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MULTIPLICATIVE BIJECTIONS OF $\mathcal{C}(\mathcal{X}, I)$

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Abstract. Let $\mathcal{X}$ be a compact Hausdorff space which satisfies the first axiom of countability, let $I = [0, 1]$ and let $\mathcal{C}(\mathcal{X},I)$ be the set of all continuous functions from $\mathcal{X}$ to I. If $\varphi : \mathcal{C}(\mathcal{X},I) \to \mathcal{C}(\mathcal{X},I)$ is a bijective multiplicative map, then there exist a homeomorphism $\mu : \mathcal{X} \to \mathcal{X}$ and a continuous map $k : \mathcal{X} \to (0, \infty)$, such that $\varphi(f)(x) = f(\mu(x))^{k(x)}$ for all $x \in \mathcal{X}$ and for all $f \in \mathcal{C}(\mathcal{X}, I)$.

1. Introduction and statement of the result

The problem we consider in this paper has been motivated by a recent result of L. Molnár in [7], where he proved a structural result for the so-called sequential isomorphisms between the sets of von Neumann algebra effects.

To explain the content of that result we recall the following notions. Let $\mathcal{A}$ be a unital C^*-algebra. The effects in $\mathcal{A}$ are the positive elements of $\mathcal{A}$ which are less than or equal to the unit of $\mathcal{A}$. The set of all effects in $\mathcal{A}$ is denoted by $E(\mathcal{A})$. If $\mathcal{A}$ equals the algebra $B(\mathcal{H})$ of all bounded linear operators on the complex Hilbert space $\mathcal{H}$, then the corresponding effects are called Hilbert space effects. These objects play a very important role in certain parts of quantum mechanics, e.g., in the quantum theory of measurement; see, for example, [2]. For more general concepts of effects we refer to the monograph [3]. Motivated by physical considerations, in their paper [5] Gudder and Nagy introduced the concept of the sequential product between effects; see also [6]. This operation is defined by

$$A \circ B = A^{1/2} B A^{1/2}, \qquad A, B \in E(\mathcal{A}).$$

In the paper [4], Gudder and Greechie described the general form of the sequential automorphisms of the set of all Hilbert space effects assuming that the underlying Hilbert space is at least three dimensional. If $\mathcal{A}$, $\mathcal{B}$ are unital C^* -algebras, then the bijective map $\varphi : E(\mathcal{A}) \to E(\mathcal{B})$ is called a sequential isomorphism if it satisfies

$$\varphi(A \circ B) = \varphi(A) \circ \varphi(B), \qquad A, B \in E(\mathcal{A}).$$

The result of Gudder and Greechie says that for Hilbert space effects every such transformation φ is implemented by either a unitary or an antiunitary operator U of the underlying Hilbert space, i.e., φ is of the form

$$\varphi(A) = UAU^*.$$

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In the paper [7] this result was significantly generalized from the case of effects in $B(\mathcal{H})$ to the case of effects in general von Neumann algebras. Namely, Molnár proved the following result.

Let $\mathcal{A}$, $\mathcal{B}$ be von Neumann algebras and let $\varphi : E(\mathcal{A}) \to E(\mathcal{B})$ be a sequential isomorphism. Then there are direct decompositions

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 \quad \text{and} \quad \mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \mathcal{B}_3$$

within the category of von Neumann algebras and there are bijective maps

$$\varphi_1 : E(\mathcal{A}_1) \to E(\mathcal{B}_1), \quad \Phi_2 : \mathcal{A}_2 \to \mathcal{B}_2, \quad \Phi_3 : \mathcal{A}_3 \to \mathcal{B}_3,$$

such that

(i) $\mathcal{A}_1$, $\mathcal{B}_1$ are commutative von Neumann algebras and the algebras $\mathcal{A}_2 \oplus \mathcal{A}_3$, $\mathcal{B}_2 \oplus \mathcal{B}_3$ have no commutative direct summands;

(ii) φ_1 is a multiplicative bijection, Φ_2 is an algebra *-isomorphism, Φ_3 is an algebra *-antiisomorphism and $\varphi = \varphi_1 \oplus \Phi_2 \oplus \Phi_3$ holds on $E(\mathcal{A})$.

Much information is available on algebra *-isomorphisms and algebra *-antiisomorphisms. For example, we know their general forms in many particular cases. In contrast to this, nothing seems to be known about the third factor in the above decomposition, i.e., about the bijective multiplicative maps between the sets of effects in commutative von Neumann algebras or, more generally, in commutative unital C^*-algebras. The aim of this paper is to somehow try to fill this gap by presenting a structural result concerning these latter transformations.

It is well known that every commutative C^*-algebra is isomorphic to the algebra of all continuous complex-valued functions on a compact Hausdorff space $\mathcal{X}$. Therefore, it is enough to consider the structure of all continuous functions from $\mathcal{X}$ to the unit interval I which we denote by $\mathcal{C}(\mathcal{X}, I)$. The main result of our paper describes the general form of all bijective multiplicative maps of $\mathcal{C}(\mathcal{X}, I)$ under the technical condition that $\mathcal{X}$ satisfies the first axiom of countability.

Theorem. *Let $\mathcal{X}$ be a compact Hausdorff space which satisfies the first axiom of countability and let $I = [0,1]$. If $\varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ is a bijective multiplicative map, then there exist a homeomorphism $\mu : \mathcal{X} \to \mathcal{X}$ and a continuous map $k : \mathcal{X} \to (0, \infty)$, such that for $x \in \mathcal{X}$,*

$$\varphi(f)(x) = f(\mu(x))^{k(x)}$$

for all $f \in \mathcal{C}(\mathcal{X}, I)$.

