A BOUND FOR RATIOS OF EIGENVALUES OF SCHRÖDINGER OPERATORS WITH SINGLE-WELL POTENTIALS

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Abstract. For Schrödinger operators with nonnegative single-well potentials ratios of eigenvalues are extremal only in the case of zero potential. To prove this, we investigate some monotonicity properties of Prüfer-type variables.

1. Introduction

Consider the Schrödinger operator

\[ -y'' + q(x)y = \lambda y \]

on the interval \([0, \pi]\) with Dirichlet boundary conditions. If \(q \in L_1(0, \pi)\) is real-valued, then the spectrum consists of a growing sequence of infinitely many points, \(\lambda_1, \lambda_2, \ldots\); see for example in [3]. Moreover, if \(q(x)\) is nonnegative, \(\lambda_n \geq n^2\) (as it is seen later, for example, from (2.7) and Lemma 2.1).

Ashbaugh and Benguria in [2] proved the bound

\[ \frac{\lambda_n}{\lambda_1} \leq n^2 \]

for nonnegative potentials. They also examined the ratio of two arbitrary eigenvalues, and found

\[ \frac{\lambda_n}{\lambda_m} \leq \left\lceil \frac{n}{m} \right\rceil^2 \]

where \(\left\lceil x \right\rceil\) denotes the smallest integer greater than or equal to \(x\). To show that this estimate is optimal, they constructed multiple-well examples which came arbitrarily near to attain the bound. They formulated the conjecture that if the potential is nonnegative and convex, then

\[ \frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad n \geq m, \]

holds. In this paper we prove more. Namely, we only need that the potential \(q \geq 0\) be single-well. This means that there is a point \(a \in [0, \pi]\) such that \(q\) is decreasing

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in $[0,a]$ and increasing in $[a,\pi]$ (see in [1]). Our proof relies on some monotonicity properties of the Prüfer-type variables $\varphi$ and $r$ from (2.3)–(2.4).

2. The main statement

Denote by $y(x,z)$ the unique solution of the initial value problem

$$-y'' + q(x)y = z^2y, \quad x \in [0,\pi], \quad z > 0,$$

(2.1)

$$y(0) = 0, y'(0) = 1,$$

(2.2)

and let us introduce Prüfer-type variables:

$$y(x,z) = \frac{r(x,z)}{z} \sin \varphi(x,z),$$

(2.3)

$$y'(x,z) = r(x,z) \cos \varphi(x,z),$$

(2.4)

$$\varphi(0,z) = 0,$$

(2.5)

where $r(x,z) > 0$, and we denote by prime the derivative with respect to $x$ (and by dot the derivative with respect to $z$). Define further

$$\psi = \frac{\varphi}{z}. $$

(2.6)

An easy computation shows that for these variables the following equations hold:

$$\varphi' = z - \frac{q}{z} \sin^2 \varphi,$$

(2.7)

$$\frac{r'}{r} = \frac{q}{z} \sin \varphi \cos \varphi.$$  

(2.8)

Remark. These formulae hold in the usual sense at the continuity points of $q$ and in both half-sided senses at the jumps of $q$: $\varphi'_\pm(x,z) = z - \frac{q(x,\pm 0)}{z} \sin^2 \varphi(x,z)$, and analogously for $r$.

It is obvious that $y = 0$ iff $\sin \varphi = 0$, hence $z^2$ is an eigenvalue iff $\varphi(\pi,z)$ is a multiple of $\pi$. Denote by $z_n$ the square root of $\lambda_n$.

Lemma 2.1. $\varphi(\pi,z_n) = n\pi$.

Proof. See equation (2.7) in Ashbaugh and Benguria [2].

Our idea is to show that (under certain conditions) $\psi(x,z)$ is a monotone increasing function in $z$, since this will imply (1.4).

Theorem 2.2. Let $q(x) \geq 0$ be monotone decreasing in $[0,x_0]$. Then $\dot{\psi}(x_0,z) \geq 0$, i.e., $\psi(x_0,z)$ is a monotone increasing function in $z > 0$. If there is a $z > 0$ with $\dot{\psi}(x_0,z) = 0$, then $q = 0$ in $(0,x_0]$.

This theorem implies various results for different boundary conditions. For example, we mention the following corollary.

