MULTIPLICATION AND DIVISION BY INNER FUNCTIONS
IN THE SPACE OF BLOCH FUNCTIONS

DANIEL GIRELA, CRISTÓBAL GONZÁLEZ, AND JOSÉ ÁNGEL PELÁEZ

(Communicated by Juha M. Heinonen)

Abstract. A subspace $X$ of the Hardy space $H^1$ is said to have the $f$-property if $h/I \in X$ whenever $h \in X$ and $I$ is an inner function with $h/I \in H^1$. We let $B$ denote the space of Bloch functions and $B_0$ the little Bloch space. Anderson proved in 1979 that the space $B_0 \cap H^\infty$ does not have the $f$-property. However, the question of whether or not $B \cap H^p$ has the $f$-property was open. We prove that for every $p \in [1, \infty)$ the space $B \cap H^p$ does not have the $f$-property.

We also prove that if $B$ is any infinite Blaschke product with positive zeros and $G$ is a Bloch function with $|G(z)| \to \infty$ as $z \to 1$, then the product $BG$ is not a Bloch function.

1. Introduction and statement of results

We denote by $\Delta$ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and by $H^p$ ($0 < p \leq \infty$) the classical Hardy spaces of analytic functions in $\Delta$ (see [6] and [9]). A function $I$, analytic in $\Delta$, is said to be an inner function if $I \in H^\infty$ and $I$ has a radial limit $I(e^{i\theta})$ of modulus one for almost every $e^{i\theta} \in \partial \Delta$.

Given a function $v \in L^\infty(\partial \Delta)$, the associated Toeplitz operator $T_v$ is defined by

$$(T_v f)(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{v(\zeta)f(\zeta)}{\zeta - z} d\zeta \quad (f \in H^1, z \in \Delta).$$

Definition 1.1. A subspace $X$ of $H^1$ is said to have the $K$-property if $T_{\psi} X \subset X$ for any $\psi \in H^\infty$.

Definition 1.2. A subspace $X$ of $H^1$ is said to have the $f$-property if $h/I \in X$ whenever $h \in X$ and $I$ is an inner function with $h/I \in H^1$.

These notions were introduced by Havin [12] and Korenblum [14]. The $K$-property implies the $f$-property: indeed, if $h \in H^1$, $I$ is inner and $h/I \in H^1$, then $h/I = T_I h$.

In addition to the Hardy spaces $H^p$ ($1 < p < \infty$) many other spaces such as the Dirichlet space $D$ ([12], [14]) and several spaces of Dirichlet type (see [7], [15] and
The spaces $BMOA$ and $VMOA$ and the spaces $Q_p$ ($0 < p < 1$) have the $K$-property. Clearly, $H^1$ has the $f$-property but an argument of duality shows that it does not possess the $K$-property. Hedenmalm proved in [13] that $VMOA \cap H^\infty$ has the $f$-property but does not have the $K$-property. More generally, it is proved in [13] that no subspace of $H^\infty$ containing the disc algebra has the $K$-property.

The first example of a space not possessing the $K$-property was given by Gurarii [11] who proved that the space of analytic functions in $\Delta$ with an absolutely convergent power series does not have the $f$-property.

Recall that if $f$ is an analytic function in $\Delta$, then $f$ is said to be a Bloch function if

$$
||f||_B \overset{def}{=} |f(0)| + \sup_{z \in \Delta} (1 - |z|^2)|f'(z)| < \infty.
$$

The space of all Bloch functions is denoted by $\mathcal{B}$. The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ such that $\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0$. Alternatively, $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm. We mention [2] for the theory of Bloch functions. Let us recall that $H^\infty \subset \mathcal{B}_0 \subset \mathcal{B}$ and $VMOA \subset \mathcal{B}_0$.

Anderson proved in [1] that $\mathcal{B}_0 \cap H^\infty$ does not have the $f$-property. Consequently, the same is true for $\mathcal{B}_0 \cap H^p$ for every $p \in [1, \infty)$. It is natural to ask the following question: Does $\mathcal{B} \cap H^p$ ($1 \leq p \leq \infty$) have the $f$- or $K$-property?

Since $H^\infty \subset \mathcal{B}$, we see that $\mathcal{B} \cap H^\infty = H^\infty$ which has the $f$-property but does not have the $K$-property. It is easy to prove the following result.

**Proposition 1.3.** If $1 \leq p < \infty$, then $\mathcal{B} \cap H^p$ does not have the $K$-property.

In fact, we can prove the following stronger result.