We believe that the same result also holds without the countability assumption, and it would be interesting to find a proof of this conjecture.

2. Proof of the Theorem

Our first auxiliary result gives us the form of multiplicative functions on the interval I.

Lemma 2.1. *Let $m : I \to I$ be a multiplicative function. Then $m(x) = 0$ for all $x \in I$, or $m(x) = 1$ for all $x \in I$, or there exists $k > 0$ such that $m(x) = x^k$ for all $x \in I$, or*

$$m(x) = \begin{cases} 0, & x = 0, \\ 1, & x \in (0,1], \end{cases}$$

or

$$m(x) = \begin{cases} 0, & x \in [0,1), \\ 1, & x = 1. \end{cases}$$

Proof. The function m is multiplicative, therefore $m(0)(m(0)-1) = 0$ which yields $m(0) = 0$ or $m(0) = 1$. If $m(0) = 1$, then $m(x) = m(x)m(0) = m(0) = 1$ for every $x \in \mathcal{X}$. In this case m is of the form

$$m(x) = 1 \quad \text{for all } x \in I.$$

Let us assume that $m(0) = 0$ and let us define $h(x) = m(e^{-x})$, $x \in [0, \infty)$. Then

$$h(x+y) = m(e^{-x-y}) = m(e^{-x})m(e^{-y}) = h(x)h(y), \quad x, y \in [0, \infty).$$

The function h is also bounded from above by 1. Therefore by [1, Chapter 3] h has one of the following forms: $h(x) = 0$ for all $x \in [0, \infty)$, or $h(x) = e^{cx}$ for all $x \in [0, \infty)$, where c is a nonpositive constant, or

$$h(x) = \begin{cases} 0, & x > 0, \\ 1, & x = 0. \end{cases}$$

For $x \in (0, 1]$ we obtain $h(-\ln x) = m(x)$ and therefore $m(x) = 0$ for all $x \in (0, 1]$, or $m(x) = e^{-c \ln x} = x^{-c}$ for all $x \in (0, 1]$, $c \leq 0$, or

$$m(x) = \begin{cases} 0, & x \in (0,1), \\ 1, & x = 1. \end{cases}$$

Considering that m sends 0 to 0, we conclude the proof of Lemma 2.1. □

From now on let $\varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ be a multiplicative bijective map. Also, let $1_{\mathcal{X}}(x) = 1$ for all $x \in \mathcal{X}$ and $0_{\mathcal{X}}(x) = 0$ for all $x \in \mathcal{X}$. We will next show that $\varphi(0_{\mathcal{X}}) = 0_{\mathcal{X}}$ and $\varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}$. Since φ is multiplicative we obtain $\varphi(0_{\mathcal{X}})(\varphi(h) - 1_{\mathcal{X}}) = 0_{\mathcal{X}}$ for every $h \in \mathcal{C}(\mathcal{X}, I)$. Suppose $\varphi(0_{\mathcal{X}})(x_0) \neq 0$ for some $x_0 \in \mathcal{X}$. Because φ is surjective there exists $h \in \mathcal{C}(\mathcal{X}, I)$ such that $\varphi(h)(x_0) \neq 1$. Since this is contrary to the upper equation we get

$$\varphi(0_{\mathcal{X}}) = 0_{\mathcal{X}}.$$

Similarly, we obtain $\varphi(1_{\mathcal{X}})(\varphi(1_{\mathcal{X}}) - 1_{\mathcal{X}}) = 0_{\mathcal{X}}$. If $\varphi(1_{\mathcal{X}})(x_0) \neq 1$ and therefore $\varphi(1_{\mathcal{X}})(x_0) = 0$ for some $x_0 \in \mathcal{X}$, then $\varphi(h)(x_0) = \varphi(1_{\mathcal{X}})(x_0)\varphi(h)(x_0) = 0$ for every $\varphi(h)$. This is a contradiction since φ is surjective. Hence,

$$\varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}.$$

Lemma 2.2. *Let $f, g \in \mathcal{C}(\mathcal{X}, I)$ with $f(x) \geq g(x)$ for all $x \in \mathcal{X}$ and $f(x) \neq 0$ for all $x \in \mathcal{X}$. Then $\varphi(f)(x) \geq \varphi(g)(x)$ for all $x \in \mathcal{X}$.*

Proof. Let $h = \frac{g}{f}$. Then $h \in \mathcal{C}(\mathcal{X}, I)$ and $hf = g$. The map φ is multiplicative, therefore $\varphi(h)\varphi(f) = \varphi(g)$ which yields $\varphi(f)(x) \geq \varphi(g)(x)$ for all $x \in \mathcal{X}$. □

In the following lemma we will need the notion of the so-called maximal open set of the function from $\mathcal{C}(\mathcal{X},I)$. Let W be a closed subset of $\mathcal{X}$ with $W \neq \mathcal{X}$ and $\mathrm{Int}W = U \neq \emptyset$. If there exists $f \in \mathcal{C}(\mathcal{X},I)$, such that $f^{-1}(1) = W$, then U will be called the maximal open set of the function f.