Corollary 2.3. Consider equation (1.1) with the Dirichlet-Neumann boundary conditions

$$y(0) = y'(\pi) = 0.$$  

(2.9)
If the potential $q$ is nonnegative and decreasing, then for the $m$-th and $n$-th eigenvalues with $m \leq n$,

$$\frac{\lambda_n}{\lambda_m} \leq \frac{(2n-1)^2}{(2m-1)^2},$$

and if for two different $m$ and $n$ equality holds, then $q = 0$ in $(0, \pi]$. The proof will be given in Section 3.

The main statement of this paper reads as follows:

**Theorem 2.4.** Consider equation (1.1) with the Dirichlet boundary conditions

$$y(0) = y(\pi) = 0.$$

If the potential $q$ is nonnegative and single-well, then for the $m$-th and $n$-th eigenvalues with $m \leq n$,

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2},$$

and if for two different $m$ and $n$ equality holds, then $q = 0$ in $(0, \pi)$. The proof will be given in Section 4.

3. The proof of Theorem 2.2

**Lemma 3.1.** If $q(x)$ is monotone decreasing in $[0, x_0]$, then (for $z > 0$) $\varphi(x_0, z)$ is a strictly monotone increasing function of $x$ in $[0, x_0]$. Moreover, $\varphi'(x, z) > 0$ for $z > 0$.

**Proof.** Fix $z$. From (2.7) if $\varphi'(\tilde{x}, z) \leq 0$, then $q(x) \geq z^2$ for $x < \tilde{x}$. Through $y'' = (q - z^2)y$, $y$ is convex, positive and increasing, so $y' > 0$, and by that, $\varphi(x, z) < \frac{\pi}{2}$ for $x < \tilde{x}$. For small $x > 0$, $\varphi'(0, z) > 0$ (see 2.7). The function $\varphi'(x, z)$ is continuous at the continuity points of $q(x)$, and (since $q(x)$ is monotone decreasing) cannot jump downward. Thus if somewhere $\varphi'(x, z)$ is negative or zero, there exists a point $x_2 \in (0, \tilde{x}]$ where $\varphi'(x_2, z) = 0$ and $\varphi'(x, z) > 0$ for $x < x_2$. Choose an arbitrary point $x_1 \in (0, x_2)$, then $0 < \varphi(x_1, z) < \frac{\pi}{2}$ in $[x_1, x_2]$ and

$$\cot \varphi(x_1, z) = \frac{1}{\sin^2 \varphi'(x_1, z)},$$

which implies $\sqrt{q(x) - z^2} < z \cot \varphi(x_2, z)$ for $x \in [x_1, x_2]$ and $\sqrt{q(x_2 - 0) - z^2} = z \cot \varphi(x_2, z)$. We show that this is not possible. Indeed, choose $x_3$ arbitrarily from
From (3.1) we get
\[
\left[ \log \left( z \cot(x, z) - \sqrt{q(x) - z^2} \right) \right]_{x_1}^{x_3}
= \int_{x_1}^{x_3} \frac{d( z \cot(x, z) - \sqrt{q(x) - z^2} )}{z \cot(x, z) - \sqrt{q(x) - z^2}}
\]
\[
= \int_{x_1}^{x_3} \frac{q(x) - z^2 - (z \cot(x, z))^2}{z \cot(x, z) - \sqrt{q(x) - z^2}} \, dx
- \int_{x_1}^{x_3} \frac{dq(x)}{2 \sqrt{q(x) - z^2}(z \cot(x, z) - \sqrt{q(x) - z^2})}
\geq \int_{x_1}^{x_3} -(z \cot(x, z) + \sqrt{q(x) - z^2}) \, dx
\]
by the monotonicity of $q$. Now $z \cot(x, z)$ is continuous and bounded in $[x_1, x_2)$, thus $-(z \cot(x, z) + \sqrt{q(x) - z^2})$ is bounded from below, and hence
\[
(3.2) \quad \left[ \log \left( z \cot(x, z) - \sqrt{q(x) - z^2} \right) \right]_{x_1}^{x_3} \geq K
\]
with $K$ independent of $x_3$. If we let $x_3$ approach $x_2$, this implies that
\[
(3.3) \quad z \cot(x_2, z) > \sqrt{q(x_2 - 0) - z^2},
\]
i.e., $\varphi(x_2, z) > 0$, a contradiction.

In the following formulae we sometimes write $\varphi(x)$ instead of $\varphi(x, z)$.

Lemma 3.2.

