A NOTE ON A SYMMETRY RESULT
FOR TRAVELING WAVES IN CYLINDERS

C. E. KENIG AND F. MERLE

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Abstract. We prove in this note that all bounded traveling waves, in cylinders, of some $N$-dimensional viscous conservation laws are symmetric.

I. The main result

In this note, we consider traveling wave solutions of the equation

\[
\frac{\partial U}{\partial t} = \Delta U - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(f_i(U))
\]

for $(x_1, x') \in \mathbb{R} \times T^{N-1}$ and $f_i \in C^2(\mathbb{R}, \mathbb{R})$. That is, we consider solutions of the form

\[
U(t, x) = u(x_1 - ct, x'),
\]

where $u$ satisfies

\[
\Delta u - (f_1(u) - cu)_{x_1} - \sum_{i=2}^{N} (f_i(u))_{x_i} = 0
\]

for $(x_1, x') \in \mathbb{R} \times T^{N-1}$.

The main result of this note is the following.

Theorem 1. Assume that $u$ is an $L^\infty$ solution of (1.2).

i) Then there are $a_\pm$ such that $\forall x' \in T^{N-1}$, $\lim_{x_1 \to \pm \infty} u(x_1, x')$ is defined and equal to $a_\pm$. In addition, the limit is uniform in $x' \in T^{N-1}$.

ii) Assume in addition that

\[
f'_1(a_\pm) - c \neq 0.
\]

Then $u(x_1, x') = v(x_1)$, where

\[
v_{x_1} - (f_1(v) - cv)_{x_1} = 0 \\
\text{for } x_1 \in \mathbb{R}.
\]

Remark. If $f_i \equiv 0$ for $i \geq 2$, or in some sense the degeneracy of $f_i$ at $a_\pm$ is of higher order than the one of $f_1$, the degeneracy condition in part (ii) of the theorem can be relaxed.
Under the assumptions of Theorem 1, we have the following.

**Theorem 2** (Liouville Theorem for (1.1)). Let \( u(x) \) be a solution of equation (1.2). Consider \( U(t,x) \) to be a solution of equation (1.1) defined for all time \( t \in \mathbb{R} \) such that for a constant \( C_0 > 0 \),

\[
\forall t \in \mathbb{R}, \quad \|U(t,x) - u(x_1 - ct)\|_{L^1} \leq C_0.
\]

Then there is an \( x_0 \in \mathbb{R} \) such that

\[
\forall t \in \mathbb{R}, \forall x \in \mathbb{R} \times T^{N-1}, \quad U(t,x) = u(x_1 - ct + x_0).
\]

From this result, we can then derive the following asymptotic stability result for equation (1.1).

**Theorem 3** (Asymptotic Stability for Traveling Waves). Let \( U(t,x) \) be a solution of equation (1.1) for \( t > 0 \), \( x \in \mathbb{R} \times T^{N-1} \) and initial data \( U(0,x) = U_0(x) \).

Assume in addition that for \( A > 0 \) and for a travelling wave \( u \),

\[
\|U_0(x_1,x') - u(x_1)\|_{L^1(\mathbb{R} \times T^{N-1})} \leq A.
\]

Then there exists a function \( g_A(t) \) depending only on \( A, u \) with \( g_A(t) \to 0 \) as \( t \to +\infty \), such that, for all such \( u \), we have

\[
\forall t > 1, \text{Inf}_{x_{10} \in \mathbb{R}} \|U(t,\cdot) - u(\cdot + x_{10} - ct)\|_{L^\infty(\mathbb{R} \times T^{N-1})} \leq g_A(t).
\]

The proofs of Theorems 2, 3 are completely similar to the corresponding ones of [1] as soon as Theorem 1 is proved. We therefore devote the rest of this paper to the proof of Theorem 1.

II. **Proof of Theorem 1**

Let \( u \) be an \( L^\infty \) solution of (1.2) and let \( C = \mathbb{R} \times T^{n-1} \). We first prove the following.

**Lemma 1.** We have

\[
\lim_{x_1 \to +\infty} u(x_1, x') = a_+,
\]

\[
\lim_{x_1 \to -\infty} u(x_1, x') = a_-,
\]

uniformly in \( x' \in T^{N-1} \), where \((a_+, a_-) \) are \((\text{Sup}_C u, \text{Inf}_C u)\) or \((\text{Inf}_C u, \text{Sup}_C u)\).