Lemma 2.3. *Let U be an open nonempty subset of $\mathcal{X}$ where $\overline{U} \neq \mathcal{X}$. Then there exists $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 1_{\mathcal{X}}$, such that $f\left(\overline{U}\right) = \{1\}$. Furthermore, for every $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 1_{\mathcal{X}}$, with $f\left(\overline{U}\right) = \{1\}$ there exists the maximal open set V of the function $\varphi(f)$.*

Proof. Let U be a nonempty open subset of $\mathcal{X}$ with $\overline{U} \neq \mathcal{X}$. Every compact Hausdorff space is normal, so by Urysohn's lemma there exists a continuous function f from $\mathcal{X}$ to I such that $f(x) = 1$ for all $x \in \overline{U}$ and $f \neq 1_{\mathcal{X}}$. Let $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 1_{\mathcal{X}}$ and $f\left(\overline{U}\right) = \{1\}$. Since $\mathcal{X}$ is normal and $U \neq \emptyset$ there exists a closed nonempty set $U_1 \subset U$. The sets U^c and U_1 are closed disjoint subsets of $\mathcal{X}$, therefore by Urysohn's lemma there exists a continuous function $g : \mathcal{X} \to I$ such that $g(x) = 0$ for all $x \in U^c$ and $g(x) = 1$ for all $x \in U_1$. Then $gf = g$ which, by the multiplicativity of the map φ, yields

$$\varphi(g)(\varphi(f) - 1_{\mathcal{X}}) = 0_{\mathcal{X}}.$$

This implies that if $\varphi(g)(x) \neq 0$, then $\varphi(f)(x) = 1$ and if $\varphi(f)(x) \neq 1$, then $\varphi(g)(x) = 0$ for $x \in \mathcal{X}$. Let us define $V = \operatorname{Int}\big(\varphi(f)^{-1}(1)\big)$. The set $\big(\varphi(g)^{-1}(0)\big)^c$ is a nonempty open set since $g \neq 0_{\mathcal{X}}$ and φ is injective. Also, $\big(\varphi(g)^{-1}(0)\big)^c \subset \varphi(f)^{-1}(1)$ and therefore by the definition of V, $V \neq \emptyset$. Observe that $\varphi(f)^{-1}(1) \neq \mathcal{X}$ since $f \neq 1_{\mathcal{X}}$ and φ is injective. This yields that V is the maximal open set of the function $\varphi(f)$. □

The next lemma is a consequence of Lemma 2.3.

Lemma 2.4. *Let U be an open nonempty subset of $\mathcal{X}$ where $\overline{U} \neq \mathcal{X}$. Then there exists $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 0_{\mathcal{X}}$, such that $f\left(\overline{U}\right) = \{0\}$. Furthermore, for every $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 0_{\mathcal{X}}$, with $f\left(\overline{U}\right) = \{0\}$ there exists an open set V such that $\varphi(f)(\overline{V}) = \{0\}$.*

Proof. By Urysohn's lemma there exists a continuous function f from $\mathcal{X}$ to I such that $f(x) = 0$ for all $x \in \overline{U}$ and $f \neq 0_{\mathcal{X}}$. Since $\mathcal{X}$ is normal there exists an open nonempty set U_1, where $\overline{U_1} \subset U$. Again by Urysohn's lemma there exists a function $g : \mathcal{X} \to I$ such that $g(x) = 0$ for all $x \in U^c$ and $g(x) = 1$ for all $x \in \overline{U_1}$. Then $gf = 0_{\mathcal{X}}$ and therefore

$$\varphi(g)\varphi(f) = 0_{\mathcal{X}}.$$

This implies that if $\varphi(g)(x) \neq 0$, then $\varphi(f)(x) = 0$ and if $\varphi(f)(x) \neq 0$, then $\varphi(g)(x) = 0$ for $x \in \mathcal{X}$. By Lemma 2.3 there exists the maximal open set V of the function $\varphi(g)$ which yields $\varphi(f)\left(\overline{V}\right) = 0$. □

Lemma 2.5. *Let U be the maximal open set of the functions f_1 and f_2. If V_1 is the maximal open set of the function $\varphi(f_1)$ and V_2 is the maximal open set of the function $\varphi(f_2)$, then $V_1 = V_2$.*

Proof. Let us assume that $V_1 \neq V_2$. Then $\overline{V_1} \neq \overline{V_2}$ because V_1 and V_2 are maximal. Without loss of generality we may assume that $\overline{V_1}$ is not a subset of $\overline{V_2}$. Then there exists a nonempty open set $V_3 \subset V_1 \cap \overline{V_2}^c$. So, $V_3 \subset V_1$ and $\overline{V_2} \cap V_3 = \emptyset$. By Urysohn's lemma and the surjectivity of φ there exists a function $\varphi(h) \neq 0_{\mathcal{X}}$ such that $\varphi(h)\left(V_3^c\right) = \{0\}$. Then on one hand we obtain $\varphi(h)\varphi(f_1) = \varphi(h)$ and therefore by the multiplicativity and injectivity of φ

$$h(f_1 - 1_{\mathcal{X}}) = 0_{\mathcal{X}}.$$

It follows that $h(x) = 0$ for all $x \in U^c$ which yields $hf_2 = h$. On the other hand we get $\varphi(h)\varphi(f_2) \neq \varphi(h)$ and therefore $hf_2 \neq h$, which means that our assumption was wrong. Therefore $V_1 = V_2$, which completes the proof. □

Lemma 2.6. *Let U_1, U_2, ..., U_n be the maximal open sets of the functions f_1, f_2, ..., f_n, respectively. Then*

$$U_1 \cap U_2 \cap ... \cap U_n \neq \emptyset$$

if and only if

$$V_1 \cap V_2 \cap ... \cap V_n \neq \emptyset,$$

where V_1, V_2, ..., V_n are the maximal open sets of the functions $\varphi(f_1)$, $\varphi(f_2)$, ..., $\varphi(f_n)$, respectively.