(3.4) \quad \dot{\varphi}(x) = \int_0^x \left( 1 + \frac{q(t)}{z^2} \sin^2 \varphi(t) \right)e^{-\int_0^t \frac{a}{2} \sin 2\varphi} \, dt.

Proof. Differentiate equation (2.7) with respect to $z$:
\[
\dot{\varphi}'(x, z) = 1 + \frac{q(x)}{z^2} \sin^2 \varphi(x) - \frac{q(x)}{z} \sin 2\varphi(x, z) \hat{\varphi}(x, z).
\]
This is a linear differential equation in $x \rightarrow \dot{\varphi}(x, z)$. Multiplying both sides by $e^{\int_0^x \frac{a}{2} \sin 2\varphi}$, we have
\[
(\dot{\varphi}(x, z)e^{\int_0^x \frac{a}{2} \sin 2\varphi})' = \left( 1 + \frac{q(x)}{z^2} \sin^2 \varphi(x, z) \right)e^{\int_0^x \frac{a}{2} \sin 2\varphi}.
\]
Using $\dot{\varphi}(0) = 0$, we get (3.4).

Remark. From (2.3) we can rewrite (3.4):
\[
\dot{\varphi}(x) = \int_0^x \left( 1 + \frac{q(t)}{z^2} \sin^2 \varphi(t) \right) \frac{r^2(t)}{r^2(x)} \, dt.
\]
From equation (3.7) it is obvious that $\varphi(x, z)$ is strictly monotone increasing in $z$. 
Corollary 3.3.

\begin{equation}
\psi(x) = \frac{2}{r^2(x)z^2} \int_0^x r^2(t) \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right). \tag{3.8}
\end{equation}

Proof.

\[ \psi(x) = \frac{\varphi(x)}{z} - \frac{\varphi(x)}{z^2} \]

\[ = \frac{1}{r^2(x)z^2} \left\{ \int_0^x r^2(t) \left[ 2(z - \varphi'(t)) + \varphi'(t) \right] dt - r^2(x)\varphi(x) \right\} \]

\[ = \frac{2}{r^2(x)z^2} \int_0^x r^2(t)(z - \varphi'(t)) - r(t)\varphi'(t) \right\} dt \]

\[ = \frac{2}{r^2(x)z^2} \int_0^x r^2 \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right). \]

\[ \square \]

From now on, we define the potential to be zero on \((x_0, \infty)\) and extend the definition of \(\varphi, x, r, \) and \(\psi\) accordingly. Then \(\varphi(x, z) \to 0\) if \(z\) is fixed and \(x \to \infty\).

Lemma 3.4. If \(0 < |\varphi| < \frac{\pi}{2}\), then \(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi > 0\).

Proof. This is a simple corollary of \(\varphi < \tan \varphi\) if \(0 < \varphi < \frac{\pi}{2}\). \(\square\)

Corollary 3.5. Let \(k \geq 0\) be an integer, \(k\pi \leq c \leq k\pi + \frac{\pi}{2}\), \(k\pi + \frac{\pi}{2} \leq d \leq (k+1)\pi\). Then for any fixed \(z\)

\begin{equation}
\int_{\varphi^{-1}(c)}^{\varphi^{-1}(c)} r^2(t) \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right) \geq -k\pi \frac{1}{2} \left[ r^2 \right]_{\varphi^{-1}(c)}^{\varphi^{-1}(c)} \tag{3.9}
\end{equation}

\begin{equation}
\int_{\varphi^{-1}(d)}^{\varphi^{-1}(d)} r^2(t) \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right) \geq -(k+1)\pi \frac{1}{2} \left[ r^2 \right]_{\varphi^{-1}(d)}^{\varphi^{-1}(d)}, \tag{3.10}
\end{equation}

and equality holds iff \(q = 0\) in the corresponding open interval.

Proof.

\[ \int_{\varphi^{-1}(c)}^{\varphi^{-1}(c)} r^2 \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right) \]

\[ = \int_{\varphi^{-1}(c)}^{\varphi^{-1}(c)} r^2 \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} (\varphi - k\pi) \sin \varphi \cos \varphi \right) \]

\[ -k\pi \int_{\varphi^{-1}(c)}^{\varphi^{-1}(c)} r^2 \left( \frac{q}{z} \sin \varphi \cos \varphi \right). \]
Now \((\varphi - k\pi)\) is between \(-\frac{\pi}{2}\) and \(\frac{\pi}{2}\), so, by the preceding lemma, the first term is positive, except when \(q = 0\). The second term is the same as in the right-hand side of (3.9) as we can easily see from (2.8).

