**HSP ≠ SHPS FOR COMMUTATIVE RINGS WITH IDENTITY**

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**Abstract.** Let \( I, H, S, P, P_s \) be the usual operators on classes of rings: \( I \) and \( H \) for isomorphic and homomorphic images of rings and \( S, P, P_s \) respectively for subrings, direct, and subdirect products of rings. If \( \mathcal{K} \) is a class of commutative rings with identity (and in general of any kind of algebraic structures), then the class \( HSP(\mathcal{K}) \) is known to be the variety generated by the class \( \mathcal{K} \). Although the class \( SHPS(\mathcal{K}) \) is in general a proper subclass of the class \( HSP(\mathcal{K}) \) for many familiar varieties \( HSP(\mathcal{K}) = SHPS(\mathcal{K}) \). Our goal is to give an example of a class \( \mathcal{K} \) of commutative rings with identity such that \( HSP(\mathcal{K}) \neq SHPS(\mathcal{K}) \). As a consequence we will describe the structure of two partially ordered monoids of operators.

1. **Introduction**

The operators we will consider are assumed to be defined on classes of algebras of the same type (e.g. groups, rings, lattices, etc.). Further on \( \mathcal{K} \) will stand for an arbitrary class of algebras of the same type. Operators \( X, Y \) are equal, in symbols \( X = Y \), if and only if \( X(\mathcal{K}) = Y(\mathcal{K}) \) for every \( \mathcal{K} \). On the set of operators we define a partial order by \( X \leq Y \) if and only if \( X(\mathcal{K}) \subseteq Y(\mathcal{K}) \) for all \( \mathcal{K} \), and the composition of operators \( X \) and \( Y \) by \( XY(\mathcal{K}) = X(Y(\mathcal{K})) \), for an arbitrary \( \mathcal{K} \). It is easy to see that the composition of operators is an associative operation, i.e., \( X(YZ) = (XY)Z \) for any class operators \( X, Y, Z \). An operator \( X \) is a closure operator if it is extensive \( (\mathcal{K} \subseteq X(\mathcal{K})) \), monotone \( (\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow X(\mathcal{K}_1) \subseteq X(\mathcal{K}_2)) \), and idempotent \( (X(X(\mathcal{K})) = X(\mathcal{K})) \).

Given a class \( \mathcal{K} \) we let \( I(\mathcal{K}) \) and \( H(\mathcal{K}) \) denote the classes of all isomorphic images and the class of all homomorphic images of algebras in \( \mathcal{K} \), respectively. Let \( S(\mathcal{K}) \), denote the class of all algebras isomorphic to subalgebras of algebras in \( \mathcal{K} \), and let \( P(\mathcal{K}), P_u(\mathcal{K}), P_s(\mathcal{K}) \) denote the classes of all algebras isomorphic to direct, ultra and subdirect products of algebras in \( \mathcal{K} \) respectively.

\( I, H, S, P, P_u, P_s \) are closure operators and all the composites (e.g. \( HS, P_uH, HSP, SHHP \)) can be thought of as forming a monoid \( \mathcal{M} \) with the composition of operators as multiplication and \( I \) as the identity element. We will be interested in two special cases. Namely, we will consider the monoid \( \mathcal{M} \) generated by the operators \( H, S, P, P_s \), as well as the monoid \( \mathcal{M}_s \) generated by the operators \( H, S, P, P_s \).

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The structure of $\mathcal{M}$, which is in fact a partially ordered monoid (in the remainder of the text po-monoid) is determined by Pigozzi in [8]. There are at most eighteen possible composites one can get starting with $I, H, S, P$. The corresponding partial ordering is shown in Figure 1.

![Diagram of partial ordering of $\mathcal{M}$]

**Figure 1.** The partial ordering of $\mathcal{M}$

The structure of $\mathcal{M}_s$ is determined in [10]. In this case there are at most 22 operators one can get as composites starting with $I, H, S, P, P_s$. The corresponding partial ordering is shown in Figure 2.