**Proof.** We first remark from standard elliptic theory that

\[
u \in C^3(\mathbb{R} \times T^{N-1}, \mathbb{R}) \quad \text{and} \quad |v|_{C^3} \leq M.
\]

Moreover, if \( u \) achieves a local maximum or a local minimum, from the strong maximum principle we have

\[
u(x_1, x') \equiv \text{constant} \equiv a,
\]

and all the conclusions of Theorem 1 hold. We now assume that \( u(x_1, x') \) is different from a constant solution.
From the fact that there are no local extrema, $\text{Inf}_{C}$ and $\text{Sup}_{C}$ are not achieved and there are $(y_{1n}, y'_{1n}) \in \mathbb{R} \times T^{N-1}$ (respectively $(z_{1n}, z'_{1n}) \in \mathbb{R} \times T^{N-1}$) such that

\begin{align}
(2.3) \quad u(y_{1n}, y'_{1n}) \xrightarrow[n \to +\infty]{} \text{Sup}_{C} u = a_+,
(2.4) \quad u(z_{1n}, z'_{1n}) \xrightarrow[n \to +\infty]{} \text{Inf}_{C} u = a_-,
\end{align}

with $|y_{1n}| \to +\infty$ and $|z_{1n}| \to +\infty$.

One can assume, eventually extracting a subsequence, that

\begin{align}
(2.5) \quad y_{1n} \to +\infty \quad \text{and} \quad y_{1n} < y_{1n+1}.
\end{align}

(The proof in the other case is identical.)

Let us now prove

\begin{align}
(2.6) \quad \text{Inf}_{x' \in T^{N-1}} u(y_{1n}, x') \xrightarrow[n \to +\infty]{} a_+.
\end{align}

Indeed, let us consider

\begin{align}
u_n(x_1, x') = u(y_{1n} + x_1, x').
\end{align}

We have

- $|u_n|_{C^2} \leq C$,
- $u_n$ is solution of equation (1.2),
- $u_n(0, y'_{1n}) \xrightarrow[n \to +\infty]{} a_+ = \text{Sup} u = \text{Sup} u_n$.

Extracting a subsequence, we have for $W : (x_1, x') \mapsto W(x_1, x')$ and $y' \in T^{N-1}$,

\begin{align}
\text{Inf}_{x' \in T^{N-1}} u(y_{1n}, x') \xrightarrow[n \to +\infty]{} W(x_1, x') \quad \text{and} \quad y' \to y',
\end{align}

where

$W$ is solution of (1.2), $\text{Sup} W \leq a_+$, $W(0, y') = a_+$.

Therefore,

\begin{align}
W = a_+
\end{align}

and since the result is true for all subsequences

\begin{align}
u(y_{1n} + x_1, x') \xrightarrow[C_{loc}^{\infty}]{} a_+,
\end{align}

then (1.7) follows.

From the fact that $u$ does not have local minimum, we have

\begin{align}
a_+ \geq \text{Inf}_{x_1 \in (y_{1n}, y_{1(n+1)})} \text{Inf}_{x' \in T^{N-1}} u(x_1, x') \xrightarrow[n \to +\infty]{} a_+.
\end{align}

Since $y_{1n} < y_{1(n+1)}$ and $y_{1n} \xrightarrow[n \to +\infty]{} +\infty$, then

\begin{align}
\lim_{x_1 \to +\infty} \left\{ \text{Inf}_{x' \in T^{N-1}} u(x_1, x') \right\}
\text{exists and equals } a_+.
\end{align}

It follows that $z_{1n} \to -\infty$ and by the same procedure,

\begin{align}
\text{Inf}_{x_1 \to -\infty} \left( \text{Sup}_{x' \in T^{N-1}} u(x_1, x') \right)
\text{exists and is equal to } a_-.
\end{align}

This concludes the proofs of Lemma \ref{lemma} and Theorem \ref{theorem} part (i).

We now assume in addition a nondegeneracy condition at $a_+, a_-$ for $f_1(u)$, namely

\begin{align}
f'_1(a_+)-c \neq 0 \quad \text{and} \quad f'_1(a_-)-c \neq 0,
\end{align}

and, for example, $a_- < a_+$. \hfill $\square$
Lemma 2. There exist \( \alpha > 0 \) and \( C_0 > 0 \) such that
\[
\begin{align*}
|u(x_1, x') - a_+| &\leq C_0 e^{-\alpha x_1}, \\
|u(x_1, x') - a_-| &\leq C_0 e^{\alpha x_1}.
\end{align*}
\]

Proof. Let us prove the first one for example. The only question is when \( x_1 \to +\infty \).

