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STRONG PROXIMINALITY AND RENORMINGS

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Abstract. We characterize finite-dimensional normed linear spaces as strongly proximinal subspaces in all their superspaces. A connection between upper Hausdorff semi-continuity of metric projection and finite dimensionality of subspace is given.

1. Introduction

For a Banach space X, we will denote by B_X and S_X, respectively, the closed unit ball and unit sphere of X. A subspace Y of a Banach space X is said to be proximinal subspace if, for every x in X, the set $P_Y(x) = \{y \in Y : d(x, Y) = \|x - y\|\}$ is nonempty. If $P_Y(x)$ is a singleton for every $x \in X$, we say that Y is a Chebyshev subspace in X. Let $f \in X^* \setminus \{0\}$. It is easy to see that f attains its norm on X (i.e., there exists $x \in S_X$ such that $f(x) = \|f\|^*$) if and only if $\ker f$ is a proximinal hyperplane in X. By James' characterization of reflexive spaces, if X is not reflexive, then there exists an $f \in X^* \setminus \{0\}$ which does not attain its norm on X. In other words, if X is nonreflexive then there exist nonproximinal hyperplanes in X and using this, W. Pollul ([8]) and I. Singer ([9]) gave a characterization of reflexive spaces through proximinality. More precisely they proved the following:

Theorem 1.1 (Theorem A)**.** *A normed linear space Y is proximinal in every superspace X if and only if it is a reflexive Banach space. Moreover, if Y is a nonreflexive Banach space, then it can be embedded isometrically as a nonproximinal closed hyperplane in another Banach space X.*

Recently a stronger notion of proximinality called "strong proximinality" was introduced and studied in [4] and [5] using strong subdifferentiablity of convex functionals (which will be defined later). It is natural to wonder what happens if we replace proximinality by the above stronger notion of proximinality in Theorem A. It turns out that only finite-dimensional spaces are strongly proximinal in all their superspaces (Theorem 2.2).

It is easy to see that if Y is strongly proximinal, then P_Y is *upper Hausdorff semi-continuous* (which will be defined later). But the converse does not hold. In general, continuity of the metric projection does not give much information about the proximinal subspace. In the second part of the note we establish a good

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connection between *upper Hausdorff semi-continuity* of the metric projection and finite dimensionality of the proximinal subspace (Theorem 2.4).

We now recall some definitions:

Let Y be a closed subspace in a Banach space X and let $x \in X$. For $\delta > 0$, consider the following set:

$$P_Y(x, \delta) = \{y \in Y : \|x - y\| < d(x, Y) + \delta\}.$$

A proximinal subspace Y is said to be strongly proximinal at x in X if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in P_Y(x, \delta)$, there exists $y' \in P_Y(x)$ such that $\|y - y'\| < \varepsilon$ or equivalently,

$$P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y.$$

Let Y be a proximinal subspace in a Banach space X, let $x \in X$ and let $\varepsilon > 0$. The metric projection P_Y is said to be *upper semi-continuous* (*u.s.c.*) at x if for every open set $V \subseteq Y$ such that $P_Y(x) \subseteq V$, there exists an open neighborhood U of x such that $P_Y(z) \subseteq V$ for all $z \in U$. P_Y is said to be *upper Hausdorff semi-continuous* (*u.H.s.c.*) at x, if for every $\varepsilon > 0$, there exists an open neighborhood U of x in X such that $P_Y(z) \subseteq P_Y(x) + \varepsilon B_Y$ for all $z \in U$. It is proved in [1, Theorem 1] that P_Y is *u.s.c.* at x if and only if P_Y is *u.H.s.c.* at x and $P_Y(x)$ is compact. P_Y is said to be *lower semi-continuous* (*l.s.c.*) at x if for each open set $V \subset Y$ with $P_Y(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $P_Y(z) \cap V \neq \emptyset$ for all $z \in U$.

2. Main results

We start with the following simple lemma.

Lemma 2.1. *Let Y be a finite-dimensional normed linear space. Then Y is a strongly proximinal subspace in all its super spaces.*

Proof. Let Y be a finite-dimensional normed linear space. Using compactness arguments we can easily prove that Y is a strongly proximinal subspace in all its superspaces. Indeed, let X be a superspace of Y, let $x \in X$ and let $\varepsilon > 0$ be given. By compactness, there exists $y \in Y$ such that $\|x - y\| = d(x, Y)$, which implies proximinality of Y at x. To prove the strong proximinality of Y, we need to show that for each $\varepsilon > 0$ and $x \in X$, there exists $\delta > 0$ such that $P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y$. If not, for every n, there exists $y_n \in P_Y(x, \frac{1}{n})$ such that $d(y_n, P_Y(x)) > \varepsilon$. Then Y being finite dimensional, $\{y_n\}$ has a subsequence converging to some $\overline{y} \in Y$. Then $\|x - \overline{y}\| = d(x, Y)$, which implies $\overline{y} \in P_Y(x)$. But $d(\overline{y}, P_Y(x)) \geq \varepsilon$, which is a contradiction. Hence Y is a strongly proximinal subspace in X. □

The following result is a version of Theorem A for strong proximinality. The proof uses a renorming technique which depends on the existence of biorthogonal systems. Renorming techniques are extensively studied in [2].

