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GROMOV HYPERBOLICITY OF THE j_G AND $\tilde{j}_G$ METRICS

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Abstract. In this note it is shown that the $\tilde{j}_G$ metric is always Gromov hyperbolic, but that the j_G metric is Gromov hyperbolic if and only if G has exactly one boundary point. As a corollary we get a new proof for the fact that the quasihyperbolic metric is Gromov hyperbolic in uniform domains.

In this note it is shown that the metric

$$\tilde{j}_G(x,y) = \frac{1}{2}\log\left(1+\frac{|x-y|}{\delta(x)}\right)\left(1+\frac{|x-y|}{\delta(y)}\right)$$

is always Gromov hyperbolic, but that the metric

$$j_G(x,y) = \log\left(1+\frac{|x-y|}{\min\{\delta(x),\delta(y)\}}\right)$$

is Gromov hyperbolic if and only if G has exactly one boundary point. Here and in the rest of this paper $G \subsetneq \mathbb{R}^n$ is open, $x, y \in G$ and $\delta(x)$ is short for $d(x, \partial G)$. The proofs of these somewhat surprising results are completely elementary. As a corollary we get a new proof for the fact that the quasihyperbolic metric is Gromov hyperbolic in uniform domains.

The $\tilde{j}_G$ and j_G metrics are well known in geometric function theory. The former was introduced by F. Gehring and B. Osgood [6]. The latter one is a modification of this metric of M. Vuorinen's [12]. Note that $\tilde{j}_G \leqslant j_G \leqslant 2\tilde{j}_G$ for every domain G. These metrics have been applied in many settings; see e.g. [13]. For instance, uniform domains can be characterized as those domains in which the j_G metric and the quasihyperbolic metric are comparable [6, Corollary 1].

In recent years, several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic, thus tapping into the results proved in the general setting (e.g. [4, 5, 7, 8]). Some specific examples of results are that the quasihyperbolic [2, 11] and the Klein-Hilbert metric [1, 9] are Gromov hyperbolic (under particular conditions on the domain of definition).

Although Gromov hyperbolicity can be defined in completely general metric spaces, most investigators have restricted their attention to geodesic metric spaces (but see e.g. [3, 10] for the general case). It is well known that there are some fundamental differences between these cases. For instance, Gromov hyperbolicity

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is preserved in geodesic spaces under rough quasi-isometries, i.e. for metrics (G,d) and (G,d') related by

$$\tfrac{1}{a}d(x,y)-b\leqslant d'(x,y)\leqslant ad(x,y)+b,\quad \forall x,y\in G$$

(e.g. [10, Theorem 3.17]). In non-geodesic spaces, on the other hand, Gromov hyperbolicity is not in general preserved under this condition—rather we have to require that our transformation be a rough isometry, i.e. satisfy the previous inequality for $a=1$. This particular difference serves to explain the somewhat counterintuitive results in this paper.

For $a,b\in\mathbb{R}$ we denote $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$. Let $x,y,w\in G$ be points in the metric space (G,d). We define the *Gromov product*

$$(x|y)_w=\tfrac{1}{2}\big(d(x,w)+d(y,w)-d(x,y)\big).$$

The space (G,d) is said to be *Gromov hyperbolic* (with base point $w\in G$) if there exists a constant $\delta\geqslant 0$ such that

$$(x|z)_w+\delta\geqslant (x|y)_w\wedge(y|z)_w$$

for all $x,y,z\in G$. If this relation holds for one $w\in G$, then it holds for every $w\in G$ with twice as large δ. This allows us to cast the definition in a more symmetric form: (G,d) is Gromov hyperbolic if and only if there exists a constant $\delta>0$ such that

$$d(x,z)+d(y,w)\leqslant (d(x,w)+d(y,z))\vee(d(x,y)+d(z,w))+2\delta$$

for all points $x,y,z,w\in G$. In the proofs we will use the latter form of the inequality.

In geodesic metric spaces this definition is equivalent to a more intuitive one based on Rips triangles (see e.g. [10, Theorems 2.34 & 2.35] in the Banach space setting) but in general we have to stick to this one. For our simple metrics, though, it turns out to be quite possible to use the definition directly.

We are now ready to prove the claims about the $\tilde{j}_G$ and j_G metrics.

