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POSITIVITY AND STRONG ELLIPTICITY

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Abstract. We consider partial differential operators $H = -\operatorname{div}(C\nabla)$ in divergence form on $\mathbf{R}^d$ with a positive-semidefinite, symmetric, matrix C of real L_∞-coefficients, and establish that H is strongly elliptic if and only if the associated semigroup kernel satisfies local lower bounds, or, if and only if the kernel satisfies Gaussian upper and lower bounds.

The classical Nash–De Giorgi [Nas], [DeG] theory analyzes positive second-order partial differential operators in divergence form, i.e., operators

$$H = -\sum_{i,j=1}^{d} \partial_i \, c_{ij} \, \partial_j, \tag{1}$$

where $\partial_i = \partial/\partial x_i$, the coefficients c_{ij} are real L_∞-functions and the matrix $C = (c_{ij})$ is assumed to be symmetric and positive-definite almost everywhere. The starting point of the theory is the strong ellipticity assumption,

$$C \geq \mu \, I > 0 \tag{2}$$

almost everywhere, and the principal conclusion is the local Hölder continuity of weak solutions of the associated elliptic and parabolic equations. In Nash's approach the Hölder continuity of the elliptic solution is derived as a corollary of continuity of the parabolic solution, and the latter is established by an iterative argument from good upper and lower bounds on the fundamental solution. Aronson [Aro] subsequently improved Nash's bounds and proved that the fundamental solution of the parabolic equation, the heat kernel, satisfies Gaussian upper and lower bounds. Specifically the kernel K of the semigroup S is a symmetric function over $\mathbf{R}^d \times \mathbf{R}^d$ satisfying bounds

$$a' \, G_{b';t}(x-y) \leq K_t(x\,;y) \leq a \, G_{b;t}(x-y) \tag{3}$$

for some $a, a', b, b' > 0$, uniformly for all $x, y \in \mathbf{R}^d$ and $t > 0$, where $G_{b;t}(x) = t^{-d/2} e^{-b|x|^2 t^{-1}}$. Background information and references can be found in the books and reviews [Dav], [DER], [Gri], [Str1], [Str2].

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In this note we observe that a converse statement is true. If H is an elliptic operator of the form (1), then the corresponding heat kernel satisfies the Aronson bounds (3) if and only if H satisfies the strong ellipticity condition (2). In fact we show that (2) and (3) are both equivalent to lower bounds $K_t(x\,;y) \geq a\,t^{-d/2}$ for all $|x-y| \leq rt^{1/2}$ and $t \in \langle 0,1]$.

In the Nash–De Giorgi theory the strong ellipticity assumption (2) is first used to give a precise definition of H as a positive self-adjoint operator on the complex Hilbert space $L_2(\mathbf{R}^d)$ through quadratic form techniques. Specifically, one defines the quadratic form h on $L_2(\mathbf{R}^d)$ by

$$h(\varphi) = \sum_{i,j=1}^{d} \int_{\mathbf{R}^d} dx\, \overline{(\partial_i\varphi)(x)}\, c_{ij}(x)\, (\partial_j\varphi)(x) \tag{4}$$

with domain $D(h) = W^{1,2}(\mathbf{R}^d) = \bigcap_{i=1}^d D(\partial_i) = D(\Delta^{1/2})$, where Δ denotes the self-adjoint Laplacian, i.e., $\Delta = -\sum_{i=1}^d \partial_i^2$, on $L_2(\mathbf{R}^d)$. Then h is positive, symmetric, densely-defined and as a direct consequence of (2) it is also closed. Therefore there is a unique, positive, self-adjoint operator H, with $D(H) \subset D(h)$, canonically associated with h. In particular $(\varphi, H\varphi) = h(\varphi)$ for all $\varphi \in D(H)$.

Since our intention is to analyze the operator H without the strong ellipticity assumption (2), the foregoing definition of H is not applicable, and one has to adopt an alternative approach. One can still introduce the form h as above, but there is no reason for the form to be closable. (For examples of non-closable h see [FOT], Theorem 3.1.6.) We circumvent this problem by a 'viscosity' method.