Theorem 2.2. *Let Y be an infinite-dimensional Banach space. Then Y can be embedded isometrically as a nonstrongly proximinal hyperplane in another Banach space.*

Proof. Let Y be a given infinite-dimensional Banach space and let Z be a closed infinite-dimensional separable subspace of Y. Let $\varepsilon > 0$ be given and let $\{(e_n, e_n^*) : n \geq 1\} \subset Z \times Z^*$ be a sequence of elements satisfying the following.

(i) $\|e_n\| = 1$ and $\|e_n^*\| < 1 + \varepsilon$ for all $n \geq 1$.
(ii) $e_k^*(e_n) = \delta_n^k$.
(iii) $\overline{span}(e_n) = Z$.
(iv) If $e_k^*(z) = 0$ for all k, then $z = 0$.
(v) There exists $c \geq 1$ such that $\|e_n\|\|e_n^*\| \leq c$ for every $n = 1, 2, \cdots$.

The above pair $\{(e_n, e_n^*) : n \geq 1\}$ is called a fundamental and total biorthogonal system. [6, Theorem 1] proves the existence of such biorthogonal systems. We now show that there exists a norm $|\cdot|$ on $X = Y \oplus \mathbb{R}$ such that $\|\cdot\|_Y = |\cdot|_{|Y}$ and Y is not strongly proximinal in $(X, |\cdot|)$. Let $e = (0, 1) \in X$, where $0 \in Y$ and $1 \in \mathbb{R}$, $x_n = (e_n, 1 - \frac{1}{n})$, $A = \{e\} \cup \{x_n : n \geq 1\}$ and $B = -A$. Let $B_X = \overline{\text{conv}}(B_Y \cup A \cup B)$. Define the norm on X such that the unit ball of X with this norm $|\cdot|$ is $B_X(|\cdot|) = B_X$. Let $Y \to Y \times \{0\}$ be a natural embedding of Y in X, and we denote this embedding by Y itself. We now make the following claims:

(i) $B_X \cap (e + Y) = \{e\}$.
(ii) $B_X(e, 1 + \varepsilon) \cap Y \not\subseteq B_Y(0, \frac{1}{4})$ for all $\varepsilon > 0$.

We complete the proof of the result assuming the claims. Claim (i) implies that $d(e, Y) = 1$ and $P_Y(e) = \{0\}$. Using claim (ii), we get

$$B_X(e, d(e, Y) + \varepsilon) \cap Y \not\subseteq P_Y(e) + \frac{1}{4} B_Y. \tag{1}$$

This shows that Y is not strongly proximinal at e. This completes the proof of the result. We now prove the claims.

Proof of claim (i). Suppose $C = B_X \cap \{e + Y\} \neq \{e\}$. Let $z (\neq e) \in C$. Now $z = (y, 1)$, where $y \in B_Y$ and $1 \in \mathbb{R}$. Since z is not in the convex hull of $B_Y \cup A \cup B$, it is the limit point of a sequence in the convex hull of $B_Y \cup A \cup B$. Thus $z = \lim_{n\to\infty} z_n$, where $z_n \in \text{conv}(B_Y \cup A \cup B)$. Now z_n is of the form $z_n = \lambda_n y_n + \mu_n a_n + \eta_n b_n$, where $y_n \in B_Y, a_n \in A, b_n \in B$ and $\lambda_n, \mu_n, \eta_n \geq 0$, $\lambda_n + \mu_n + \eta_n = 1$. We also have $z_n = (y_n', \alpha_n)$. Clearly for all n, $\alpha_n \leq \mu_n$. Since $\lim_{n\to\infty} \alpha_n = 1$, we have $\lim_{n\to\infty} \mu_n = 1$ and $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \eta_n = 0$.