Proof. Suppose $U_1 \cap U_2 \cap ... \cap U_n \neq \emptyset$. Because of Lemma 2.5 we may assume without loss of generality that $f_i(x) \neq 0$ for all $x \in \mathcal{X}$ and for all $i \in \{1, 2, ..., n\}$. The finite intersection of open sets is an open set, so by Urysohn's lemma there exist a function $h \in \mathcal{C}(\mathcal{X}, I)$ and an open set $U \subset U_1 \cap U_2 \cap ... \cap U_n$, where $h(x) = 0$ for every $x \in (U_1 \cap U_2 \cap ... \cap U_n)^c$ and U is the maximal open set of h. This yields

$$h(x) \leq f_i(x) \qquad \text{for all } x \in \mathcal{X}, \quad i = 1, 2, ..., n,$$

and therefore by Lemma 2.2

$$\varphi(h)(x) \leq \varphi(f_i)(x) \qquad \text{for all } x \in \mathcal{X}, \quad i = 1, 2, ..., n. \tag{2.1}$$

By Lemma 2.3 there exist the maximal open set V of the function $\varphi(h)$ and the maximal open sets V_1, V_2, ..., V_n of the functions $\varphi(f_1)$, $\varphi(f_2)$, ..., $\varphi(f_n)$, respectively. From (2.1) we may conclude that

$$\emptyset \neq V \subset V_1 \cap V_2 \cap ... \cap V_n.$$

This implication is also true in the converse direction since φ^{-1} has the same properties as φ. □

In the next step we will construct the homeomorphism $\mu : \mathcal{X} \to \mathcal{X}$.

Almost to the end of the proof we will assume that $|\mathcal{X}| > 1$. For the point $x_0 \in \mathcal{X}$ let A_{x_0}, $\overline{A_{x_0}} \neq \mathcal{X}$ be an arbitrary open neighbourhood of x_0. Then by Urysohn's lemma there exists an open neighbourhood U of the point x_0, $\overline{U} \subset A_{x_0}$, and a function $f \in \mathcal{C}(\mathcal{X}, I)$, where U is the maximal open set of f. Let $\mathcal{U}_{A_{x_0}}$ be the family of all pairs (U, f), where U is the maximal open set of f, $x_0 \in U$ and $\overline{U} \subset A_{x_0}$. Then $x_0 \in \bigcap_{(U,f) \in \mathcal{U}_{A_{x_0}}} U$. Let $x_1 \in \mathcal{X}$, $x_1 \neq x_0$. Then there exist open sets A_1, A_2 such that $A_1 \cap A_2 = \emptyset$ and $x_0 \in A_1$, $x_1 \in A_2$. Again by Urysohn's lemma there exists $(U, f) \in \mathcal{U}_{A_{x_0}}$ with $\overline{U} \subset A_1 \cap A_{x_0}$. So, $U \cap A_2 = \emptyset$ and hence $x_1 \notin \bigcap_{(U,f) \in \mathcal{U}_{A_{x_0}}} U$. This gives us

$$\bigcap_{(U,f) \in \mathcal{U}_{A_{x_0}}} U = \{x_0\}.$$

By Lemma 2.3 there exists for every $(U, f) \in \mathcal{U}_{A_{x_0}}$ the maximal open set V of the function $\varphi(f)$. Let $\mathcal{V}_{A_{x_0}}$ be the family of all $(V, \varphi(f))$, where $(U, f) \in \mathcal{U}_{A_{x_0}}$ and V is the maximal open set of $\varphi(f)$. We will next show that there exists a point $x_1 \in \mathcal{X}$ such that

$$\bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} V = \{x_1\}.$$

Let us first assume that $\bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} \overline{V} = \emptyset$. Then $\bigcup_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} \overline{V}^c = \mathcal{X}$. Each open covering in a compact space has a finite subcovering, hence there exist

$$(V_1, \varphi(f_1)), (V_2, \varphi(f_2)), ..., (V_n, \varphi(f_n)) \in \mathcal{V}_{A_{x_0}}$$

such that $\overline{V_1}^c \cup \overline{V_2}^c \cup ... \cup \overline{V_n}^c = \mathcal{X}$. This yields

$$\overline{V_1} \cap \overline{V_2} \cap ... \cap \overline{V_n} = \emptyset.$$

Let (U_1, f_1), (U_2, f_2), ..., $(U_n, f_n) \in \mathcal{U}_{A_{x_0}}$. Since $U_1 \cap U_2 \cap ... \cap U_n \neq \emptyset$ we obtain by Lemma 2.6 that $V_1 \cap V_2 \cap ... \cap V_n \neq \emptyset$, a contradiction. We have proved that

$$\bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} \overline{V} \neq \emptyset.$$