Let l be the closed quadratic form associated with the Laplacian Δ, i.e.,

$$l(\varphi) = \sum_{i=1}^{d} \|\partial_i\varphi\|_2^2 = \|\Delta^{1/2}\varphi\|_2^2 \tag{5}$$

with $D(l) = D(\Delta^{1/2})$. Then for each $\varepsilon \in \langle 0,1]$ define h_ε by $D(h_\varepsilon) = D(h) = D(l)$ and

$$h_\varepsilon(\varphi) = h(\varphi) + \varepsilon\, l(\varphi),$$

where h denotes the form given by (4). Since h is positive, the form h_ε satisfies the strong ellipticity condition

$$h_\varepsilon(\varphi) \geq \varepsilon\, l(\varphi) \tag{6}$$

for all $\varphi \in D(h)$. In addition it satisfies the upper bounds

$$h_\varepsilon(\varphi) \leq (1 + \|C\|)\, l(\varphi), \tag{7}$$

where $\|C\|$ is the essential supremum of the matrix norm of $C(x) = (c_{ij}(x))$. It follows immediately from (6) and (7) that h_ε is closed. Therefore there is a positive self-adjoint operator H_ε canonically associated with h_ε. The operator H_ε is the strongly elliptic operator with coefficients $C + \varepsilon I$. But $\varepsilon \mapsto h_\varepsilon(\varphi)$ decreases monotonically as ε decreases for each $\varphi \in D(h)$. Therefore it follows from a result of Kato, [Kat] Theorem VIII.3.11, that the H_ε converge in the strong resolvent sense, as $\varepsilon \to 0$, to a positive self-adjoint operator H_0, which we will refer to as the **viscosity operator** with coefficients $C = (c_{ij})$. The strong resolvent convergence also implies that the positive contractive semigroups $S^{(\varepsilon)}$ generated by the H_ε converge strongly to the semigroup $S^{(0)}$ generated by H_0. Therefore $S^{(0)}$ is positive and contractive on $L_2(\mathbf{R}^d)$.

Let h_0 denote the form associated with H_0, i.e., $D(h_0) = D(H_0^{1/2})$ and $h_0(\varphi) = \|H_0^{1/2}\varphi\|_2^2$. There is an alternative method of defining h_0 which shows that it has more universal significance.

One may associate with any positive quadratic form h a unique maximal closable minorant h_r, i.e., h_r is the largest closable positive quadratic form which is majorized by h (see [Sim2], [Dal]). Then h_0 is the closure of h_r. In particular, if h is closable, then h_0 is its closure. In addition, h_0 is the largest closed positive quadratic form which is majorized by h. Consequently $D(h) \subseteq D(h_0)$. One may characterize $D(h_0)$ as the vector space of all $\varphi \in L_2$ for which there are $\varphi_1, \varphi_2, \ldots \in D(h)$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in L_2 and $\liminf_{n\to\infty} h(\varphi_n) < \infty$. Moreover, $h_0(\varphi)$ equals the minimum of all $\liminf_{n\to\infty} h(\varphi_n)$, where the minimum is taken over all $\varphi_1, \varphi_2, \ldots \in D(h)$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in L_2. (See [Sim1], Theorem 3.) In convergence theory h_0 is variously called the lower semi-continuous regularization of h [EkT], page 10, or the relaxed form [Dal], page 28.

The following theorem gives a precise formulation of the characterizations of strong ellipticity mentioned above. Other characterizations are given in Proposition 2.

Theorem 1. *Let H_0 be the viscosity operator with coefficients $C = (c_{ij})$ and $K^{(0)}$ the distribution kernel of the positive contraction semigroup $S^{(0)}$ generated by H_0. The following conditions are equivalent:*

I. *There is a $\mu > 0$ such that $C \geq \mu I$ almost everywhere.*

II. *There are $a, r > 0$ such that for all $t \in \langle 0, 1]$ one has*

$$K_t^{(0)}(x\,;y) \geq a\, t^{-d/2}$$

for almost every $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$ with $|x - y| \leq rt^{1/2}$.