The other part of the lemma can be proved in the same way:

\[
\int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}(d)} r^2(\frac{q}{z}\sin^2 \varphi - \frac{q}{z}\varphi \sin \varphi \cos \varphi)
\]

\[
= \int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)} r^2(\frac{q}{z}\sin^2 (\varphi - (k + 1)\pi) - \frac{q}{z}(\varphi - (k + 1)\pi) \sin \varphi \cos \varphi)
\]

\[-(k + 1)\pi \int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)} r^2(\frac{q}{z}\sin \varphi \cos \varphi)
\]

\[\geq -(k + 1)\pi \int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)} rr' = -(k + 1)\frac{\pi}{2} \left[ r^2 \right]_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)}.
\]

\[\square\]

Corollary 3.6. Let \(0 \leq C \leq \frac{\pi}{2}\), \(0 \leq D \leq \pi\). Then

\[(3.11) \quad \int_{0}^{\varphi^{-1}(C)} r^2(t)(\frac{q}{z}\sin^2 \varphi - \frac{q}{z}\varphi \sin \varphi \cos \varphi) \geq 0,
\]

\[(3.12) \quad \int_{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)}^{\varphi^{-1}(k\pi + \frac{\pi}{2})} r^2(t)(\frac{q}{z}\sin^2 \varphi - \frac{q}{z}\varphi \sin \varphi \cos \varphi)
\]

\[\geq -(k + 1)\pi \frac{\pi}{2} \left[ r^2 \right]_{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)}^{\varphi^{-1}(k\pi + \frac{\pi}{2})},
\]

and equality holds iff \(q = 0\) in the corresponding open interval.

(3.11), follows from (3.10) with \(k = 0\). If \(D \leq \frac{\pi}{2}\), then (3.12) is the same as (3.10).

If not, it is the sum of (3.10) with \(d = (k + 1)\pi\) and (3.9) with \(c = k\pi + \frac{\pi}{2} + D\) (and with \(k\) replaced by \(k + 1\)).

\[\square\]

Lemma 3.7. \(r(\varphi^{-1}(k\pi + 3\frac{\pi}{2})) \leq r(\varphi^{-1}(k\pi + \frac{\pi}{2}))\), if \(k = 0, 1, 2, ..., \). Moreover, the function \(r\) is monotone increasing between \(\varphi^{-1}(k\pi)\) and \(\varphi^{-1}(k\pi + \frac{\pi}{2})\) and is monotone decreasing between \(\varphi^{-1}(k\pi + \frac{\pi}{2})\) and \(\varphi^{-1}((k + 1)\pi)\).

Proof. Since the logarithmic function is strictly increasing, it is enough to prove

\[(3.13) \quad \left[ \log r^2 \right]_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}(k\pi + 3\frac{\pi}{2})} \leq 0.
\]
The monotonicity of \( \log r^2 \), (hence of \( r \)) follows from the sign of its derivative, \( \frac{q}{z} \sin 2\varphi \). By substituting \( u = \varphi(x) \):

\[
\int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{2u'}{r} = \int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{q}{z} \sin 2\varphi
\]

\[
= \int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{q(x) \sin 2\varphi(x)}{z^2 - q(x) \sin^2 \varphi(x)} \varphi'(x) \, dx
\]

\[
= \int_{k\pi + \frac{\pi}{2}}^{(k+1)\pi} \frac{q(\varphi^{-1}(u)) \sin 2u}{z^2 - q(\varphi^{-1}(u)) \sin^2 u} \, du.
\]

Note that \( \sin 2u < 0 \) for \( k\pi + \frac{\pi}{2} < u < (k+1)\pi \), while the denominator is always positive, as we have seen in Lemma 3.1. Hence if we replace \( q \) by its minimum, \( q(\varphi^{-1}((k+1)\pi)) \), the value of the fraction will increase:

\[
\int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{2u'}{r} \leq \int_{k\pi + \frac{\pi}{2}}^{(k+1)\pi} \frac{q(\varphi^{-1}((k+1)\pi)) \sin 2u}{z^2 - q(\varphi^{-1}((k+1)\pi)) \sin^2 u} \, du
\]

\[
= \left[ -\ln(z - \frac{q(\varphi^{-1}((k+1)\pi))}{z} \sin^2 u) \right]_{k\pi + \frac{\pi}{2}}^{(k+1)\pi}.
\]

