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ON THE NUMBER OF DIFFERENT PRIME DIVISORS OF ELEMENT ORDERS

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ABSTRACT. We prove that the number of different prime divisors of the order of a finite group is bounded by a polynomial function of the maximum of the number of different prime divisors of the element orders. This improves a result of J. Zhang.

1. INTRODUCTION

Given a finite group G, let $\rho(G)$ be the number of different prime divisors of $|G|$ and let $\alpha(G)$ be the maximum number of different prime divisors of the orders of the elements of G. It was proved by J. Zhang in [6] that if G is solvable, then $\rho(G)$ is bounded by a quadratic function of $\alpha(G)$ and that for arbitrary G, $\rho(G)$ is bounded by a superexponential function of $\alpha(G)$. The result for solvable groups was improved by T. M. Keller in [3], where he proved that $\rho(G)$ is bounded by a linear function of $\alpha(G)$. The purpose of this short note is to provide a proof of a better bound in the case of arbitrary finite groups.

Theorem A. *There exist universal (explicitly computable) constants C_1 and C_2 such that for every finite group $G > 1$ the inequality*

$$\rho(G) \leq C_1 \alpha(G)^4 \log \alpha(G) + C_2$$

holds.

The proof relies on the classification of simple groups, as in Zhang's paper. Actually, the case where we improve on Zhang's argument is in the case of alternating groups. This polynomial bound has been used in [4].

2. PROOF

First, we prove that for simple groups there is an essentially cubic bound. We begin with the alternating groups.

Lemma 2.1. *There exists a constant C_1 such that $\rho(A_n) \leq C_1 \alpha(A_n)^2$ for every $n \geq 5$.*

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Proof. Let p_j be the jth prime number. Let k be the maximum integer such that

$$4+\sum_{j=2}^{k} p_j \leq n.$$

It is clear that the elements of A_n that can be written as the product of two 2-cycles, one p_2-cycle, one p_3-cycle,..., one p_{k-1}-cycle and one p_k-cycle, with all these cycles pairwise disjoint, are divisible by $\alpha(A_n)=k$ different primes. It follows from p. 190 of [5], for instance, that $p_j \leq 10j\log j$. Therefore

$$\alpha(A_n) \geq \max\{l \mid 4+10\sum_{j=2}^{l} j\log j \leq n\} \geq \max\{l \mid 4+10l^2\log l \leq n\} = t.$$

In particular, we have that $n < 4+10(t+1)^2\log(t+1)$. By p. 160 of [5], for instance, we have that $\rho(A_n)$ is bounded by a quadratic function of t. The result follows. $\square$

All the inequalities that appear in this proof have reversed inequalities of the same order of magnitude. This implies that there exists a constant K_1 such that $\rho(A_n) \geq K_1\alpha(A_n)^2$ for every $n \geq 5$.

Next, we consider the simple groups of Lie type.

Lemma 2.2. *There exists a constant C_2 such that $\rho(G) \leq C_2\alpha(G)^3\log\alpha(G)$ whenever G is a simple group of Lie type.*

Proof. It suffices to argue as in the proof of Lemma 5 of [6], using the proof of Lemma 2.1 instead of the proof of Lemma 4 of [6]. $\square$

Now, we are ready to prove Theorem A.

Proof of Theorem A. We know by [3] that there exists $n_0 > 1$ such that if H is solvable and $\alpha(H) \geq n_0$, then $\rho(H) < 5\alpha(H)$. We consider groups G with $\alpha(G)=k \geq n_0$ and we want to prove that $\rho(G) \leq Ck^4\log k$, where $C = 10\max\{C_1, C_2, C_3, 5\}$ and C_3 is defined in such a way that $\rho(G) \leq C_3k^3$ whenever $\alpha(G)=k<n_0$ or G is sporadic.

Let G be a minimal (nonsolvable) counterexample. We define the series $1 = S_0 \leq R_1 < S_1 < R_2 < S_2 < \cdots < R_m < S_m \leq R_{m+1} = G$ as follows: R_1 is the largest normal solvable subgroup of G, and for any $i \geq 1$, S_i/R_i is the socle of G/R_i and R_{i+1}/S_i is the largest normal solvable subgroup of G/S_i. Note that for $i \geq 1$, S_i/R_i is a direct product of nonabelian simple groups.

We claim that $m \leq 5k$. In order to see this, we will first prove that there exists a prime divisor q_i of $|S_i/R_i|$ that is coprime to $|G/S_i||R_i|$ for $i=1,\ldots,m$. This argument is due to Zhang [6]. Let P be a Sylow 2-subgroup of S_i. By the Frattini argument, $G = S_iN_G(P)$. Put $T = R_iN_G(P)$. Then T is a proper subgroup of G. If every prime divisor of $|S_i/R_i|$ divides $|G/S_i||R_i|$, then we would have $\rho(T)=\rho(G)$. Since the theorem holds for T, it also holds for G. This contradiction implies that such q_i exists.

Now, let Q_m be a q_m-Sylow subgroup of G. We have that Q_m acts coprimely on R_m and using Glauberman's Lemma (Lemma 13.8 of [2]), we deduce that there exists $Q_{m-1} \in \mathrm{Syl}_{q_{m-1}}(R_m)$ that is Q_m-invariant. Now, we consider the action of

$Q_{m-1}Q_m$ on R_{m-1} and conclude that there exists a $Q_{m-1}Q_m$-invariant Sylow q_{m-2}-subgroup of G. In this way, we build a solvable subgroup $H = Q_m Q_{m-1} \dots Q_1$. By [3], we have that $m \le 5\alpha(H) \le 5\alpha(G)$, as claimed.

Using Lemmas 2.1 and 2.2 together with [3], one can see that

$$\rho(S_i/S_{i-1}) \le (C/5)k^3 \log k.$$

Finally we deduce that

$$\rho(G) \le m \cdot \max_i \rho(S_i/S_{i-1}) \le Ck^4 \log k.$$

This contradiction completes the proof. □

After this paper was submitted for publication, Keller informed us that he had independently obtained this bound. The result appears stated, but without proof, in Remark 16.19(b) of [1].

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