Let us next assume that

$$\bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} V = \emptyset.$$

Then there exist $x_\lambda \in \bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} \overline{V}$ and $(V_\lambda, \varphi(f_\lambda)) \in \mathcal{V}_{A_{x_0}}$, where $x_\lambda \in \overline{V_\lambda}$ and $x_\lambda \notin V_\lambda$. Let $(U_\lambda, f_\lambda) \in \mathcal{U}_{A_{x_0}}$. By Urysohn's lemma there exist a function f_μ and an open set U_μ, $\overline{U_\mu} \subset U_\lambda$, where $(U_\mu, f_\mu) \in \mathcal{U}_{A_{x_0}}$ and $f_\mu\left(U_\lambda^c\right) = \{0\}$. This yields $f_\mu f_\lambda = f_\mu$ and therefore

$$\varphi(f_\mu)(\varphi(f_\lambda) - 1_{\mathcal{X}}) = 0_{\mathcal{X}}.$$

If $\varphi(f_\lambda)(x) \neq 1$, then $\varphi(f_\mu)(x) = 0$, $x \in \mathcal{X}$, which implies by the continuity of $\varphi(f_\mu)$ that

$$\varphi(f_\mu)\left(V_\lambda^c\right) = \{0\}.$$

Therefore, since $x_\lambda \in V_\lambda^c$, we obtain that $x_\lambda \notin \overline{V_\mu}$ where V_μ is the maximal open set of $\varphi(f_\mu)$. This is a contradiction, since $(V_\mu, \varphi(f_\mu)) \in \mathcal{V}_{A_{x_0}}$ and $x_\lambda \in \bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} \overline{V}$.

We have proved that

$$\bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} V \neq \emptyset.$$

Let us now assume that there exist $x_1, x_2 \in \mathcal{X}$, $x_1 \neq x_2$, such that $\{x_1, x_2\} \subset \bigcap_{(V,\varphi(f)) \in \mathcal{V}_{A_{x_0}}} V$. Let V' and V'' be disjoint open neighbourhoods of the points x_1 and x_2. There exists by Urysohn's lemma and the surjectivity of φ a function $\varphi(h_1)$ with maximal open set V_1, where $\overline{V_1} \subset V'$, $x_1 \in V_1$, and $\varphi(h_1)\left(V'^c\right) = \{0\}$. Similarly, there exists a function $\varphi(h_2)$ with maximal open set V_2, where $\overline{V_2} \subset V''$, $x_2 \in V_2$, and $\varphi(h_2)\left(V''^c\right) = \{0\}$. Clearly, $\overline{V_1} \cap \overline{V_2} = \emptyset$, $\varphi(h_1)\varphi(h_2) = 0_{\mathcal{X}}$ and hence by the multiplicativity and injectivity of φ,

$$h_1 h_2 = 0_{\mathcal{X}}. \tag{2.2}$$

By Lemma 2.3 and (2.2) there exist U_1 and U_2 which are the maximal open sets of h_1 and h_2 such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. Without loss of generality we may assume that $x_0 \notin \overline{U_2}$. Since $\mathcal{X}$ is normal we may find an open neighbourhood U_3, $\overline{U_3} \subset A_{x_0}$ of the point x_0 such that

$$U_2 \cap U_3 = \emptyset.$$

By Urysohn's lemma there exists a function h_3 and an open set U_4 such that $\overline{U_4} \subset U_3$, where U_4 is the maximal open set of h_3, $x_0 \in U_4$, and $h_3\left(U_3^c\right) = \{0\}$. So, $(U_4, h_3) \in \mathcal{U}_{A_{x_0}}$. This yields that there exists V_3 where $(V_3, \varphi(h_3)) \in \mathcal{V}_{A_{x_0}}$. Then on one hand we establish that

$$x_2 \in V_3 \cap V_2,$$

and on the other hand, since $U_4 \cap U_2 = \emptyset$, we obtain by Lemma 2.6

$$V_3 \cap V_2 = \emptyset,$$

which is a contradiction. Therefore

$$\bigcap_{(V,\varphi(f))\in\mathcal{V}_{A_{x_0}}} V = \{x_1\}.$$

Next, we will prove that this intersection is independent of the selection of an open neighbourhood of the point x_0. Now let B_{x_0}, $\overline{B_{x_0}} \neq \mathcal{X}$, $B_{x_0} \neq A_{x_0}$, be an open neighbourhood of x_0 and let $\bigcap_{(V,\varphi(f))\in\mathcal{V}_{B_{x_0}}} V = \{x_2\}$. Suppose $C_{x_0} = A_{x_0} \cap B_{x_0}$ and let $\bigcap_{(V,\varphi(f))\in\mathcal{V}_{C_{x_0}}} V = \{x_3\}$. Clearly, if $(U, f) \in \mathcal{U}_{C_{x_0}}$, then $(U, f) \in \mathcal{U}_{A_{x_0}}$ and $(U, f) \in \mathcal{U}_{B_{x_0}}$. Hence $(V, \varphi(f)) \in \mathcal{V}_{A_{x_0}}$ and $(V, \varphi(f)) \in \mathcal{V}_{B_{x_0}}$ for all $(V, \varphi(f)) \in \mathcal{V}_{C_{x_0}}$. This yields

$$x_1 = x_3 = x_2.$$

Now let $\psi : \mathcal{X} \to \mathcal{X}$ be the function for which $x_0 \mapsto x_1$. We will prove that ψ is a homeomorphism. Let $x_a \neq x_b$, $x_a, x_b \in \mathcal{X}$. Then there exist functions f_a and f_b and disjoint maximal open sets U_a and U_b of f_a, f_b, respectively, that are neighbourhoods of the points x_a, x_b. Let V_a, V_b be the corresponding maximal open sets of the functions $\varphi(f_a)$, $\varphi(f_b)$. Therefore $(V_a, \varphi(f_a)) \in \mathcal{V}_{A_{x_a}}$ and $(V_b, \varphi(f_b)) \in \mathcal{V}_{B_{x_b}}$. Since $U_a \cap U_b = \emptyset$ we get by Lemma 2.6 $V_a \cap V_b = \emptyset$. This yields

$$\{\psi(x_a)\} = \bigcap_{(V,\varphi(f))\in\mathcal{V}_{A_{x_a}}} V \neq \bigcap_{(V,\varphi(f))\in\mathcal{V}_{B_{x_b}}} V = \{\psi(x_b)\}.$$

