

COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA AND q -GAMMA FUNCTIONS

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ABSTRACT. We give an infinite family of functions involving the gamma function whose logarithmic derivatives are completely monotonic. Each such function gives rise to an infinitely divisible probability distribution. Other similar results are also obtained for specific combinations of the gamma and q -gamma functions.

1. INTRODUCTION

We first recall some definitions and basic facts. A function f is completely monotonic if for all n , $(-1)^n f^{(n)}(x) \geq 0$ on $(0, \infty)$; see Feller [12] and Widder [27] for properties of completely monotonic functions. Bernstein's theorem asserts that f is completely monotonic if and only if $f(x) = \int_{\mathbb{R}} e^{-xt} d\mu(t)$ where μ is a positive measure supported on a subset of $[0, \infty)$. A probability measure is called infinitely divisible if for every n , $n = 1, 2, \dots$, there exists a probability measure μ_n such that μ is an n -fold convolution of μ_n . The Lévy-Khinchin theorem characterizes infinitely divisible measures supported on a subset of $[0, \infty)$ as those measures whose Laplace transform is of the form $f(x) = e^{-h(x)}$ and h' is completely monotonic. This holds in a wider sense without the standard normalization $f(0) = 1$, and also for $f(0) = \infty$ (cf. [8]). We remind the reader that any function $f(x) = e^{-h(x)}$ is completely monotonic if h' is completely monotonic. We call such functions $f(x)$ *logarithmically completely monotonic* [8, 24, 25]. Berg [8] points out that these functions are the same as those studied by Horn [20] under the name *infinitely divisible completely monotonic functions*. The subject of deriving inequalities for combinations of gamma functions is an old subject and some of the early works are by Kazarinoff [23], Watson [26], and Gautschi [14], [15]. These papers provide bounds for quotients of products of gamma functions. In the 1980s Ismail et al., [21], [9], realized that some of the inequalities are consequences of the complete monotonicity of functions of similar structure. We continue to witness progress in this area as can be seen from the more recent works [1, 2, 3, 4, 6, 22] (see also [5]). This paper is a further contribution to the subject.

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In Section 2 we prove Theorems 1.1 and 1.2 below. Before stating them we need some notation. Let S_n be the symmetric group over n symbols, a_1, a_2, \dots, a_n . Let O_n and E_n be the sets of odd and even permutations over n symbols, respectively. Finally, let $P_{n,k}$ ($k = 1, \dots, n$) be the set of all vectors $\mathbf{m} = (m_1, \dots, m_k)$ whose components are natural numbers such that $1 \leq m_\nu < m_\mu \leq n$ for $1 \leq \nu < \mu \leq k$; and let $P_{n,0}$ be the empty set.

Theorem 1.1. *Let $a_1 > a_2 > \dots > a_n > 0$ and define*

$$(1.1) \quad F(x) = \frac{\prod_{\sigma \in E_n} [\Gamma(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \dots + (n-1)a_{\sigma(n)})]}{\prod_{\sigma \in O_n} [\Gamma(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \dots + (n-1)a_{\sigma(n)})]}.$$

Then $F(x - a_2 - 2a_3 - \dots - (n-1)a_n)$ is logarithmically completely monotonic.

Observe that

$$\sum_{k=1}^n (k-1)(a_{\sigma(k)} - a_k) \geq 0$$

holds for any permutation σ when $a_1 > a_2 > \dots > a_n > 0$. This follows from Theorem 368 of Section 10.2 in Hardy, Littlewood, and Pólya [19]. Therefore $F(x - a_2 - 2a_3 - \dots - (n-1)a_n)$ is defined for $x > 0$.

Theorem 1.2 below is more general than Theorem 1.1. It is also a generalization of a result in Bustoz and Ismail [9] which corresponds to the case $n = 2$ in Theorem 1.2. The Ismail-Bustoz result is that the function

$$(1.2) \quad g(x) = \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)}$$

is completely monotonic for any $a, b \geq 0$.

Theorem 1.2. *For any $a_k > 0$ ($k = 1, \dots, n$), define*

$$(1.3) \quad F_n(x) = \frac{\Gamma(x) \prod_{k=1}^{\lfloor n/2 \rfloor} \left[\prod_{\mathbf{m} \in P_{n,2k}} \Gamma(x + \sum_{j=1}^{2k} a_{m_j}) \right]}{\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \left[\prod_{\mathbf{m} \in P_{n,2k-1}} \Gamma(x + \sum_{j=1}^{2k-1} a_{m_j}) \right]}.$$

Then $F_n(x)$ is logarithmically completely monotonic. Any product of functions of the type (1.3) with different parameters a_k is logarithmically completely monotonic as well.

