^+ - x^-$, let

$$x = \int_{-\infty}^{\infty} \lambda de_\lambda$$

be the spectral resolution of x. Set

$$p = \int_{-\infty}^{0} de_\lambda.$$

Then according to (3.2) one gets $\tau(xp) \geq 0$. On the other hand we have $xp = -x^-$; hence $\tau(x^-) \leq 0$, and since $x^- \geq 0$ and τ is faithful we infer that $x^- = 0$. Therefore $x = x^+ \geq 0$. $\square$

From Lemma 3.2 and Theorem 3.1 we find the following.

Theorem 3.5. *Let $T : L^1(M, \tau) \to L^1(M, \tau)$ be a positive contraction such that $|T(x)| \leq T(|x|)$ for every $x \in L^1(M, \tau)$, $x = x^*$. Assume that there exits no nonzero $y \in L^1(M, \tau)$, $y \geq 0$, such that $Ty = y$. If for $z \in L^1(M, \tau)$ the sequence $(T^n z)$ converges weakly to some element of $L^1(M, \tau)$, then $\lim_{n\to\infty} \|T^n z\| = 0$. In particular, if T is mixing, then T is completely mixing.*

Proof. As in the proof of Theorem 3.1 we assume that

$$\lim_{n\to\infty} \|T^n z\| = \alpha > 0.$$

Define $\lambda : M_{sa} \to \mathbb{R}$ by

$$\lambda(x) = L((\tau(x|T^n z|)_{n\in\mathbb{N}}))$$

for every $x \in A$. Using the same argument as in the proof of Theorem 3.1 one can show that there exists a nonzero positive element $y \in L^1(M, \tau)$ such that

$$\lambda_n(x) = \tau(yx), \quad \forall x \in M.$$

Here λ_n is the normal part of λ.

From the property of T we infer

$$\begin{aligned} \tau(Tyx) &= \tau(yT^*x) \\ &= L((\tau(|T^n z|T^*x))_{n\in\mathbb{N}}) \\ &= L((\tau(T|T^n z|x))_{n\in\mathbb{N}}) \\ &\geq L((\tau(|T^{n+1} z|x))_{n\in\mathbb{N}}) = \tau(yx) \end{aligned}$$

for all $x \geq 0$. Hence, for every $x \geq 0$ we have

$$\tau((Ty - y)x) \geq 0.$$

According to Lemma 3.4 we infer that $Ty \geq y$. Since T is a contraction one gets $Ty = y$. But this contradicts the assumption of the theorem. □

Remark 3.2. The proved theorem is a noncommutative analog of Theorem 1.1. Certain similar results have been obtained in [19], [10] for quantum dynamical semigroups in von Neumann algebras.

Corollary 3.6. *Let $\alpha : M \to M$ be a normal Jordan automorphism such that there exits no nonzero $y \in L^1(M, \tau)$, $y \geq 0$, such that $\alpha^* y = y$, where α^* is the conjugate operator to α. If for $z \in L^1(M, \tau)$ the sequence $((\alpha^*)^n z)$ converges weakly to some element of $L^1(M, \tau)$, then $\lim\limits_{n\to\infty} \|(\alpha^*)^n z\| = 0$.*

The proof immediately comes from Theorem 3.5 since for Jordan automorphisms the equality $|\alpha(x)| = \alpha(|x|)$ is valid for all $x \in M_{sa}$ (see [7]).

Remark 3.3. Note that an analogous theorem has recently been proved by A. Katz [16] for automorphisms of an arbitrary von Neumann algebra. Corollary 3.6 extends his result to Jordan automorphisms of finite von Neumann algebras. Here it should also be noted that linear mappings of von Neumann algebras which satisfy the condition $|\alpha(x)| = \alpha(|x|)$ have been studied in [21].

Remark 3.4. It should be noted that Theorems 3.1 and 3.5 are not valid if a von Neumann algebra is semi-finite. Indeed, let $B(\ell_2)$ be the algebra of all linear bounded operators on Hilbert space ℓ_2. Let $\{\phi_n\}$, $n \in \mathbb{N}$, be a standard basis of ℓ_2, i.e.

$$\phi_n = (\underbrace{0, \cdots, 0, 1}_{n}, 0 \cdots).$$

The matrix units of $B(\ell_2)$ can be defined by

$$e_{ij}(\xi) = (\xi, \phi_i)\phi_j, \quad \xi \in \ell_2, \ i, j \in \mathbb{N}.$$

A trace on $B(\ell_2)$ is defined by

$$\tau(x) = \sum_{k=1}^{\infty} (x\phi_k, \phi_k).$$

By ℓ_∞ we denote a maximal commutative subalgebra generated by elements $\{e_{ii} : i \in \mathbb{N}\}$. Let $E : B(\ell_2) \to \ell_\infty$ be the canonical conditional expectation (see [23]). Define a map $s : \ell_\infty \to \ell_\infty$ as follows: for every element $a \in \ell_\infty$, $a = \sum\limits_{k=1}^{\infty} a_k e_{kk}$ put

$$s(a) = \sum_{k=1}^{\infty} a_k e_{k+1,k+1}.$$

Define $T : B(\ell_2) \to B(\ell_2)$ as $T(x) = s(E(x))$, $x \in B(\ell_2)$. It is clear that T is positive and $\tau(T(x)) \le \tau(x)$ for every $x \in L^1(B(\ell_2), \tau) \cap B(\ell_2)$, $x \ge 0$. Hence, T is a positive L^1-contraction. But for this T there is no nonzero x such that $Tx = x$. Moreover, for every $y \in L^1(B(\ell_2), \tau)$ we have $\lim\limits_{n\to\infty} \|T^n y\|_1 \ne 0$.

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