We will consider commutative rings with identity as structures in the language $R = \langle R, +, \cdot, -, 0, 1 \rangle$. Let $R_c$ denote the variety of all commutative rings with identity. When we restrict domains of the operators in $\mathcal{M}$ and $\mathcal{M}_s$ to be subclasses of $R_c$, we get the so-called monoids of the variety of commutative rings with identity generated by the corresponding operators. The monoid of the variety $R_c$ generated by $H, S, P$ is a homomorphic image of $\mathcal{M}$, and it is denoted by $\mathcal{M}(R_c)$. Similarly, we will consider $\mathcal{M}_s(R_c)$ the monoid of the variety $R_c$ generated by $H, S, P, P_s$. The monoids of operators $H, S, P$ are described for the following varieties: groups in [7], metabelian groups in [1], commutative semigroups in [8], pseudocomplemented lattices in [5], implicative semilattices in [6], varieties generated by a quasiprimal algebra in [9]. The account of the various results is also given in [4].

For many varieties there is actually a great deal of collapsing among the operators which implies that for such a variety its monoid of operators $H, S, P$ is a proper...
Figure 2. The partial ordering of $\mathcal{M}_s$

homomorphic image of $\mathcal{M}$. For example, if $A$ is the variety of Abelian groups, then for every $K \subseteq A$ we have $HS(K) = SH(K)$, i.e., $HS = SH$ on $A$. Hence in case of Abelian groups we have $HSP = SHPS$. For commutative rings we will see that $HSP \neq SHPS$. So, on one hand, we have that in general $HSP \neq SHPS$, but if we restrict our attention to the Abelian groups we have $HSP = SHPS$. On the other hand, there are relations which are true in general, such as $HSP = HP_s$ and $SPHS = P_s HS$.

2. Results on commutative rings with identity

We recall the following result from [8].

Theorem 2.1 (see [8 Theorem 4]). Let $\mathcal{V}$ be a variety. A necessary and sufficient condition for the monoid of operators $H, S, P$ of the variety $\mathcal{V}$ to be isomorphic to $\mathcal{M}$ is that there exist classes $K_1, K_2 \subseteq \mathcal{V}$ satisfying the non-inclusions

\begin{align*}
(1) & \quad HSP(K_1) \nsubseteq SHPS(K_1), \\
(2) & \quad HP(K_2) \nsubseteq SPHS(K_2).
\end{align*}
Now we are ready to show

**Theorem 2.2.** There exists a class $\mathcal{K}$ of commutative rings with identity satisfying the non-inclusions (1) and (2). Consequently, the monoid of operators $H$, $S$, $P$ is full, i.e., $\mathcal{M}(\mathcal{R}_c) \cong M$.

**Proof.** Let $\mathcal{K} = \{\mathbb{Z}_p \mid p \in P\}$ where $P$ is the set of all primes. Let $\mathcal{U}$ be a nonprincipal ultrafilter over $P$ and let $F = \prod_{p \in P} \mathbb{Z}_p/\mathcal{U}$. Since $\mathbb{Z}_p$ is a field for $p \in P$, by Loš’s theorem $F$ is also a field. So we have $F \in P_u(\mathcal{K}) \subseteq HP(\mathcal{K})$. On the other hand, since the only subrings and homomorphic images of $\mathbb{Z}_p$’s are trivial (either $\mathbb{Z}_p$ or a zero ring up to isomorphism) we have $SPHS(\mathcal{K}) = SP(\mathcal{K})$. The ring $F$ being a field is simple and therefore subdirectly irreducible. Hence, if $F$ were in $SPHS(\mathcal{K})$, then $F$ would be in $S(\mathcal{K})$ which is a contradiction. So, $F \in HP(\mathcal{K})$ and $F \notin SPHS(\mathcal{K})$ which shows (2).

To prove (1) we need the following results.