So, $\psi(x_a) \neq \psi(x_b)$ which means that ψ is injective. Since φ^{-1} has the same properties as φ it follows that ψ is also surjective.

Now let $V \subset \mathcal{X}$ be the maximal open set of some function $\varphi(f)$ and let x_v be an arbitrary point in V. Let U be the corresponding maximal open set of the function f. The set U is then a neighbourhood of the point $\psi^{-1}(x_v)$. So, $\psi^{-1}(V) \subset U$. Similarly, for each $x_u \in U$ we can conclude that $\psi(x_u) \in V$ which yields $\psi(U) \subset V$ and therefore

$$\psi^{-1}(V) = U.$$

We have proved that the inverse image of any maximal open set of some function $\varphi(f)$ is the maximal open set of the function f.

Now let $A \subset \mathcal{X}$ be any nonempty open set. We may find, by Urysohn's lemma, that for every $a \in A$ a function f_a and the maximal open set V_a of f_a, $\overline{V_a} \subset A$, such that V_a is a neighbourhood of the point a. Therefore $A = \bigcup_{a\in A} V_a$ which gives us

$$\psi^{-1}(A) = \bigcup_{a\in A} \psi^{-1}(V_a).$$

The inverse image of any maximal open set of some function $\varphi(f)$ is the maximal open set of the function f, hence $\psi^{-1}(A)$ is an open set.

This yields that ψ is a continuous function. Since $\mathcal{X}$ is a compact Hausdorff space and ψ is a continuous bijection, we can conclude that ψ is a homeomorphism. Let us denote the homeomorphism

$$\mu = \psi^{-1}.$$

In the conclusion of the proof of the theorem we will need another two auxiliary results. We will show that the same result as in Lemma 2.2 is also valid locally.

Lemma 2.7. *Let $f, g \in \mathcal{C}(\mathcal{X},I)$. Suppose there exists an open neighbourhood U of $\mu(x_1)$, $x_1 \in \mathcal{X}$, where $f(x) = g(x)$ for all $x \in U$ and there is no $x \in \mathcal{X}$ such that $f(x) = g(x) = 0$. Then $\varphi(f)(x_1) = \varphi(g)(x_1)$.*

Proof. Let $j = \max\{f, g\}$. Then $j \in \mathcal{C}(\mathcal{X},I)$. Let $h_1 = \frac{g}{j}$ and $h_2 = \frac{f}{j}$. Suppose $h_1 \neq 1_{\mathcal{X}}$ and $h_2 \neq 1_{\mathcal{X}}$. Then there exist maximal open sets $U_1 \supset U$ and $U_2 \supset U$ of the functions h_1 and h_2, respectively, which are neighbourhoods of $\mu(x_1)$. Hence, $\varphi(h_1)(x_1) = \varphi(h_2)(x_1) = 1$. Also, since φ is multiplicative, $\varphi(h_1)\varphi(j) = \varphi(g)$ and $\varphi(h_2)\varphi(j) = \varphi(f)$. Therefore $\varphi(j)(x_1) = \varphi(g)(x_1)$ and $\varphi(j)(x_1) = \varphi(f)(x_1)$, hence

$$\varphi(g)(x_1) = \varphi(f)(x_1).$$

Clearly, if $h_1 = h_2 = 1_{\mathcal{X}}$, then $f = g$ and therefore $\varphi(f) = \varphi(g)$. If we assume that $h_1 = 1_{\mathcal{X}}$ and $h_2 \neq 1_{\mathcal{X}}$, then $g = j$ and $\varphi(h_2)(x_1) = 1$. So, $\varphi(j)(x_1) = \varphi(g)(x_1)$, $\varphi(j)(x_1) = \varphi(f)(x_1)$ and therefore

$$\varphi(f)(x_1) = \varphi(g)(x_1).$$

Similarly, we obtain the same result if $h_1 \neq 1_{\mathcal{X}}$ and $h_2 = 1_{\mathcal{X}}$. □

Lemma 2.8. *Let $f, g \in \mathcal{C}(\mathcal{X},I)$ such that $f(\mu(x_1)) > g(\mu(x_1))$, $x_1 \in \mathcal{X}$, and there is no $x \in \mathcal{X}$ such that $f(x) = g(x) = 0$. Then $\varphi(f)(x_1) \geq \varphi(g)(x_1)$.*

Proof. By the continuity of the functions f and g there exists an open neighbourhood U of $\mu(x_1)$, where $f(x) > g(x)$ for all $x \in U$. Let $j = \max\{f, g\}$. Then by Lemma 2.2 $\varphi(j)(x) \geq \varphi(f)(x)$ and $\varphi(j)(x) \geq \varphi(g)(x)$ for all $x \in \mathcal{X}$. Also, $f(x) = j(x)$ for all $x \in U$ and therefore by Lemma 2.7 $\varphi(f)(x_1) = \varphi(j)(x_1)$. Hence, $\varphi(f)(x_1) \geq \varphi(g)(x_1)$. □

Now we are in the position to conclude the proof of the theorem. First, we will discuss the form of the image of any constant function.