**Theorem 1.4.** If $1 \leq p < \infty$, then $\mathcal{B} \cap H^p$ does not have the $f$-property.

Next we shall consider products of the form $B \cdot f$ with $B \in H^\infty$ and $f \in \mathcal{B}$ but before doing so it is convenient to recall some definitions and results. If a sequence of points $\{a_n\}$ in $\Delta$ satisfies the Blaschke condition $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$, the corresponding Blaschke product $B$ is defined as

$$
B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n \bar{z}}.
$$

Such a product is analytic in $\Delta$, and, in fact, it is an inner function. If there exists a positive constant $\delta$ such that $\prod_{n \neq m} \left| \frac{a_n - a_m}{1 - a_n \bar{a_m}} \right| \geq \delta$, for all $n$, we say that the sequence $\{a_n\}$ is uniformly separated and that $B$ is an interpolating Blaschke product. Equivalently,

$$
B \text{ is an interpolating Blaschke product } \iff \inf_{n \geq 1} (1 - |a_n|^2)|B'(a_n)| > 0.
$$

Thus no interpolating Blaschke product belongs to $\mathcal{B}_0$. Sarason [20] proved that $\mathcal{B}_0$ contains infinite Blaschke products. Other constructions of such products were given by Stephenson in [21] and Bishop in [3], where a description of all $H^\infty$-functions in $\mathcal{B}_0$ is also given.

We recall that a function $f$ which is meromorphic in $\Delta$ is a normal function in the sense of Lehto and Virtanen if

$$
\sup_{z \in \Delta} (1 - |z|^2) \frac{|f'(z)|}{1 + |f(z)|^2} < \infty.
$$


We refer to [2] and [17] for the theory of normal functions. Certainly, any Bloch function is normal. Using (1.1) we can deduce the following result.

**Proposition 1.5.** If $B$ is an interpolating Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^{\infty}$ and $G$ is an analytic function in $\Delta$ with $G(a_n) \to \infty$, as $n \to \infty$, then the function $f = B \cdot G$ is not normal (and, hence, it is not a Bloch function).

Proposition 1.5 has been used by several authors (see e.g. [10], [5], [22], [23], [10] and [4]) to construct distinct classes of non-normal functions. We can prove a result of this kind dealing with Blaschke products with zeros in a radius but not necessarily interpolating.

**Theorem 1.6.** Let $B$ be an infinite Blaschke product whose sequence of zeros $\{a_n\}$ is contained in the radius $(0, 1)$ and let $G$ be a Bloch function such that $G(z) \to \infty$, as $z \to 1$. Then the function $f = B \cdot G$ is not a Bloch function.

2. **Division by inner functions**

Even though Proposition 1.3 follows from Theorem 1.4, we shall give a direct proof of it. We shall use the following easy lemma.

**Lemma 2.1.** The space $B_0 \cap H^p$ ($1 \leq p < \infty$) is the closure of the polynomials in $B \cap H^p$, that is, for any $f \in B_0 \cap H^p$ there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ which converges to $f$ both in the Bloch norm and in the $H^p$-norm.

**Proof.** Take $f \in B_0 \cap H^p$ with $1 \leq p < \infty$. For $0 < r < 1$, set $f_r(z) = f(rz)$ ($z \in \Delta$). Also set $r_n = 1 - \frac{1}{n}$, $n = 1, 2, \ldots$. Using Theorem 2.1 of [2] and Theorem 2.6 of [6], we see that

\[
\max (\|f - f_{rn}\|_B, \|f - f_{rn}\|_{H^p}) \to 0, \quad \text{as } n \to \infty.
\]

For each $n$, $f_{rn}$ is analytic in $\{|z| \leq 1\}$, hence, we can find a polynomial $P_n(z)$ such that $|f_{rn}(z) - P_n(z)| < \frac{1}{n}$ for $|z| \leq 1$. Note that if $g \in H^\infty$, then $\|g\|_B \leq 2\|g\|_{H^\infty}$ (see p. 13 of [2]) and $\|g\|_{H^p} \leq \|g\|_{H^\infty}$. This and (2.1) give

\[
\max (\|f - P_n\|_B, \|f - P_n\|_{H^p}) \\
\leq \max (\|f - f_{rn}\|_B, \|f - f_{rn}\|_{H^p}) + \frac{2}{n} \to 0, \quad \text{as } n \to \infty.
\]

**Proof of Proposition 1.3.** Take $p \in [1, \infty)$ and suppose that $B \cap H^p$ has the $K$-property. Then for any inner function $I$, we would have that $T_I(B \cap H^p) \subset B \cap H^p$. Using the closed graph theorem, it would follow that $T_I$ is continuous from $B \cap H^p$ into itself. Now, bearing in mind that $T_I$ maps polynomials into polynomials and using Lemma 2.1 we would deduce that $T_I(B_0 \cap H^p) \subset B_0 \cap H^p$ for any inner function $I$. Hence, $B_0 \cap H^p$ would possess the $f$-property. But, as mentioned above, this is not true.