**Claim 2.3.** $HSP(\mathcal{K}) = \mathcal{R}_c$.

**Claim 2.4.** If $R$ is a ring such that $Char(R) = 2$ and $R \in SHP(\mathcal{K})$, then $R \in HSP(\mathbb{Z}_2)$.

To prove Claim 2.3 suppose that $HSP(\mathcal{K})$ is a proper subvariety of $\mathcal{R}_c$. Then there is an identity $p = q$ where $p, q \in Z[x_1, \ldots, x_l]$ for some $l$ such that $\mathcal{R}_c \models p = q$ and $HSP(\mathcal{K}) \models p = q$ or equivalently there is a polynomial $0 \neq f \in Z[x_1, \ldots, x_l]$ such that $\mathcal{R}_c \models f = 0$ and $HSP(\mathcal{K}) \models f = 0$. Since $\mathbb{Z}_p \models f = 0$ for all $p \in P$ we have $\prod_{p \in P} \mathbb{Z}_p/\mathcal{U} \models f = 0$. Therefore we have the field $\prod_{p \in P} \mathbb{Z}_p/\mathcal{U}$ of characteristic zero satisfying $f = 0$ which implies that $f$ is to be the zero polynomial. This contradiction proves Claim 2.3.

To prove Claim 2.4 assume that $R \in SHP(\mathcal{K})$ and let $R \leq M \in HP(\mathcal{K})$. Since $Char(R) = 2$ and our rings are with identity, we will have $Char(M) = 2$. Let $\alpha : \prod_{i \in I} R_i \rightarrow M$ be an epimorphism, where $R_i \in \mathcal{K}$ for all $i \in I$. Let us denote by $J = \{i \in I \mid R_i = \mathbb{Z}_2\}$ and let $\mathbf{R}_i = R_i$ if $i \notin J$ and $\mathbf{R}_i = 0$, a zero ring, if $i \in J$. We claim that $\prod_{i \in I} \mathbf{R}_i \subseteq Ker(\alpha)$. If $x \in \prod_{i \in I} \mathbf{R}_i$, then $\alpha(2x) = 2\alpha(x) = 0$ and hence $2x \in Ker(\alpha)$. Since $R_i \in \mathcal{K} - \{\mathbb{Z}_2\}$ are fields, in $R_i$ we will have $2^{-1}$, the inverse of 2. Now define an element $2^{-1} \in \prod_{i \in I} R_i$ as follows: if $i \notin J$, then $2^{-1}(i) = 0$, otherwise $2^{-1}(i) = 2^{-1}$. Hence $2^{-1}(2x) = x \in Ker(\alpha)$ because $Ker(\alpha)$ is an ideal. Therefore, $\prod_{i \in I} \mathbf{R}_i \subseteq Ker(\alpha)$. Since $\prod_{i \in I} \mathbf{R}_i$ is also an ideal of the ring $\prod_{i \in I} R_i$, we have

$$M \cong \prod_{i \in I} R_i/\text{Ker}(\alpha) \cong (\prod_{i \in I} R_i/\prod_{i \in I} \mathbf{R}_i)/(\text{Ker}(\alpha)/\prod_{i \in I} \mathbf{R}_i)$$

and hence

$$M \in H(\prod_{i \in I} R_i/\prod_{i \in I} \mathbf{R}_i) = H(\prod_{i \in I} (R_i/\mathbf{R}_i)) = H(\mathbb{Z}_2) \subseteq HP(\mathbb{Z}_2).$$

Now $R \in S(M)$ implies $R \in SHP(\mathbb{Z}_2) = HSP(\mathbb{Z}_2)$ which proves the claim.

For $n \geq 2$, the Galois field $GF(2^n) \in HSP(\mathcal{K})$, and if $GF(2^n)$ were in $SHPS(\mathcal{K})$, then $GF(2^n) \in HSP(\mathbb{Z}_2)$. Since $HSP(\mathbb{Z}_2) \models x^2 = x$, we would have $GF(2^n)$ is idempotent for $n \geq 2$ which is not true. Finally, we have that $GF(2^n) \in HP(\mathcal{K})$ for $n \geq 2$, and $GF(2^n) \notin SHPS(\mathcal{K})$. □
To prove that the monoid of operators $H, S, P, P_s$ of the variety $R_c$ is full we will use the following.

**Lemma 2.5** (see [10] Lemma 3.1). Let $K$ be any variety. Then $M_s(K) \cong M_s$ if and only if there exist classes of algebras $K_1, K_2 \subseteq K$ satisfying the non-inclusions

1. $HSP(K_1) \not\subseteq SHPS(K_1)$,
2. $SHP(K_2) \not\subseteq P_sHPS(K_2)$.