Let $x_0 \in \mathcal{X}$ and let $m_0 : I \to I$ be the function defined in the following way:

$$m_{x_0}(c) = \varphi(c)(x_0),$$

where on one hand $c \in I$ and on the other hand $c \in \mathcal{C}(\mathcal{X},I)$ is a constant function. Then for $c_i, c_j \in I$

$$m_{x_0}(c_i c_j) = \varphi(c_i c_j)(x_0) = \varphi(c_i)(x_0)\ \varphi(c_j)(x_0) = m_{x_0}(c_i)\ m_{x_0}(c_j),$$

hence m_{x_0} is a multiplicative function from I to I. Let $x \in \mathcal{X}$. Then we obtain by Lemma 2.1 and since $\varphi(0_{\mathcal{X}}) = 0_{\mathcal{X}}$ and $\varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}$ that

$$\varphi(c)(x) = \begin{cases} 0, & c \in [0,1), \\ 1, & c = 1, \end{cases} \tag{2.3}$$

or

$$\varphi(c)(x) = \begin{cases} 0, & c = 0, \\ 1, & c \in (0,1], \end{cases} \tag{2.4}$$

or there exists $k(x) > 0$ such that

$$\varphi(c)(x) = c^{k(x)} \quad \text{for all } c \in [0,1]. \tag{2.5}$$

Let X_1 be the set of all $x \in \mathcal{X}$ that satisfy (2.5), let X_2 be the set of all $x \in \mathcal{X}$ that satisfy (2.3), and let X_3 be the set of all $x \in \mathcal{X}$ that satisfy (2.4). Note that

X_1, X_2, X_3 are disjoint and $\mathcal{X} = X_1 \cup X_2 \cup X_3$. Let us prove that if $x \in X_1$, then there exists $k(x) > 0$ such that for every $f \in \mathcal{C}(\mathcal{X},I)$

$$\varphi(f)(x) = f(\mu(x))^{k(x)}.$$

We will discuss several different options. Let $x_1 \in X_1$ and first let $f \in \mathcal{C}(\mathcal{X},I)$ be any function for which $f(\mu(x_1)) \in (0,1)$. Then there exists $\epsilon > 0$ such that $(f(\mu(x_1)) - \epsilon, f(\mu(x_1)) + \epsilon) \subset (0,1)$. Since

$$f(\mu(x_1)) - \epsilon < f(\mu(x_1)) < f(\mu(x_1)) + \epsilon,$$

we obtain by Lemma 2.8 that

$$\varphi(f(\mu(x_1)) - \epsilon)(x_1) \leq \varphi(f)(x_1) \leq \varphi(f(\mu(x_1)) + \epsilon)(x_1).$$

Since $x_1 \in X_1$ we have $\varphi(f(\mu(x_1)) + \epsilon)(x_1) = (f(\mu(x_1)) + \epsilon)^{k(x_1)}$, $k(x_1) > 0$, and $\varphi(f(\mu(x_1)) - \epsilon)(x_1) = (f(\mu(x_1)) - \epsilon)^{k(x_1)}$. So,

$$(f(\mu(x_1)) - \epsilon)^{k(x_1)} \leq \varphi(f)(x_1) \leq (f(\mu(x_1)) + \epsilon)^{k(x_1)}$$

for every $\epsilon > 0$ for which $(f(\mu(x_1)) - \epsilon, f(\mu(x_1)) + \epsilon) \subset (0,1)$. This yields

$$\varphi(f)(x_1) = f(\mu(x_1))^{k(x_1)}, \quad k(x_1) > 0,$$

for every $f \in \mathcal{C}(\mathcal{X},I)$, where $f(\mu(x_1)) \in (0,1)$.

Now let $f(\mu(x_1)) = 0$. Then $\varphi(f(\mu(x_1))+\epsilon)(x_1) = (f(\mu(x_1))+\epsilon)^{k(x_1)}$, $k(x_1) > 0$, $\epsilon \in (0,1)$. By Lemma 2.8 we obtain

$$\varphi(f)(x_1) \leq (f(\mu(x_1)) + \epsilon)^{k(x_1)} = \epsilon^{k(x_1)}$$

and therefore $\varphi(f)(x_1) = 0$.

If $f(\mu(x_1)) = 1$, then $\varphi(f(\mu(x_1)) - \epsilon)(x_1) = (f(\mu(x_1)) - \epsilon)^{k(x_1)}$, $k(x_1) > 0$, $\epsilon \in (0,1)$, and similarly as before we obtain $\varphi(f)(x_1) = 1$.