Theorem 1. *The metric $\tilde{j}_G$ is Gromov hyperbolic for every domain $G\subsetneq\mathbb{R}^n$. The constant satisfies $\delta\leqslant\log 3$.*

Proof. The inequality that we have to prove is

$$\tilde{j}_G(x,z)+\tilde{j}_G(y,w)\leqslant(\tilde{j}_G(x,w)+\tilde{j}_G(y,z))\vee(\tilde{j}_G(x,y)+\tilde{j}_G(z,w))+\log 9.$$

After multiplying by 2 and taking the exponential function of both sides this becomes

$$\begin{aligned}\frac{1}{81}&\left(1+\frac{|x-z|}{\delta(x)}\right)\left(1+\frac{|x-z|}{\delta(z)}\right)\left(1+\frac{|y-w|}{\delta(y)}\right)\left(1+\frac{|y-w|}{\delta(w)}\right)\\ &\leqslant\left(1+\frac{|x-w|}{\delta(x)}\right)\left(1+\frac{|x-w|}{\delta(w)}\right)\left(1+\frac{|y-z|}{\delta(y)}\right)\left(1+\frac{|y-z|}{\delta(z)}\right)\\ &\quad\vee\left(1+\frac{|x-y|}{\delta(x)}\right)\left(1+\frac{|x-y|}{\delta(y)}\right)\left(1+\frac{|z-w|}{\delta(z)}\right)\left(1+\frac{|z-w|}{\delta(w)}\right).\end{aligned}$$

We assume by symmetry that $|x-z|\geqslant|y-w|$. By swapping the letters x and z and/or y and w we may further assume that $|x-w|\leqslant\min\{|y-z|,|x-y|,|z-w|\}$. In order to simplify the equations we introduce some abbreviations (see Figure 1):

$$|x-w|=a,|x-y|=b,|y-z|=c,|z-w|=d,|y-w|=e,|x-z|=f.$$

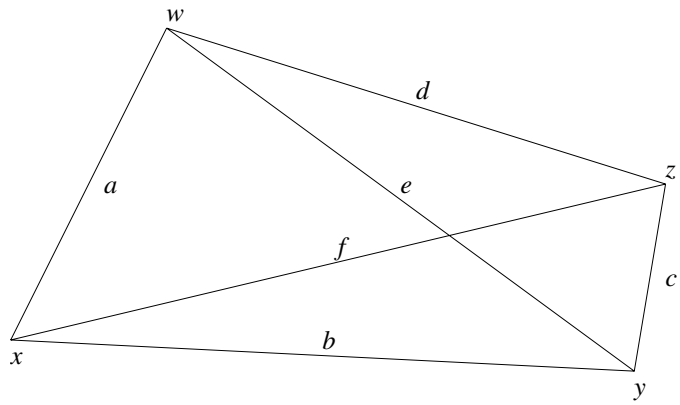

FIGURE 1.

Then we further write x instead of $1/\delta(x)$, etc. The previous inequality becomes

$$\begin{aligned}\tfrac{1}{81}(1+fx)(1+fz)(1+ey)(1+ew) &\leqslant (1+ax)(1+aw)(1+cy)(1+cz)\\ &\quad\vee (1+bx)(1+by)(1+dz)(1+dw).\end{aligned} \tag{1}$$

Suppose first that $f \leqslant 3(a \wedge c)$, and recall that $e \leqslant f$. Then

$$(1+ax)/(1+fx) \geqslant (1+ax)/(1+3ax) \geqslant 3^{-1}.$$

A similar inequality holds for each of the other terms before the $\vee$ in inequality (1), hence we get

$$(1+fx)(1+fz)(1+ey)(1+ew) \leqslant 3^4(1+ax)(1+aw)(1+cy)(1+cz).$$

If instead $f \leqslant 3(b \wedge d)$, then we get a similar inequality involving the terms after the $\vee$ in inequality (1).