In addition, note that $F_n(x) = F_{n-1}(x)/F_{n-1}(x+a_n)$ if F_n and F_{n-1} are defined in (1.3) by the same parameters a_k . Also $\lim_{a_n \rightarrow \infty} F_n(x) = F_{n-1}(x)$, $n > 2$.

Here are some consequences of Theorem 1.2.

Corollary 1.3. *For $a > 0$ and $n = 1, 2, \dots$, let*

$$(1.4) \quad L_n(x) = \frac{\prod_{k=0}^{\lfloor n/2 \rfloor} \Gamma_{(2k)}^{(n)}(x + 2ka)}{\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \Gamma_{(2k-1)}^{(n)}(x + (2k-1)a)}.$$

Then $L_n(x)$ is logarithmically completely monotonic.

Corollary 1.4. *The function*

$$(1.5) \quad \frac{\Gamma(x)\Gamma(x+\alpha+\beta)\Gamma(x+\lambda+\alpha)\Gamma(x+\lambda+\beta)}{\Gamma(x+\alpha)\Gamma(x+\beta)\Gamma(x+\lambda)\Gamma(x+\lambda+\alpha+\beta)}$$

with nonnegative parameters α, β, λ is logarithmically completely monotonic.

Corollary 1.5. *Let*

$$(1.6) \quad f(x) = \frac{\Gamma(x)\Gamma(x+\alpha+\beta+\gamma+\delta)}{\Gamma(x+\lambda+\alpha+\beta+\gamma+\delta)\Gamma(x+\lambda)} \times \frac{\Gamma(x+\lambda+\alpha+\gamma)\Gamma(x+\lambda+\beta+\delta)}{\Gamma(x+\alpha+\delta)\Gamma(x+\beta+\gamma)}.$$

Then $f(x)$ is logarithmically completely monotonic when $\min\{\lambda, \alpha, \beta, \gamma, \delta\} \geq 0$ and $(\beta - \alpha)(\delta - \gamma) > 0$.

Note that the function $F_n(x)$ reduces to $L_n(x)$ when $a_k = a$ ($k = 1, \dots, n$). Function (1.5) is a function of the type (1.3) for $n = 3$. According to our observation, the limiting case $\lambda \rightarrow \infty$ of the function appearing in (1.5) is function (1.3) for $n = 2$, i.e. function (1.2) of Bustoz and Ismail [9]. Function (1.6) is a product of two functions of the type (1.3), namely: $F_2(x + \alpha + \gamma)$ with $a_1 = \delta - \gamma$ and $a_2 = \beta - \alpha$ (if $\delta > \gamma$ and $\beta > \alpha$), and $F_3(x)$ with $a_1 = \alpha + \gamma$, $a_2 = \beta + \delta$, and $a_3 = \lambda$. There is an interesting connection between function (1.6) and the bi-Hermitian form of the general inequalities for fractional integrals and some values of the gamma function [17] (see also [16, 18] for related results).

The q -gamma function is, [13],

$$(1.7) \quad \Gamma_q(z) := (1 - q)^{1-z} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+z}}, \quad 0 < q < 1.$$

The q -gamma function extends the gamma function in the sense that

$$(1.8) \quad \lim_{q \rightarrow 1^-} \Gamma_q(z) = \Gamma(z).$$

There are few known monotonicity properties of the q -gamma function; see [21] and [22]. One of the results proved in Ismail and Muldoon's work [22] is that both

$$(1.9) \quad \frac{\prod_{k=1}^n \Gamma_q(x + a_k)}{\Gamma_q^n(x + \bar{a})}, \quad \bar{a} := \frac{1}{n} \sum_{k=1}^n a_k,$$

and

$$(1.10) \quad \frac{\Gamma_q^{n-1}(x) \Gamma_q(x + a_1 + a_2 + \dots + a_n)}{\prod_{k=1}^n \Gamma_q(x + a_k)}$$

are logarithmically completely monotonic for $0 < q \leq 1$ and $a_k > 0$. In Section 3 we extend Theorems 1.1 and 1.2 to the q -gamma function and generalize the results from [22].

2. PROOFS OF THEOREMS 1.1 AND 1.2

A basic tool used in the proofs of Theorems 1.1 and 1.2 is the integral representation

$$(2.1) \quad \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^1 \frac{1 - t^{z-1}}{1 - t} dt, \quad \Re z > 0,$$

where γ is the Euler constant, [11, p. 16].