Next, we will prove that $X_1 = \mathcal{X}$. Let us assume that X_2 is a nonempty set and let $x_0 \in X_2$. First, we will show that there exists for each open neighbourhood U of x_0 a point $x \in U \cap X_1$. Let U be a nonempty open set such that $U \subset X_2$ and $c \in (0,1)$. Then by (2.3) $\varphi(c)(x) = 0$ for all $x \in U$. Since φ^{-1} has the same properties as φ there exists by Lemma 2.4 an open set U' such that $c(x) = 0$ for all $x \in U'$, a contradiction. So, there does not exist a nonempty open set U where $U \subset X_2$ and similarly by Lemma 2.3 there does not exist a nonempty open set $U \subset X_3$. Let us now assume that there is an open neighbourhood U_0 of x_0 such that $U_0 \subset X_2 \cup X_3$. Since $\mathcal{X}$ satisfies the first axiom of countability there exists an at most countable family $\{U_n; n \in \mathbb{N}, U_n \subset U_0\}$ of neighbourhoods of x_0 where for each open set G, $x_0 \in G$, there is some $U_n \subset G$. Let

$$V_k = \bigcap_{i=1}^{k} U_i.$$

Then $x_0 \in V_k$, V_k is an open set for every $k \in \mathbb{N}$ and $V_1 \supset V_2 \supset V_3 \supset \ldots$. We may find $x_1 \in V_1$, $x_2 \in V_2$, ..., where $x_i \in X_3$ for all $i \in \mathbb{N}$. If V is an arbitrary open neighbourhood of x_0, then there exists U_i such that $U_i \subset V$. This yields $\{x_i, x_{i+1}, x_{i+2}, \ldots\} \subset V_i \subset U_i \subset V$ hence

$$x_0 = \lim_{i \to \infty} x_i.$$

Note that the sequence $\{x_i\}$ converges only to x_0 since $\mathcal{X}$ is a Hausdorff space. Let $c \in (0,1)$. Then by (2.4) $\varphi(c)(x_i) = 1$ for all $i \in \mathbb{N}$. By the continuity of the function $\varphi(c)$ we obtain

$$\varphi(c)(x_0) = \lim_{i\to\infty} \varphi(c)(x_i) = 1,$$

but $\varphi(c)(x_0) = 0$ since $x_0 \in X_2$, a contradiction. We may conclude that for each open neighbourhood U of x_0 there exists $x \in U \cap X_1$.

If we define V_i, $i \in \mathbb{N}$, in a similar way as before, then there exist $x_1 \in V_1$, $x_2 \in V_2, ...$, where $x_i \in X_1$ for all $i \in \mathbb{N}$ and $x_0 = \lim\limits_{i\to\infty} x_i$. Let $c \in (0,1)$. By the continuity of the function $\varphi(c)$ we obtain

$$\varphi(c)(x_0) = \lim_{i\to\infty} \varphi(c)(x_i) = \lim_{i\to\infty} c^{k(x_i)}, \quad k(x_i) > 0.$$

Since $\varphi(c)(x_0) = 0$ we may conclude that $k(x_i) \to \infty$ when $i \to \infty$. So, there exists $i_0 \in \mathbb{N}$ such that $k(x_j) > 1$ for all $j \geq i_0$.

Now let $f_r : \{\mu(x_j),\ j \in \mathbb{N},\ j \geq i_0\} \cup \{\mu(x_0)\} \to [0,1]$ be the function defined in the following way:

$$f_r(\mu(x_j)) = \begin{cases} 1, & j = 0 \text{ or } j \geq i_0 \text{ and } j \text{ odd}, \\ 1 - \frac{1}{k(x_j)}, & j \geq i_0 \text{ and } j \text{ even}. \end{cases}$$

Since μ is continuous we obtain $\mu(x_0) = \lim\limits_{j\to\infty} \mu(x_j)$. The space $\mathcal{X}$ is first countable, therefore for an arbitrary $A \subset \mathcal{X}$, $\overline{A} = \{x \in \mathcal{X};\ x \text{ is a limit of a sequence from } A\}$. This yields that $\{\mu(x_j), j \geq i_0\} \cup \{\mu(x_0)\}$ is a closed subset of $\mathcal{X}$. By Tietze theorem, since f_r is continuous, there exists a continuous extension $f : \mathcal{X} \to I$. Furthermore, since $\varphi(f)$ is continuous, we get $\varphi(f)(x_0) = \lim\limits_{j\to\infty} \varphi(f)(x_j) = \lim\limits_{j\to\infty} f(\mu(x_j))^{k(x_j)}$, $k(x_j) > 0$. But

$$\lim_{j\to\infty} f(\mu(x_{2j-1}))^{k(x_{2j-1})} = 1$$

and

$$\lim_{j\to\infty} f(\mu(x_{2j}))^{k(x_{2j})} = \lim_{j\to\infty} \left(1 - \frac{1}{k(x_{2j})}\right)^{k(x_{2j})} = e^{-1},$$

a contradiction. This yields that X_2 is the empty set. Similarly, we obtain that X_3 is also empty. It follows that

$$X_1 = \mathcal{X}.$$

We have proved that for $|\mathcal{X}| > 1$ and $x \in \mathcal{X}$ there exists $k(x) > 0$ such that

$$\varphi(f)(x) = f(\mu(x))^{k(x)}.$$

Since for every constant function $c \in (0,1)$ we have $\varphi(c)(x) = c^{k(x)}$ and $\varphi(c)$ is a continuous function, it follows that the function $k : \mathcal{X} \to (0,\infty)$ is continuous.

Finally, let $|\mathcal{X}| = 1$. Then $\varphi(f)(x) = m(f)$, where $m : I \to I$. The function m is multiplicative and bijective, so by Lemma 2.1,

$$\varphi(f)(x) = f(x)^k,\ k > 0.$$

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