Now we can prove

**Theorem 2.6.** There exist classes $K_1$ and $K_2$ of commutative rings with identity satisfying the non-inclusions (3) and (4). Consequently, the monoid of operators $H, S, P, P_s$ is full, i.e., $M_s(R_c) \cong M$.

**Proof.** We have already showed the non-inclusion (3) in Theorem 2.2. To show (4) we will use the following class: let $P^* = \{ p \in P \mid -1 \text{ is a square in } Z_p \}$ and let $K^* = \{ Z_p \mid p \in P^* \}$. The set $P^*$ is infinite because odd primes $p$ such that $p \equiv 1 \mod 4$ satisfy $p = a^2 + b^2$ for some $a$ and $b$ and hence $1 + (b/a)^2 = 0$ in $Z_p$. Since $P^*$ is an infinite subset of primes $HP(K^*)$ contains fields of characteristic zero. This implies that $Q \in SHP(K^*)$, where $Q$ is the field of rationals. On the other hand, $P_sHPS(K^*) = P_sHP(K^*)$ and the property “$-1$ is a square” is preserved under direct products and homomorphic images. Now, if $F$ is a field and $F \in P_sHP(K^*)$, then $F \in HP(K^*)$ because $F$ being a field is a subdirectly irreducible ring. So, it will also hold that “$-1$ is a square in $F$” which implies that $Q$ cannot be in $P_sHPS(K^*) = P_sHP(K^*)$. This shows the non-inclusion (4) and completes the proof.

**Appendix**

The following result was obtained by the second author and added to the paper on October 25, 2004.

The aim of this appendix is to give a simpler example for the inequality $HSP \neq SHPS$. To show that $HSP \neq SHPS$ in Theorem 2.2 we used an infinite class of finite rings. An advantage of that example is that it simultaneously showed both non-inclusions required in Theorem 2.1. We will prove that one can take the ring $Z_8$ to show that $HSP(Z_8) \neq SHPS(Z_8)$.

**Proposition 2.7.** If $Z_8 = (Z_8, +, \cdot, -, 0, 1)$ denotes the commutative ring with identity of integers mod 8, then $HSP(Z_8) \neq SHPS(Z_8)$.

**Proof.** The Variety generated by $Z_8$ is defined by the following identities:

1. $x^2 - x(y^2 - y)(z^2 - z) = 0$,
2. $(x^2 - x)^2 = 2(x^2 - x)$.

This variety is residually large and Willard described in [11] all countable subdirectly irreducibles in $V = HSP(Z_8)$. We will use the fact that $V$ has arbitrary large finite subdirectly irreducibles. Let $R \in V$ be a finite subdirectly irreducible ring such that $|R| > 8$. If $R \in SHPS(Z_8) = SHP(Z_8)$, then $R \in SHP_{fin}(Z_8)$, where $P_{fin}$ denotes the operator of taking finite products. Therefore, $R \leq (Z_8)^n/I$ for some $n \in N$, and some ideal $I$ of $(Z_8)^n$. Since commutative rings with identity have the Fraser Horn property (FHP for short) we have that $I = I_1 \times I_2 \times \cdots \times I_n$.
for some ideals $I_1, \ldots, I_n$ of $\mathbb{Z}_8$. This gives us $(\mathbb{Z}_8)^n/I \cong \mathbb{Z}_8/I_1 \times \cdots \times \mathbb{Z}_8/I_n \in P_{fin} H(\mathbb{Z}_8) = P_{fin}((\mathbb{Z}_8, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_1))$. Finally, $R \in SP_{fin}((\mathbb{Z}_8, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_1))$ and $R$ being subdirectly irreducible give us $R \in S((\mathbb{Z}_8, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_1))$. Contradiction, since we assumed $|R| > 8$.

The previous proposition can easily be generalized to

\textbf{Proposition 2.8.} Let $A$ be a finite algebra. If the variety $HSP(A)$ satisfies the FHP and contains a finite subdirectly irreducible algebra $B$ such that $|B| > |A|$, then $HSP(A) \neq SHPS(A)$. In particular, if $A$ generates a residually large variety satisfying the FHP, then $HSP(A) \neq SHPS(A)$.

\textbf{References}


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