The other part of the integral can be handled in an analogous way except that this time \( \sin 2u > 0 \) on the interval in question and we replace \( q \) by its maximum:

\[
(3.14) \quad \int_{\varphi^{-1}(k\pi + \frac{3\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{2u'}{r} \leq \left[ -\ln(z - \frac{q(\varphi^{-1}((k+1)\pi))}{z} \sin^2 u) \right]_{k\pi + \frac{\pi}{2}}^{(k+1)\pi}.
\]

Summing up,

\[
(3.15) \quad \int_{\varphi^{-1}(k\pi + \frac{3\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{2u'}{r} \leq \left[ -\ln(z - \frac{q(\varphi^{-1}((k+1)\pi))}{z} \sin^2 u) \right]_{k\pi + \frac{\pi}{2}}^{(k+1)\pi} = 0.
\]

\[\square\]
Proof of Theorem 2.2 If \( \varphi(x_0, z) \leq z \), then the statement of the theorem immediately follows from (3.8) and (3.11). If not, let \( \varphi(x_0, z) = \frac{z}{2} + k\pi + D \), with \( 0 \leq D \leq \pi \):

\[
\int_0^{\varphi^{-1}(\frac{z}{2} + k\pi + D)} r^2(t) \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right) dt
\]

\[
= \int_0^{\varphi^{-1}(\frac{z}{2})} r^2(t) \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right) dt
\]

\[
+ \sum_{i=1}^{k} \int_{\varphi^{-1}(i\pi + \frac{z}{2})}^{\varphi^{-1}(i\pi + \frac{z}{2} + D)} r^2(t) \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right) dt
\]

\[
+ \int_{\varphi^{-1}(k\pi + \frac{z}{2})}^{\varphi^{-1}(k\pi + \frac{z}{2} + D)} r^2(t) \left( \frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi \right) dt.
\]

Here every term is nonnegative by Corollary 3.6 and Lemma 3.7 and if their sum is zero, then \( q \) has to be zero in the whole interval \((0, x_0)\). \( \square \)

Proof of Corollary 2.3 By the current boundary conditions,

\[
z_n \psi(\pi, z_n) = (n - \frac{1}{2})\pi.
\]

Let \( m \) be less than \( n \). Then \( \frac{(2m-1)\pi}{2z_m} = \psi(\pi, z_m) \leq \psi(\pi, z_n) = \frac{(2n-1)\pi}{2z_n} \), and thus \( z_m \leq \frac{2n-1}{2m-1} \) and \( \frac{z_m}{z_n} = \frac{2n-1}{2m-1} \). If equality holds, then \( \psi(\pi, z_m) = \psi(\pi, z_n) \), and by Theorem 2.2 this implies that \( q = 0 \) in \((0, \pi)\). \( \square \)

4. The Proof of Theorem 2.4

Let the potential \( q(x) \) be monotone decreasing in \([0, a]\) and monotone increasing in \([a, \pi]\). Denote by \( \tilde{q}(x) \) the reverse of the potential, i.e., \( \tilde{q}(x) = q(\pi - x) \). Then \( y(\pi - x, z_n) \) is an eigenfunction for the potential \( \tilde{q}(x) \). Moreover, define

\[
\tilde{y}(x, z_n) = (-1)^{n+1} \frac{y(\pi - x, z_n)}{r(\pi, z_n)},
\]

\[
\tilde{r}(x, z_n) = \frac{r(\pi - x, z_n)}{r(\pi, z_n)},
\]

and

\[
\tilde{\varphi}(x, z_n) = n\pi - \varphi(\pi - x, z_n).
\]

Then

\[
\tilde{y}(0, z_n) = 0,
\]

\[
\tilde{y}'(0, z_n) = 1,
\]

which means that \( \tilde{y}(x, z_n) \) is the solution of the initial value problem (2.1)–(2.2) with \( \tilde{q} \) instead of \( q \). It is also simple that

\[
\tilde{y}(x, z_n) = \frac{\tilde{r}(x, z_n)}{z_n} \sin \tilde{\varphi}(x, z_n),
\]
Proof of Theorem 2.4. Consider the function \( \Psi(z) = \psi(a, z) + \psi(\pi - a, z) \). This is, by Theorem 2.2, the sum of two monotone increasing functions. By (4.3),

\[
(5.1) \quad \tilde{\varphi}(n, z_n) = n\pi.
\]