(i) Let us first introduce
\[
w(x_1) = \frac{1}{\text{vol}(T^{N-1})} \int_{T^{N-1}} (a_+ - u(x_1, x')) \, dx'.
\]
We have
\[
w \geq 0.
\]

Averaging the equation (2.2) over \( T^{N-1} \) and using the periodic boundary conditions, we obtain that \( w \) satisfies the following equation \( \forall x_1 \in \mathbb{R} \):
\[
w_{x_1} - \frac{\partial}{\partial x_1} \left[ \frac{1}{\text{vol}(T^{N-1})} \int_{T^{N-1}} (f_1(u(x_1, x')) - cu(x_1, x')) \, dx' \right] = 0
\]
or equivalently
\[
w_{x_1} - \left[ \frac{1}{\text{vol}(T^{N-1})} \int_{T^{N-1}} (f_1(u(x_1, x')) - cu(x_1, x')) \, dx' \right] = C_0.
\]
Define \( \beta = f'_1(a_+) - c \) and \( \gamma = f_1(a_+) - ca_+ \). We then have by linearization of the nonlinear term at \( a_+ \) in (2.10):
\[
\forall x_1 \in \mathbb{R}, \quad |w_{x_1} - \beta w - C_0 + \gamma| \leq C \sup_{x' \in T^{N-1}} |u(x_1, x') - a_+|^2.
\]
Since \( w(x_1, x') \to 0 \) as \( x_1 \to +\infty \), we have \( C_0 - \gamma = 0 \), and
\[
\forall x_1 \in \mathbb{R}, \quad |w_{x_1} - \beta w| \leq C(\sup_{x' \in T^{N-1}} |u(x_1, x') - a_+|).
\]

(ii) Relation between \( u \) and \( w \): We now apply the Harnack principle as \( x_1 \to +\infty \) to \( u \), and we get: there is a \( C > 0 \) such that for \( x_1 \geq 0 \),
\[
\sup_{x' \in T^{N-1}} (a_+ - u(\bar{x}_1, x')) \leq C \inf_{x' \in T^{N-1}} (a_+ - u(\hat{x}_1, x'))
\]
and, in particular, \( \forall x_1 \geq 0 \),
\[
\sup_{x' \in T^{N-1}} (a_+ - u(x_1, x')) \leq Cw(x_1).
\]
In particular, it is enough to prove the exponential decay for \( w(x_1) \) to reach the conclusion of the lemma.

(iii) Exponential decay of \( w \): We have from (2.11) and (2.12),
\[
\forall x_1 \geq 0, \quad |w_{x_1} - \beta w| \leq Cw^2.
\]
Since \( w \to 0 \) as \( x_1 \to +\infty \) and \( w > 0 \) we have \( \beta < 0 \) and for \( x_1 \) large,
\[
-\frac{3}{2} \beta w \leq w_{x_1} \leq -\frac{\beta}{2} w,
\]
which concludes the proof of Lemma 2 by integration in \( x_1 \).

We are now able to conclude the proof of Theorem 1 part (ii). We argue by contradiction: Assume there is \( x_1^0 \in \mathbb{R} \) such that for some \( x_0', x_1' \in T^{N-1}, u(x_1^0, x_0') \neq u(x_1^0, x_1') \).
Then using the periodicity, there are $x'_2, x'_3 \in T^N$ such that
\[ u(x'_1, x'_2) = u(x'_1, x'_3) \]
and
\[ \nabla_{x'} u(x'_1, x'_3) \neq \nabla_{x'} u(x'_1, x'_3). \]
Then
\[ H(t, x) = u(x_1 - ct, x_2 + x') - u(x_1 - ct, x_3 + x') \]
is the difference of two solutions of equation (1.1) such that
\begin{enumerate}
  \item $\forall t \in \mathbb{R}$, $\|H(t)\|_{L^1(C)} \equiv$ constant,
  \item $H(0, (0, 0)) = 0$ and $\nabla_{x'} H(0, (0, 0)) \neq 0$.
\end{enumerate}
Applying Lemma 2.9 of [1], we obtain
\[ \|H(1)\|_{L^1(C)} < \|H(0)\|_{L^1(C)}, \]
which is a contradiction.
(Of course, multiplying $u(x_1, x_2 + x') - u(x_1, x_3 + x')$ by sign $[u(x_1, x_2 + x') - u(x_1, x_3 + x')]$ yields a contradiction, by the same calculation as in the proof of Lemma 2.9 of [1].) This concludes the proof of Theorem 1.  

\section*{References}