REMARKS ON A PAPER BY CHAO-PING CHEN AND FENG QI

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Abstract. In a recent paper, Chao-Ping Chen and Feng Qi (2005) established sharp upper and lower bounds for the sequence

\[ P_n := \frac{\Gamma(n + \frac{1}{4})}{\sqrt{\pi} \Gamma(n + \frac{3}{4})} \]

We show that their result follows easily from a theorem of G. N. Watson published in 1959. We also show that the main result of Chen and Qi’s paper is a special case of a more general inequality which admits a very short proof.

1. Introduction and results

Let

\[ P_n := \frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n} \]

This sequence appears in Wallis’s well-known product formula of approximation of \( \pi \) and in various other topics of analysis and number theory. In the recent paper [2], Chao-Ping Chen and Feng Qi proved the following inequality:

\[ \frac{1}{\sqrt{\pi(n + 1/4 - 1)}} \leq P_n < \frac{1}{\sqrt{\pi(n + 1/4)}} \quad n = 1, 2, \ldots. \]

The constants \( \frac{4}{\pi} - 1 \) and \( \frac{1}{4} \) are the best possible. Inequality (1.1) improves some earlier results dealing with estimates of the sequence \( P_n \).

Here we observe that (1.1) follows easily from a known result due to G. N. Watson [15]. Indeed, it is shown in [15] that the function

\[ \theta(x) = \left( \frac{\Gamma(x + 1)}{\Gamma(x + \frac{3}{4})} \right)^2 - x \]

is strictly decreasing on \((-1/2, \infty)\). Applying this result, together with the observation that \( \theta(1) = 4/\pi - 1 \), \( \lim_{x \to \infty} \theta(x) = 1/4 \) and \( P_n = \frac{\Gamma(n + 1/4)}{\sqrt{\pi} \Gamma(n + 3/4)} \), we obtain the double inequality (1.1).

The purpose of this note is to show that (1.1) is a special case of a more general inequality, which in fact, admits a very short proof. Let \( 0 < \alpha < 1 \) and

\[ d_n(\alpha) := \frac{(1 - \alpha)n}{n!}. \]
As usual, \((a)_k\) denotes the Pochhammer symbol, defined by
\[
(a)_0 = 1, \quad \text{and} \quad (a)_k = a(a+1) \ldots (a+k-1) = \frac{\Gamma(k+a)}{\Gamma(a)}, \quad \text{for } k = 1, 2, \ldots .
\]
Clearly, \(d_n(1/2) = P_n\). It is also well known and easy to see that
\[
\frac{1}{\Gamma(1 - \alpha)(n+1)^\alpha} < d_n(\alpha) < \frac{1}{\Gamma(1 - \alpha)n^\alpha}.
\]
Here we show that this inequality can be improved as follows.

**Theorem 1.1.** For all natural numbers \(n\), we have
\[
\frac{1}{\Gamma(1 - \alpha)(n+c_2)^\alpha} \leq d_n(\alpha) < \frac{1}{\Gamma(1 - \alpha)(n+c_1)^\alpha},
\]
where the constants
\[
c_1 = c_1(\alpha) = \frac{1 - \alpha}{2} \quad \text{and} \quad c_2 = c_2(\alpha) = \frac{1}{\Gamma(2 - \alpha)} \frac{1}{1/\alpha} - 1
\]
are the best possible.

It is clear that for \(\alpha = 1/2\), inequality (1.3) coincides with (1.1).

Inequalities such as (1.2) and (1.3) are of particular importance in certain problems on positive trigonometric sums and positive sums of Gegenbauer polynomials, having \(d_k(\alpha)\) as a sequence of coefficients. See the recent papers \([9]\) and \([10]\). We also refer to the paper \([8]\) for some interesting estimates of ratios of gamma functions. Some other proofs of (1.1) and related results can be found in the recent articles \([3]\), \([4]\), \([5]\), \([6]\) and \([7]\). Compare also the papers \([12]\), \([13]\) and \([14]\) for different proofs of (1.1) and various inequalities for the sequence \(P_n\).

2. **Proof of (1.3)**

For the proof of (1.3) we define
\[
Q_n(\alpha) := \left\{ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \right\}^{1/\alpha} - n.
\]

Using the asymptotic formula
\[
\frac{x^{b-a} \Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), \quad x \to \infty
\]
(see \([1]\), p. 615, for the complete form of this formula), we easily verify that
\[
\lim_{n \to \infty} Q_n(\alpha) = \frac{1 - \alpha}{2} = c_1.
\]
Obviously \(c_2 = Q(1, \alpha)\).

The required inequality (1.3) follows from the fact that the sequence \(Q_n(\alpha)\) is strictly decreasing. This, in turn, is an immediate consequence of a result of N. Elezović, C. Giordano and J. Pečarić, \([8]\) who showed that the function
\[
x \mapsto \left( \frac{\Gamma(x+t)}{\Gamma(x+s)} \right)^{1/(t-s)} - x
\]
is convex and decreasing on \((-r, \infty)\) if \(|t-s| < 1\), where \(s, t\) are given positive numbers and \(r = \min(s, t)\).

This completes the proof of (1.3). \(\square\)
References


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