Thus we can assume, without loss of generality, that $z_n \in conv(A)$. Then $z_n = \beta_n e + \sum_{k\geq 1} \lambda_{k,n} x_k = (y_n', \alpha_n)$ with $\beta_n + \sum_{k\geq 1} \lambda_{k,n} = 1$ for all n. Thus we have $\alpha_n = \beta_n + \sum_{k\geq 1}(1 - \frac{1}{k})\lambda_{k,n}$. Hence $1 - \alpha_n = \sum_{k\geq 1} \frac{1}{k}\lambda_{k,n}$. Therefore, for all $k \geq 1, \lambda_{k,n} \leq k(1 - \alpha_n)$. Since $\alpha_n \to 1$, we have $\lim_{n\to\infty} \lambda_{k,n} = 0$ for all $k \geq 1$.

$\lim_{n\to\infty} z_n = z$ implies that $\lim_{n\to\infty} y_n' = P(z) \in Y$, where $P : Y \oplus \mathbb{R} \to Y$ is the natural projection. Now $y_n' = \sum_{k\geq 1} \lambda_{k,n} e_k$ and for all $k \geq 1$, $\langle e_k^*, P(z)\rangle = \lim_{n\to\infty} \langle e_k^*, y_n'\rangle = \lim_{n\to\infty} \lambda_{k,n} = 0$. Hence $P(z) = 0$ and $z = e$, which proves claim (i).

Proof of claim (ii). Let $0 < \varepsilon < 1$, and choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \in [\varepsilon, 3\varepsilon]$. Let $\varepsilon_n = \frac{1}{n}, n \geq 1$. For each $n \geq 1$, $(-e_n, 1 - \varepsilon_n) \in B_X$. For each $\lambda \in [0, 1]$, one has

$$(-\lambda e_n, -1 + \lambda\varepsilon_n) = \lambda(-e_n, 1 - \varepsilon_n) + (1 - \lambda)(0, -1) \in B_X.$$

This implies that

$$(-\lambda(1 + \varepsilon)e_n, 1 + (1 + \varepsilon)(-1 + \lambda\varepsilon_n)) \in B_X(e, 1 + \varepsilon).$$

Choose λ such that $(1+\varepsilon)\lambda = \frac{\varepsilon}{\varepsilon_n}$. Then $1+(1+\varepsilon)(-1+\lambda\varepsilon_n) = 0$, and hence $(\frac{-\varepsilon}{\varepsilon_n}e_n, 0) \in B_X(e, 1+\varepsilon) \cap Y$. Since $\|\frac{\varepsilon}{\varepsilon_{n_0}}e_{n_0}\| = \frac{\varepsilon}{\varepsilon_n} \geq \frac{1}{3}$, we have $(\frac{-\varepsilon}{\varepsilon_n}e_n, 0) \notin B_Y(0, \frac{1}{4})$. Thus

$$B_X(e, 1+\varepsilon) \cap Y \not\subseteq B_Y(0, \frac{1}{4}) \text{ for all } \varepsilon > 0.$$

This completes the proof of claim (ii) and the proof of the theorem. □

Before going further, we define strong subdifferentiability (SSD) of convex functionals. Let F be a real-valued convex function defined on a Banach space X. We say that F is SSD at $x \in X$ if the one-sided limit

$$\lim_{t \to 0^+} \frac{F(x+th) - F(x)}{t}$$

exists uniformly for $h \in S_X$.

It is was proved in [5] (Theorem 2.5) that for any $f \in X^* \setminus \{0\}$, $\ker f$ is strongly proximinal in X if and only if $\|\cdot\|^*$ is SSD at f. Now we can give another proof of Theorem 2.2 using SSD of the dual norm.

Alternative Proof of Theorem 2.2. Consider the Banach space X constructed in the above proof, and let $F = (0^*, 1) \in X^*$, where 0^* is the zero of Y^*. Then $Y = \ker F$. A careful analysis of the first part of the above proof actually shows that F attains its norm only at e and moreover the dual norm of X is Gâteaux differentiable at F. We have that $(e_n, 1-1/n) \in B_X$ and $F(e_n, 1-1/n) \to 1$. But $(e_n, 1-1/n)$ does not converge to e. Then by Lemma 1.1 of [5], dual norm of X is not strongly subdiffentiable at F, which implies that $\ker F = Y$ is not strongly proximinal by Theorem 2.5 of [5]. Moreover let us note that $P_Y(x) = x - F(x)e$ for all $x \in X$. So Y is a Chebyshev subspace in X with Lipschitz metric projection, but Y is not a strongly proximinal subspace in X. □

Remark 2.3. It was noted in Theorem 3 of [7] that, if Y is a proximinal hyperplane, then P_Y is *u.H.s.c.* and *l.s.c.* By Theorem 2.2, for every infinite-dimensional Banach space Y, there exists a superspace X of Y such that P_Y is *u.s.c.* on X, but Y is not strongly proximinal subspace in X. It is not clear if Y is an infinite-dimensional Banach space, or whether Y can be embedded in another space X with codimension 2 such that P_Y is not *u.H.s.c.* on X. In the following theorem, we show that Y is an infinite-dimensional Banach space if and only if Y can be embedded as a Chebyshev hyperplane in another Banach space X such that $P_{(Y \oplus_{\ell_\infty} Y)}$ is not *u.H.s.c.* on $X \oplus_{\ell_\infty} X$. This version of the statement for u.s.c. is proved in [1]. In a Banach space X, we denote by $B_X[x, \varepsilon]$ ($B(x, \varepsilon)$, resp.) the closed ball (open ball, resp.) centered at x with radius $\varepsilon > 0$.