Lemma 2.1. *Let α_k and β_k ($k = 1, \dots, n$) be real numbers such that $\sum_{k=1}^n \alpha_k = 0$ and $\beta_k \geq 0$ for all k . Then*

$$(2.2) \quad u(x) = \prod_{k=1}^n \Gamma^{\alpha_k}(x + \beta_k)$$

is logarithmically completely monotonic if and only if

$$(2.3) \quad v(t) = \sum_{k=1}^n \alpha_k t^{\beta_k} \geq 0$$

for all $t \in (0, 1]$.

Note that functions of the form (2.3) are called Müntz polynomials. Also note that Lemma 2.1 can be generalized for functions of the type

$$u(x) = \exp \left\{ \int_E \log \Gamma[x + \beta(y)] d\alpha(y) \right\},$$

where $\alpha(y)$ is a finite measure on a set E , $\alpha(E) = 0$, and a nonnegative-valued function $\beta(y)$ satisfies the following condition: $\int_E t^{\beta(y)} d\alpha(y) \geq 0$ for all $t \in (0, 1]$. In the next section a similar remark can be made on a generalization of Lemma 3.1.

Proof of Lemma 2.1. Since $u(x) > 0$ on $(0, \infty)$ we let $h(x) = -\log u(x)$. Clearly the integral representation (2.1) implies

$$(2.4) \quad h'(x) = \int_0^1 \frac{t^{x-1}}{1-t} v(t) dt.$$

□

Proof of Theorem 1.1. We use Lemma 2.1. The function $v(t)$ defined by (2.3) equals

$$\sum_{\sigma \in E_n} t^{a_{\sigma(2)} + 2a_{\sigma(3)} + \dots + (n-1)a_{\sigma(n)}} - \sum_{\sigma \in O_n} t^{a_{\sigma(2)} + 2a_{\sigma(3)} + \dots + (n-1)a_{\sigma(n)}}.$$

Hence

$$v(t) = \sum_{\sigma \in S_n} \text{sign}(\sigma) t^{a_{\sigma(2)}} t^{2a_{\sigma(3)}} \dots t^{(n-1)a_{\sigma(n)}}$$

or

$$v(t) = \mathbf{V}(t^{a_1}, t^{a_2}, \dots, t^{a_n}),$$

where \mathbf{V} stands for a Vandermonde determinant. Therefore

$$v(t) = \prod_{1 \leq j < k \leq n} [t^{a_k} - t^{a_j}] \geq 0$$

for $t \in (0, 1]$. □

Proof of Theorem 1.2. In this case

$$v(t) = \sum_{k=0}^n (-1)^k \sum_{\mathbf{m} \in P_{n,k}} t^{\sum_{j=1}^k a_{m_j}}.$$

Thus

$$v(t) = 1 - \sum_{j=1}^n t^{a_j} + \sum_{\mathbf{m} \in P_{n,2}} t^{a_{m_1} + a_{m_2}} - \dots + (-1)^n t^{\sum_{j=1}^n a_j},$$

and we have $v(t) = \prod_{k=1}^n (1 - t^{a_k}) \geq 0$. □

3. q -GAMMA FUNCTION RESULTS

In Lemma 3.1 we use functions of the form (2.3) as they have already been used in Lemma 2.1.

Lemma 3.1. *Let α_k and β_k ($k = 1, \dots, n$) be real numbers such that $\sum_{k=1}^n \alpha_k = 0$ and $\beta_k \geq 0$ for all k . Then*

$$(3.1) \quad u(x, q) = \prod_{k=1}^n \Gamma_q^{\alpha_k}(x + \beta_k)$$

is logarithmically completely monotonic in x ($q \in (0, 1)$) if

$$v(t) = \sum_{k=1}^n \alpha_k t^{\beta_k} \geq 0$$

for all $t \in (0, 1]$.

The analogue of (2.1) to be used here is

$$(3.2) \quad \frac{\partial_x \Gamma_q(x)}{\Gamma_q(x)} = -\log(1 - q) + \log q \sum_{m=0}^{\infty} \frac{q^{m+x}}{1 - q^{m+x}},$$

which follows from (1.7).