Before embarking into the proof of Theorem 1.4 let us note that if $f$ is an analytic function in $\Delta$ we shall set

\[
M_\infty(r, f) = \sup_{|z| \leq r} |f(z)|, \quad 0 < r < 1.
\]
Proof of Theorem 1.4. Let $B$ be an infinite Blaschke product in $B_0$ whose sequence of zeros $\{a_n\}_{n=1}^\infty$ contains a subsequence which tends to 1. Set

$$\varphi(r) = \sup_{r \leq |z| < 1} (1 - |z|^2)|B'(z)|, \quad \phi(r) = \frac{1}{\varphi(r)}, \quad 0 \leq r < 1.$$  \hfill (2.2)

Using (2.2), (2.4), (2.5) and bearing in mind that $f$ exist two positive constants $C_1$ and $C_2$ such that

$$ (1 - |z|^2)|f'(z)| \leq (1 - |z|^2)|B'(z)| |F(z)| + (1 - |z|^2)|F'(z)||B(z)| \leq C_1\varphi(|z|)\phi(|z|) + C_2 = C_1 + C_2, $$  \hfill (2.3)

which, since $B \in BMOA$, implies that the same is true for $f$. Consequently, we have proved that

$$ f \in B \cap H^p, \quad 1 \leq p < \infty.$$  \hfill (2.4)

Now let $\{a_{n_k}\}_{k=1}^\infty$ be a subsequence of the sequence $\{a_n\}$ such that

$$ a_{n_k} \to 1, \quad \text{as } k \to \infty,$$  \hfill (2.5)

and

$$ \{a_{n_k}\}_{k=1}^\infty \text{ is uniformly separated.}$$  \hfill (2.6)

Let $B_1$ be the Blaschke product whose sequence of zeros is $\{a_{n_k}\}_{k=1}^\infty$ and set

$$ B_2 = \frac{B}{B_1}, \quad g = \frac{f}{B_2} = B_1F.$$  \hfill (2.7)

It is clear that $B_2$ is a Blaschke product and that $g \in H^p$ for all $p < \infty$. Next we are going to prove that $g \notin B$. Clearly, this implies that $B \cap H^p (1 \leq p < \infty)$ does not possess the $f$-property. Since $\{a_{n_k}\}$ is uniformly separated, there exists $A > 0$ such that

$$ (1 - |a_{n_k}|^2)|B_1'(a_{n_k})| \geq A, \quad \text{for all } k.$$  \hfill (2.8)
Then, using (2.9), (2.6), (2.11) and Proposition 1.5 we deduce that $g = f/B_2$ is not a Bloch function. \hfill \Box

3. Multiplication by Blaschke products with zeros in a radius

Proof of Theorem 1.6. Let $\{a_n\}$, $B$ and $G$ be as in the statement of Theorem 1.6 and set $g = BG$. Take $\alpha \in \Delta \setminus (-1, 1)$ such that $\alpha$ is not in the set $\{B(a) : a \in \Delta, B'(a) = 0\}$. Clearly, $B((0, 1)) \subset (-1, 1)$ and, hence, $\alpha$ is not a cluster point of $B|_{(0,1)}$. Then, using a result of Marshall and Sarason (see [18]), we deduce that the Frostman shift $B_\alpha$ defined by

\begin{equation}
B_\alpha(z) = \frac{B(z) - \overline{\alpha}B(z)}{1 - \overline{\alpha}B(z)}, \quad z \in \Delta,
\end{equation}

is an interpolating Blaschke product. Thus, if $\{a_n\}_{n=1}^\infty$ is the sequence of zeros of $B_\alpha$, there exists a positive constant $\delta$ such that

\begin{equation}
\left(1 - |a_n|^2\right)|B'_\alpha(a_n)| \geq \delta, \quad \text{for all } n.
\end{equation}

Also, using Theorem 6.1 on p. 75 of [9], we easily see that $b_n \to 1$, as $n \to \infty$. Note that there exists two positive constants $A_1$ and $A_2$ such that

$$A_1|B'_\alpha(z)| \leq |B'(z)| \leq A_2|B'_\alpha(z)|, \quad z \in \Delta.$$ 

This and (3.2) give

$$1 - |b_n|^2)|B'_\alpha(b_n)| \geq ||B'(b_n)||B(b_n)| - ||G'(b_n)||B(b_n)| - \|G\|_\Omega$$

$$\geq A_1(1 - |b_n|^2)|B'_\alpha(b_n)||B(b_n)| - ||G\|_\Omega$$

$$\geq A_1\delta|G(b_n)| - ||G\|_\Omega \to \infty, \quad \text{as } n \to \infty.$$ 

Consequently, $g$ is not a Bloch function. \hfill \Box

References


Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
E-mail address: girela@uma.es

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
E-mail address: cmge@uma.es

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
E-mail address: pelaez@anamat.cie.uma.es