III. *$K_t^{(0)}$ is a bounded function satisfying the Aronson Gaussian bounds* (3).

The implication I⇒III follows by the Nash–Aronson estimates and obviously III⇒II. Therefore the proof of the theorem is reduced to establishing that II⇒I. The proof is based on a variation of an argument of Carlen, Kusuoka and Stroock [CKS] which requires a different formulation of strong ellipticity.

The following proposition gives several related characterizations of strong ellipticity in terms of the forms h, h_0 and the corresponding operators. It is well known (see for example Folland [Fol], Theorem 7.17) that strong ellipticity is equivalent to a Gårding inequality, and this may be expressed in terms of either form.

Proposition 2. *Let H_0 be the viscosity operator with coefficients $C = (c_{ij})$. Moreover, let h and l be the forms given by* (4) *and* (5) *with common domain $\bigcap_{i=1}^d D(\partial_i)$, and let h_0 denote the form associated with H_0. The following conditions are equivalent:*

I. *The form h is closed.*

II. $h = h_0$.

III. *There is a $\mu > 0$ such that $C \geq \mu I$ almost everywhere.*

IV. *There is a $\mu > 0$ such that $h \geq \mu\, l$.*

V. *There are $\mu > 0$ and $\nu \geq 0$ such that $h \geq \mu\, l - \nu\, I$.*

VI. *There is a $\mu > 0$ such that $H_0 \geq \mu\, \Delta$ in the quadratic form sense.*

VII. *There are $\mu > 0$ and $\nu \geq 0$ such that $H_0 \geq \mu\, \Delta - \nu\, I$ in the quadratic form sense.*

Proof. We shall prove that III⇒IV⇒I⇒II⇒VII⇒V⇒III and IV⇒VI⇒VII.

The implication III⇒IV is trivial. Since $h \leq \|C\|\, l$ and l is closed, the implication IV⇒I is straightforward. The implication I⇒II is trivial.

If $h = h_0$, then the vector space $D(\Delta^{1/2}) = D(h) = D(h_0)$ is a Banach space with respect to the norm $\varphi \mapsto (h_0(\varphi) + \|\varphi\|_2^2)^{1/2}$. But $D(\Delta^{1/2})$ with the graph norm is also a Banach space. In addition both Banach spaces are continuously embedded in $L_2(\mathbf{R}^d)$. Hence there is a $c > 0$ such that $\|\Delta^{1/2}\varphi\|_2^2 + \|\varphi\|_2^2 \leq c\,(h_0(\varphi) + \|\varphi\|_2^2)$ for all $\varphi \in D(\Delta^{1/2})$ as a consequence of the closed graph theorem. Therefore one deduces VII.

The implication VII⇒V is evident since $h_0 \leq h$.

The implication V⇒III follows the proof of Theorem 7.17 in [Fol]. Let $\varphi \in C_c^\infty(\mathbf{R}^d)$, $k \in \mathbf{R}$ and $\xi \in \mathbf{R}^d$. Define $\varphi_k \in D(h) \cap D(l)$ by $\varphi_k(x) = e^{ikx.\xi}\varphi(x)$. Then one calculates that

$$\lim_{k\to\infty} k^{-2} h(\varphi_k) = \int_{\mathbf{R}^d} dx\, |\varphi(x)|^2\, (\xi, C(x)\xi).$$

Moreover,

$$\lim_{k\to\infty} k^{-2}(\mu\, l(\varphi_k) - \nu\, \|\varphi_k\|_2^2) = \mu \int_{\mathbf{R}^d} dx\, |\varphi(x)|^2\, |\xi|^2.$$

Since $h(\varphi_k) \geq \mu\, l(\varphi_k) - \nu\|\varphi_k\|_2^2$ one deduces that

$$\int_{\mathbf{R}^d} dx\, |\varphi(x)|^2\, (\xi, C(x)\xi) \geq \mu \int_{\mathbf{R}^d} dx\, |\varphi(x)|^2\, |\xi|^2.$$

Then one concludes that $C \geq \mu I$ almost everywhere. This proves the implication V⇒III.