Proof of Lemma 3.1. Clearly

$$\begin{aligned} \frac{\partial_x u(x, q)}{u(x, q)} &= \sum_{k=1}^n \alpha_k \frac{\partial_x \Gamma_q(x + \beta_k)}{\Gamma_q(x + \beta_k)} \\ &= \log q \sum_{k=1}^n \alpha_k \sum_{m=0}^{\infty} \frac{q^{m+x+\beta_k}}{1 - q^{m+x+\beta_k}} \\ &= \log q \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} q^{(m+x)l} v(q^l) \leq 0, \end{aligned}$$

which is a discrete analogue of the integral representation arising in the proof of Lemma 2.1. The rest of the proof is obvious. □

The functions in (1.9) and (1.10) satisfy the condition of Lemma 3.1. Indeed, in the first case $v(t) = \sum_{k=1}^n t^{a_k} - nt^{\bar{a}}$, and one can use the arithmetic mean–geometric mean inequality to show that $v(t)$ is not negative on $t \in (0, 1]$. In the second case $v(t) = n - 1 + t^{\sum_{k=1}^n a_k} - \sum_{k=1}^n t^{a_k}$. The nonnegativity of this function on $t \in (0, 1]$ is a consequence of two facts: $v'(t) < 0$ and $v(1) = 0$.

Theorem 3.2. *Let $a_1 > a_2 > \dots > a_n > 0$ and define*

$$(3.3) \quad F(x, q) = \frac{\prod_{\sigma \in E_n} [\Gamma_q(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \dots + (n - 1)a_{\sigma(n)})]}{\prod_{\sigma \in O_n} [\Gamma_q(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \dots + (n - 1)a_{\sigma(n)})]}.$$

Then $F(x - a_2 - 2a_3 - \dots - (n - 1)a_n, q)$ is a logarithmically completely monotonic function of x .

Proof of Theorem 3.2. The proof is similar to the proof of Theorem 1.1, but now we use Lemma 3.1. Clearly the function $v(t)$ in this case equals the Vandermonde determinant $\mathbf{V}(t^{a_1}, t^{a_2}, \dots, t^{a_n})$. Thus

$$v(t) = \prod_{1 \leq j < k \leq n} [t^{a_k} - t^{a_j}] \geq 0. \quad \square$$

Theorem 3.3. Let $a_k > 0$ ($k = 1, \dots, n$). Define

$$(3.4) \quad F_n(x, q) = \frac{\Gamma_q(x) \prod_{k=1}^{[n/2]} \left[\prod_{\mathbf{m} \in P_{n,2k}} \Gamma_q(x + \sum_{j=1}^{2k} a_{m_j}) \right]}{\prod_{k=1}^{[(n+1)/2]} \left[\prod_{\mathbf{m} \in P_{n,2k-1}} \Gamma_q(x + \sum_{j=1}^{2k-1} a_{m_j}) \right]}.$$

Then $F_n(x, q)$ is a logarithmically completely monotonic function of x . Any product of functions (3.4) with different parameters a_k is a logarithmically completely monotonic function of x .

Proof of Theorem 3.3. The proof is similar to the proof of Theorem 1.2. We use Lemma 3.1. The function $v(t) = \prod_{k=1}^n (1 - t^{a_k}) \geq 0$. □

Corollary 3.4. For $a > 0$ and $n = 1, 2, \dots$, let

$$(3.5) \quad L_n(x, q) = \frac{\prod_{k=0}^{[n/2]} \Gamma_q^{\binom{n}{2k}}(x + 2ka)}{\prod_{k=1}^{[(n+1)/2]} \Gamma_q^{\binom{n}{2k-1}}(x + (2k - 1)a)}.$$

Then $L_n(x, q)$ is a logarithmically completely monotonic function of x .

Corollary 3.5. The function

$$(3.6) \quad \frac{\Gamma_q(x)\Gamma_q(x + \alpha + \beta)\Gamma_q(x + \lambda + \alpha)\Gamma_q(x + \lambda + \beta)}{\Gamma_q(x + \alpha)\Gamma_q(x + \beta)\Gamma_q(x + \lambda)\Gamma_q(x + \lambda + \alpha + \beta)}$$

with nonnegative parameters α, β, λ is logarithmically completely monotonic in x .

Corollary 3.6. Let

$$(3.7) \quad f(x, q) = \frac{\Gamma_q(x)\Gamma_q(x + \alpha + \beta + \gamma + \delta)}{\Gamma_q(x + \lambda + \alpha + \beta + \gamma + \delta)\Gamma_q(x + \lambda)} \times \frac{\Gamma_q(x + \lambda + \alpha + \gamma)\Gamma_q(x + \lambda + \beta + \delta)}{\Gamma_q(x + \alpha + \delta)\Gamma_q(x + \beta + \gamma)}.$$

Then $f(x, q)$ is logarithmically completely monotonic in x when $\min\{\lambda, \alpha, \beta, \gamma, \delta\} \geq 0$ and $(\beta - \alpha)(\delta - \gamma) > 0$.

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