Let \( m \) be less than \( n \). Then \( \frac{mn}{m} = \Psi(z_m) \leq \Psi(z_n) = \frac{n\pi}{z_n} \), and thus \( \frac{z_m}{z_n} \leq \frac{m}{n} \) and \( \frac{z_m}{z_n} \leq \frac{n}{m} \). If equality holds, then \( \Psi(z_m) = \Psi(z_n) \), hence \( \psi(a, z_m) = \psi(a, z_n) \) and \( \psi(\pi - a, z_m) = \psi(\pi - a, z_n) \). By Theorem 2.2 this implies that \( q = 0 \) in \((0, a]\) and \( \tilde{q} = 0 \) in \((0, \pi - a]\), i.e., \( q = 0 \) in \((0, \pi)\). \( \square \)

5. Remarks

Remark 1. If the potential is not monotone decreasing, then \( \psi \) might not increase in \( z \) at some point. For example, let \( q \) be zero in \([0, \frac{\pi}{3}]\) and 1 otherwise. Pick \( z = 3/2 \). We can easily see that

\[
(5.1) \quad y(x, 3/2) = \begin{cases} 2 \frac{\sin 3}{3} x & \text{if } x \leq \frac{2}{3} \pi, \\ -\frac{2}{\sqrt{5}} \sin \frac{5}{2} (x - \frac{2}{3} \pi) & \text{otherwise}, \end{cases}
\]

\[
(5.2) \quad y'(x, 3/2) = \begin{cases} \cos \frac{3}{2} x & \text{if } x \leq \frac{2}{3} \pi, \\ -\cos \frac{5}{2} (x - \frac{2}{3} \pi) & \text{otherwise}. \end{cases}
\]

From (2.7)

\[
(5.3) \quad \varphi'(x) \geq \frac{3}{2} - \frac{2}{3} > 0.
\]

It can be easily checked that \( \varphi(\frac{2}{3} \pi, \frac{3}{2}) = \pi, \varphi(\pi, \frac{3}{2}) \approx 4.27083 < \frac{3}{2} \pi \). Combining this with (2.8), we get

\[
(5.4) \quad r(x) = \exp\left(\int_0^x \frac{2}{z} \sin \varphi \cos \varphi \right) = \begin{cases} \exp\left(\int_0^{\varphi(x)} \frac{1}{\frac{3}{2} - \frac{2}{3} \sin^2 v} \, dv \right) & \text{if } x \leq \frac{2}{3} \pi, \\ \exp\left(\int_0^{\varphi(x)} \frac{1}{\frac{3}{2} - \frac{2}{3} \sin^2 v} \, dv \right) & \text{otherwise}; \end{cases}
\]

hence

\[
\varphi(\pi) = \frac{2}{r^2(\pi) z^2} \int_0^\pi r^2 \left( \frac{2}{z} \sin^2 \varphi - \frac{2}{\pi} \varphi \sin \varphi \cos \varphi \right)
\]

\[
= \frac{2}{r^2(\pi) z} \int_0^\pi r^2 \left( \frac{2}{z} \sin^2 \varphi - \frac{2}{\pi} \varphi \sin \varphi \cos \varphi \right) \varphi',
\]

and substituting \( \varphi(x) = u \) to this integral, we can numerically compute it (using Maple or Mathematica):

\[
(5.5) \quad \int_{\pi}^{4.27083} \exp\left(\int_0^u \frac{2}{\frac{3}{2} - \frac{2}{3} \sin^2 v} \, dv \right) \left( \frac{2}{\frac{3}{2} \sin^2 u - \frac{2}{3} \sin u \cos u} \right) \, du \approx -0.811
\]

which means that \( \psi \) is not growing at \( z = \frac{3}{2} \).
Remark 2. We could ask whether the sum of the $\psi(\pi)$’s belonging to the potential $q(x)$ and $q(\pi - x)$ is a monotone increasing function in $z$. But the last example shows that this is not necessarily the case either. Indeed,

\begin{equation}
\dot{\psi}(\pi) \approx \frac{2}{r^2(\pi)z^2}(-0.811) \approx -0.292.
\end{equation}

In a similar manner we can compute $\dot{\psi}(x, \frac{3}{2})$ belonging to $q(\pi - x)$, which approximately equals 0.0306, so the sum is also negative.

References