Next, if IV is valid, then $\mu\, l$ is a closed positive quadratic form with $\mu\, l \leq h$. Hence $\mu\, l \leq h_0$ and VI is valid. The implication VI⇒VII is obvious. □

As a final preliminary to the proof of the missing implication in Theorem 1 we need some information on Dirichlet forms [FOT], [BoH].

It is easy to verify that $h(|\varphi|) \leq h(\varphi)$ and $h(0 \vee \varphi \wedge \mathbb{1}) \leq h(\varphi)$ for all real valued $\varphi \in D(h)$. If $\varphi \in D(h_0)$ is real valued, then there are $\varphi_1, \varphi_2, \ldots \in D(h)$ such that $\lim \varphi_n = \varphi$ in L_2 and $h_0(\varphi) = \lim h(\varphi_n)$. Then $\lim |\varphi_n| = |\varphi|$ in L_2 and $\liminf h(|\varphi_n|) \leq \liminf h(\varphi_n) = h_0(\varphi)$. So $|\varphi| \in D(h_0)$ and $h_0(|\varphi|) \leq h_0(\varphi)$. Similarly, $0 \vee \varphi \wedge \mathbb{1} \in D(h_0)$ and $h_0(0 \vee \varphi \wedge \mathbb{1}) \leq h_0(\varphi)$. Therefore h_0 is a Dirichlet form and $S^{(0)}$ extends to a positive contraction semigroup on all the L_p-spaces, which we will also denote by $S^{(0)}$. It then follows from the positivity and contractivity that the semigroup $S^{(0)}$ satisfies

$$0 \leq S_t^{(0)}\, \mathbb{1} \leq \mathbb{1} \tag{8}$$

for all $t > 0$ on $L_\infty(\mathbf{R}^d)$. (In fact one can prove that $S_t^{(0)}\, \mathbb{1} = \mathbb{1}$, but this is not straightforward (see [ERSZ], Proposition 3.6), and it is not necessary in the sequel.)

By the contractivity of $S^{(0)}$ and spectral theory one has

$$h_0(\varphi) \geq t^{-1}(\varphi, (I - S_t^{(0)})\varphi)$$

for all $\varphi \in D(h_0)$ and $t > 0$. But one deduces from (8) that

$$\|\varphi\|_2^2 = (\varphi, \varphi) \geq (S_t^{(0)}\, \mathbb{1}, |\varphi|^2) = (|\varphi|^2, S_t^{(0)}\, \mathbb{1})$$

for all $t > 0$, where $(\,\cdot\,,\,\cdot\,)$ denotes the duality between L_p and L_q. Then it follows from self-adjointness of $S_t^{(0)}$ and (8) that

$$h_0(\varphi) \geq (2t)^{-1}\Big((S_t^{(0)}\mathbb{1}, |\varphi|^2) + (|\varphi|^2, S_t^{(0)}\mathbb{1}) - (\varphi, S_t^{(0)}\varphi) - (S_t^{(0)}\varphi, \varphi)\Big)$$

for all $\varphi \in D(h_0)$ and $t > 0$. This gives a related estimate in terms of the distribution kernel.

Let $\varphi, \chi \in C_c^\infty(\mathbf{R}^d) \subset D(h) \subseteq D(h_0)$. Suppose that $0 \leq \chi \leq 1$ and $\chi = 1$ on the support of φ. Then since $S^{(0)}$ is positive, $S_t^{(0)}\chi \leq S_t^{(0)}\mathbb{1}$ and

$$\begin{aligned} h_0(\varphi) &\geq (2t)^{-1}\Big((S_t^{(0)}\chi, \chi|\varphi|^2) + (\chi|\varphi|^2, S_t^{(0)}\chi) - (\chi\varphi, S_t^{(0)}\chi\varphi) - (S_t^{(0)}\chi\varphi, \chi\varphi)\Big) \\ &= (2t)^{-1}\int_{\mathbf{R}^d\times\mathbf{R}^d} d(x,y)\, K_t^{(0)}(x\,;y)\,\chi(x)\,\chi(y)\,|\varphi(x)-\varphi(y)|^2 \end{aligned}$$

for all $t > 0$. This is the starting point of the Carlen–Kusuoka–Stroock argument to establish that II$\Rightarrow$I in Theorem 1.