So only the case when $f \geqslant 3[(a \wedge c) \vee (b \wedge d)]$ remains. We already assumed that $a \leqslant c$. By the triangle inequality we find that $f \leqslant a+d \leqslant f/3+d$, hence $d \geqslant 2f/3$. So $f \ngeqslant 3(a \vee d)$, which means that $b < d$ and $f \geqslant 3(a \vee b)$. The inequality $d \geqslant 2f/3$ gives us

$$(1+dz) \geqslant 2(1+fz)/3.$$

So inequality (1) follows if we show that

$$(1+fx)(1+ey)(1+ew) \leqslant 54(1+bx)(1+by)(1+dw).$$

We have $e \leqslant a+b \leqslant 2b$, hence $(1+ey) \leqslant 2(1+by)$. Thus the inequality reduces to

$$(1+fx)(1+ew) \leqslant 27(1+bx)(1+dw). \tag{2}$$

By the triangle inequality we have

$$w^{-1} = \delta(w) \leqslant |x-w| + \delta(x) = a + x^{-1},$$

hence $w \geqslant x(1+ax)^{-1}$. Since $d \geqslant f-a \geqslant 3b-a \geqslant 2b \geqslant e$ we find that the function

$$w \mapsto \frac{1+ew}{1+dw}$$

is decreasing, hence

$$\frac{1+ew}{1+dw} \geqslant \frac{1+ex(1+ax)^{-1}}{1+dx(1+ax)^{-1}} = \frac{1+(a+e)x}{1+(a+d)x}.$$

Using this in (2) gives

$$(1+fx)(1+(a+e)x) \leqslant 27(1+bx)(1+(a+d)x).$$

But $a+e \leqslant 2a+b \leqslant 3b$ and $f \leqslant 2(a+d)$, so this inequality is clear. □

Remark 2. Note that the previous proof in fact uses only that the Euclidean metric is a distance. Thus the result generalizes directly to the Banach space setting.

Theorem 3. *Let $G \subset \mathbb{R}^n$ be an open set. Then j_G is Gromov hyperbolic if and only if G has a single boundary point.*

Proof. Suppose first that G has at least two boundary points. Let u and v be two distinct spherically accessible boundary points. Let $\epsilon > 0$ and let u, x and w be collinear (in this order and with $x, w \in G$) and suppose $|u-x| = \epsilon^2$ and $|x-w| = \epsilon$ (this is possible for ϵ small enough). Similarly, let v, y and z be collinear (in this order and with $y, z \in G$) and suppose $|v-y| = \epsilon^2$ and $|y-z| = \epsilon$. Then when ϵ is small compared to $\min\{1, |u-v|\}$, we have

$$\begin{aligned} j_G(x,z) = \log\left(1+\frac{|x-z|}{\epsilon^2}\right) &\geqslant \log\left(\frac{|u-v|-|x-u|-|z-v|+\epsilon^2}{\epsilon^2}\right) \\ &\geqslant \log\left(\frac{|u-v|-2\epsilon}{\epsilon^2}\right). \end{aligned}$$

Using similar estimates for the other points, we get

$$\begin{aligned} j_G(x,z)+j_G(y,w) &\geqslant 2\log\left(\frac{|u-v|-2\epsilon}{\epsilon^2}\right), \\ j_G(x,w)+j_G(y,z) &= 2\log\left(1+\frac{\epsilon}{\epsilon^2}\right), \\ j_G(x,y)+j_G(z,w) &\leqslant \log\left(\frac{|u-v|+3\epsilon^2}{\epsilon^2}\right)+\log\left(\frac{|u-v|+3\epsilon}{\epsilon}\right). \end{aligned}$$

For small ϵ we find that

$$\begin{aligned} &\max\left\{j_G(x,w)+j_G(y,z), j_G(x,y)+j_G(z,w)\right\} \\ &\qquad \leqslant \log\left(\frac{|u-v|+3\epsilon^2}{\epsilon^2}\right)+\log\left(\frac{|u-v|+3\epsilon}{\epsilon}\right). \end{aligned}$$

Therefore

$$\begin{aligned} &j_G(x,z)+j_G(y,w) - \max\left\{j_G(x,y)+j_G(z,w), j_G(x,w)+j_G(y,z)\right\} \\ &\qquad \geqslant \log\left(\frac{|u-v|-2\epsilon}{\epsilon^2}\right)-\log\left(\frac{|u-v|+3\epsilon}{\epsilon}\right) \to \infty \end{aligned}$$

as $\epsilon \to 0$. Hence j_G is not Gromov hyperbolic.