Theorem 2.4. *Let Y be an infinite-dimensional Banach space. Then there exist superspaces X and X_1 of Y such that:*

(i) *P_Y is u.s.c. (hence u.H.s.c.) on X but $P_{(Y \oplus_{\ell_\infty} Y)}$ is not u.H.s.c. on $X \oplus_{\ell_\infty} X$.*
(ii) *P_Y is not u.s.c. on X_1, but it is u.H.s.c. on X_1.*

Proof of (i). Let $(X, |\cdot|)$ be the superspace of $(Y, \|\cdot\|)$ constructed in Theorem 2.2. Then Y is a Chebyshev subspace and is not a strongly proximinal subspace in X. Since Y is a proximinal hyperplane with P_Y compact, P_Y is *u.s.c.* Indeed

P_Y is not only *u.s.c.* but in fact (trivially) single-valued and Lipschitz. Let $x_0 = (e, e) \in X \oplus_{\ell_\infty} X$, where as before $e = (0, 1)$, $0 \in Y$ and $1 \in \mathbb{R}$. We have that $P_{(Y \oplus_\infty Y)}(x_0) = \{0\}$. Now we claim that $P_{(Y \oplus_{\ell_\infty} Y)}$ is not *u.H.s.c.* at x_0. Suppose not. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$P_{(Y \oplus_{\ell_\infty} Y)}(z_0) \subseteq \{0\} + \varepsilon B_{(Y \oplus_{\ell_\infty} Y)} \, \forall \, z_0 = (z_1, z_2) \in B_{X \oplus_{\ell_\infty} X}(x_0, \delta),$$

i.e.,

$$(B_X[z_1, d(z_0)] \cap Y) \oplus_{\ell_\infty} (B_X[z_2, d(z_0)] \cap Y) \subseteq \varepsilon B_Y \oplus_{\ell_\infty} \varepsilon B_Y,$$

where $d(z_0) = d(z_0, Y \oplus_{\ell_\infty} Y)$. Let $z_0 = (z_1, z_2) \in B_{X \oplus_{\ell_\infty} X}(x_0, \delta)$ such that $d(x_0) < d(z_2, Y)$. For example, let us take $z_0 = (e, (1 + \frac{\delta}{3})e)$. Thus $z_2 = (1 + \frac{\delta}{3})e$. By *u.H.s.c.* of $P_{(Y \oplus_{\ell_\infty} Y)}$ at x_0, we have that $P_{(Y \oplus_{\ell_\infty} Y)}(z_0) \subseteq \varepsilon B_{(Y \oplus_{\ell_\infty} Y)}$. In particular,

$$B_X[e, d(z_0)] \cap Y \subseteq P_Y(e) + \varepsilon B_Y.$$

We have that $P_Y(e, \delta/8) \subseteq B_X[e, 1 + \delta/8] \cap Y \subseteq B_X[e, d(z_0)] \cap Y$, so

$$P_Y(e, \delta/8) \subseteq B_X[e, d(z_0)] \cap Y \subseteq P_Y(e) + \varepsilon B_Y.$$

This implies that Y is strongly proximinal at e, which is a contradiction.

Proof of (ii). Let $X = Y \oplus_{\ell_\infty} \mathbb{R}$, and let $Y \to Y \oplus_{\ell_\infty} \{0\} = Y_0$ be the natural embedding of Y into X. Then Y_0 is a proximinal hyperplane in X, and since $dim(Y) = +\infty$, $P_{Y_0}((0, 1)) = B_Y \oplus_{\ell_\infty} \{0\}$ which is not compact. Hence by [1, Theorem 1], P_{Y_0} is not *u.s.c.* Since Y_0 is proximinal hyperplane in X, *u.H.s.c.* of P_{Y_0} follows by [7, Theorem 3]. □

Remark 2.5. The classical result on extension of norms (Lemma II.8.1, [2]) together with Theorem 2.2 shows that if X is an arbitrary Banach space and Y is a closed, infinite-dimensional proper supspace of X, then there is an equivalent norm on Z for which Y is not strongly proximinal subspace.

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