End of proof of Theorem 1. Choose a smooth positive function ρ with support in $\langle -r, r\rangle$ such that $\rho \leq 1$ and $\rho(x) = 1$ for all $x \in \mathbf{R}^d$ with if $|x| \leq r/2$. Then the previous estimate gives

$$h_0(\varphi) \geq (2t)^{-1}\int_{\mathbf{R}^d\times\mathbf{R}^d} d(x,y)\, K_t^{(0)}(x\,;y)\,\rho(|x-y|^2t^{-1})\,\chi(x)\,\chi(y)\,|\varphi(x)-\varphi(y)|^2$$

for $\varphi \in C_c^\infty(\mathbf{R}^d)$, $t > 0$ and $\chi \in C_c^\infty(\mathbf{R}^d)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on the support of φ. Then it follows from Condition II that

$$h_0(\varphi) \geq a\,(2t)^{-1}\int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy\, t^{-d/2}\,\rho(|x-y|^2t^{-1})\,\chi(x)\,\chi(y)\,|\varphi(x)-\varphi(y)|^2.$$

But the left-hand side is independent of the choice of χ, so by the monotone convergence theorem

$$h_0(\varphi) \geq a\,(2t)^{-1}\int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy\, t^{-d/2}\,\rho(|x-y|^2t^{-1})\,|\varphi(x)-\varphi(y)|^2$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$ and $t \in \langle 0, 1]$. Therefore if $\widehat{\varphi}$ denotes the Fourier transform of φ, then

$$\begin{aligned} h_0(\varphi) &\geq a\,t^{-1}\int_{\mathbf{R}^d} dx\, t^{-d/2}\,\rho(|x|^2t^{-1})\int_{\mathbf{R}^d} d\xi\,|\widehat{\varphi}(\xi)|^2\,(1-\cos\xi.x) \\ &= a\,t^{-1}\int_{\mathbf{R}^d} dx\,\rho(|x|^2)\int_{\mathbf{R}^d} d\xi\,|\widehat{\varphi}(\xi)|^2\,(1-\cos t^{1/2}\xi.x) \\ &= 2a\int_{\mathbf{R}^d} d\xi\,|\widehat{\varphi}(\xi)|^2\int_{\mathbf{R}^d} dx\,\rho(|x|^2)\,t^{-1}\sin^2(2^{-1}t^{1/2}\xi.x) \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$ and $t \in \langle 0, 1]$. Thus in the limit $t \to 0$ one has

$$h_0(\varphi) \geq 2^{-1}a\int_{\mathbf{R}^d} d\xi\,|\widehat{\varphi}(\xi)|^2\int_{\mathbf{R}^d} dx\,\rho(|x|^2)\,(\xi.x)^2 = \mu\int_{\mathbf{R}^d} d\xi\,|\widehat{\varphi}(\xi)|^2\,|\xi|^2 = \mu\,l(\varphi)$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$ with $\mu > 0$. Then since $h_0 \leq h$, by the discussion preceding Theorem 1, one has $h(\varphi) \geq \mu\,l(\varphi)$ for all $\varphi \in C_c^\infty(\mathbf{R}^d)$. But the coefficients of h are bounded and $C_c^\infty(\mathbf{R}^d)$ is a core for $W^{1,2}(\mathbf{R}^d) = D(l) = D(h)$. So $h(\varphi) \geq \mu\,l(\varphi)$ for all $\varphi \in D(l)$. Thus Condition IV in Proposition 2 is satisfied. But this is equivalent

to Condition III of the proposition which is just a repetition of the strong ellipticity hypothesis, Condition I in Theorem 1. □

Although much work in recent years has been devoted to the derivation of Gaussian upper bounds on semigroup kernels, Theorem 1 demonstrates that Gaussian lower bounds are in fact the important feature in understanding the general behaviour of the kernels. The local small time lower bounds in Condition II of the theorem encapsulate all the information contained in the Aronson upper and lower bounds. The lower bounds reflect the correct small t behaviour, and this is enough to derive the behaviour, of the semigroup and its kernel for all t.