It remains to consider the case when G has exactly one boundary point. Assume without loss of generality that this boundary point is the origin. Let $x, y, z, w \in G$. To show that $\mathbb{R}^n \setminus \{0\}$ is Gromov hyperbolic with $\delta = \log 3$ we have to show (again applying the exponential function to both sides) that

$$\begin{aligned} \frac{1}{9}\left(1+\frac{|x-z|}{|x|\wedge|z|}\right)\left(1+\frac{|y-w|}{|y|\wedge|w|}\right) &\leqslant \left(1+\frac{|x-w|}{|x|\wedge|w|}\right)\left(1+\frac{|y-z|}{|y|\wedge|z|}\right) \\ &\quad \vee \left(1+\frac{|x-y|}{|x|\wedge|y|}\right)\left(1+\frac{|z-w|}{|z|\wedge|w|}\right). \end{aligned}$$

We always have

$$(|x| \vee |z|) - (|x| \wedge |z|) \leqslant |x - z| \leqslant |x| + |z| \leqslant 2(|x| \vee |z|).$$

Thus

$$\frac{|x| \vee |z|}{|x| \wedge |z|} \leqslant 1 + \frac{|x - z|}{|x| \wedge |z|} \leqslant 3\frac{|x| \vee |z|}{|x| \wedge |z|}.$$

On the other hand, it is clear that

$$\frac{|x| \vee |z|}{|x| \wedge |z|} = \frac{|x|}{|z|} \vee \frac{|z|}{|x|}.$$

So we see that it is enough to show that

$$\left(\frac{|x|}{|z|} \vee \frac{|z|}{|x|}\right)\left(\frac{|y|}{|w|} \vee \frac{|w|}{|y|}\right) \leqslant \left(\frac{|x|}{|w|} \vee \frac{|w|}{|x|}\right)\left(\frac{|y|}{|z|} \vee \frac{|z|}{|y|}\right) \vee \left(\frac{|x|}{|y|} \vee \frac{|y|}{|x|}\right)\left(\frac{|z|}{|w|} \vee \frac{|w|}{|z|}\right).$$

By symmetry we assume without loss of generality that $|x| \leqslant |y| \wedge |z| \wedge |w|$, and by scale invariance that $|x| = 1$. The inequality to prove becomes

$$\frac{|z||y|}{|w|} \vee \frac{|z||w|}{|y|} \leqslant \frac{|w||y|}{|z|} \vee \frac{|w||z|}{|y|} \vee \frac{|y||z|}{|w|} \vee \frac{|y||w|}{|z|}.$$

This is obvious. □

The proof of the following lemma is due to M. Bonk.

Lemma 4. *Let X be a Gromov hyperbolic space and let Y be geodesic. If X and Y are roughly quasi-isometric, then Y is Gromov hyperbolic.*

Proof. By [3, Theorem 4.1] X embeds isometrically into a geodesic Gromov hyperbolic metric space $\tilde{X}$. By composition we have a rough quasi-isometry f from Y to $\tilde{X}$. Fix three points $x, y, z \in Y$ and consider a geodesic triangle T spanned by them. This triangle maps onto a roughly quasigeodesic triangle in $\tilde{X}$. Let $\tilde{T}$ denote a geodesic triangle joining $f(x)$, $f(y)$ and $f(z)$ in $\tilde{X}$.

Choose a point $f(w) \in f(T)$. By geodesic stability (see e.g. [3, Section 3]) there exists a point $\tilde{w}$ on $\tilde{T}$ within bounded distance of w. Since $\tilde{T}$ is a geodesic triangle in a Gromov hyperbolic space there exists a point $\tilde{u}$ on the union of the opposite sides of $\tilde{T}$ within bounded distance of $\tilde{w}$, by the Rips condition. Then using geodesic stability again we find a point $f(u)$ on $f(T)$ which is within bounded distance of $\tilde{u}$. So $f(w)$ and $f(u)$ are within bounded distance from each other. Since f is a rough quasi-isometry, this means that u is within bounded distance of w in Y. Thus Y satisfies the Rips condition, and is Gromov hyperbolic. □

Using the previous result we can easily reprove the Gromov hyperbolicity of the quasihyperbolic metric, which was proven in Euclidean spaces by Bonk, Heinonen and Koskela [2, Theorem 1.11] and in abstract Banach spaces by Väisälä [11, Theorems 2.11 & 3.24].

Corollary 5. *If $G \subset \mathbb{R}^n$ is a uniform domain, then it is Gromov hyperbolic with respect to the quasihyperbolic metric.*

Proof. The quasihyperbolic metric is geodesic, the $\tilde{j}_G$-metric is Gromov hyperbolic and they are bilipschitz equivalent in uniform domains, so the claim follows by the previous lemma. □

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