It is also interesting to note that in quite general circumstances (see, for example, [Cou]) the Gaussian upper bounds suffice to prove that Gaussian lower bounds are equivalent to Hölder continuity of the kernel. In particular each of the equivalent conditions of the theorem implies that the semigroup kernel is Hölder continuous.

It is also possible to extend the theorem to the setting of subelliptic operators on Lie groups. Let $a_1, \ldots, a_d$ be a vector space basis for the Lie algebra $\mathfrak{g}$ of a Lie group G. For all $i \in \{1, \ldots, d\}$ let A_i be the infinitesimal generator of the one parameter group $t \mapsto L(\exp(-ta_i))$, where L is the left regular representation in $L_2(G)$. For all $i, j \in \{1, \ldots, d\}$ let $c_{ij} \in L_\infty(G)$ and suppose that the matrix (c_{ij}) is real, symmetric and positive-definite almost everywhere. One can define as above a viscosity operator H_0 corresponding to the formal expression $-\sum_{i,j=1}^{d} A_i\, c_{ij}\, A_j$.

Next let $d' \leq d$ and suppose that $a_1, \ldots, a_{d'}$ generate the Lie algebra $\mathfrak{g}$. Associated to $a_1, \ldots, a_{d'}$ one can define a modulus $|\cdot|'$ on G and a local dimension $D' \in \mathbf{N}$, i.e., $\mathrm{Vol}\{g \in G : |g|' < \rho\} \asymp \rho^{D'}$ for $\rho \in \langle 0, 1]$. Then one has the following theorem.

Theorem 3. *Let H_0 be the viscosity operator with coefficients (c_{ij}) and let $K^{(0)}$ be the distribution kernel of the positive contraction semigroup $S^{(0)}$ generated by H_0. The following conditions are equivalent:*

I. *There is a $\mu > 0$ such that*

$$(\varphi, H_0\varphi) \geq \mu \sum_{i=1}^{d'} \|A_i\varphi\|_2^2$$

for all $\varphi \in \bigcap_{i=1}^{d'} D(A_i)$.

II. *There are $a, r > 0$ such that for all $t \in \langle 0, 1]$ one has*

$$K_t^{(0)}(g\,;h) \geq a\, t^{-D'/2}$$

for almost every $(g, h) \in G \times G$ with $|gh^{-1}|' \leq rt^{1/2}$.

III. *There are $a, a', b, b', \omega, \omega' > 0$ such that*

$$a'\, t^{-D'/2} e^{-\omega' t} e^{-b'(|gh^{-1}|')^2 t^{-1}} \leq K_t^{(0)}(g\,;h) \leq a\, t^{-D'/2} e^{\omega t} e^{-b(|gh^{-1}|')^2 t^{-1}}$$

for all $t > 0$ and $g, h \in G$.

The implication I⇒III is in [ElR2], the implication III⇒II is trivial and the implication II⇒I is as in the proof of Theorem 1. But instead of the scaling of ρ used in the above proof one has to use the maps γ_t as in [ElR1], Section 3. We omit the technical details.

Finally the situation is quite different for second-order real divergence form operators which are degenerate [ERSZ]. Then the kernel is positive but not necessarily

strictly positive even if the operator is subelliptic. One may construct examples for which the kernel vanishes on the loci of degeneracy. In particular one cannot expect any type of Gaussian lower bound. Nevertheless subellipticity and a condition of uniform strict positivity suffice to deduce Gaussian upper bounds which incorporate the correct large t behaviour (see [ERSZ] for details).

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