ON QUASI-COMPLETE INTERSECTIONS OF CODIMENSION 2

YOUNGOOK CHOI

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Abstract. In this paper, we prove that if $X \subset \mathbb{P}^n$, $n \geq 4$, is a locally complete intersection of pure codimension 2 and defined scheme-theoretically by three hypersurfaces of degrees $d_1 \geq d_2 \geq d_3$, then $H^1(\mathbb{P}^n, I_X(j)) = 0$ for $j < d_3$ using liaison theory and the Arapura vanishing theorem for singular varieties. As a corollary, a smooth threefold $X \subset \mathbb{P}^5$ is projectively normal if $X$ is defined by three quintic hypersurfaces.

1. Introduction

Let $X$ be a nondegenerate projective variety of codimension $e$ and degree $d$ in $\mathbb{P}^n$, defined over a complex number field $\mathbb{C}$. $X$ is said to be the scheme-theoretic intersection of $k+1$ homogeneous equations $f_1, f_2, \ldots, f_{k+1}$ if the saturation of an ideal $J = (f_1, \ldots, f_{k+1})$ is equal to the homogeneous ideal $I_X = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^N, I_X(j))$ of $X$. $X$ is called a quasi-complete intersection (qci for short) if $X$ is a scheme-theoretic intersection of $e = \text{codim } X + 1$ equations. In their paper ([3], Proposition 1), Beorchia and Ellia showed that if $X$ is a smooth codimension 2 qci, which is not a complete intersection, of three hypersurfaces of degrees $d_1, d_2, d_3$, then the integers $d_i$ are determined by $X$; that is, if $X$ is a qci of three other hypersurfaces of degrees $d_1', d_2', d_3'$, then $d_i = d_i'$ for $i = 1, 2, 3$ up to order. This result is generalized by Bresinsky, Schenzel, and St"uckrad ([5]) for irreducible and reduced (not necessarily smooth) qci’s of codimension 2 which are not complete intersections.

Definition 1.1. A variety $X \subset \mathbb{P}^n$ is called a qci of type $(d_1, d_2, d_3)$ if $X$ is a codimension two, quasi-complete intersection of three hypersurfaces of degrees $d_1, d_2, d_3$ with $d_1 \geq d_2 \geq d_3$ and general 3-dimensional plane sections of $X$ are smooth. A variety whose homogeneous ideal $I_X$ is generated by $e+1$ equations is called an almost complete intersection.

This paper is concentrated on the cohomological properties of quasi-complete intersections of codimension two. $X \subset \mathbb{P}^n$ is said to be $t$-normal if the cohomology group $H^1(\mathbb{P}^n, I_X(t)) = 0$. By the Serre vanishing theorem, $X$ is $t$-normal for all $t \gg 0$, and it would be very interesting to find an explicit lower bound $B(d,e)$ in terms of its degree $d$ and codimension $e$ such that $X$ is $t$-normal for all $t \geq B(d,e)$. For example, S. Kwak ([13]) showed that any smooth threefold $X \subset \mathbb{P}^5$ of degree...
$d$ is $t$-normal for all $t \geq d - 4$ and that this bound is sharp as the Palatini scroll of degree 7 shows. In their paper ([4], Proposition 1), Bertram-Ein-Lazarsfeld gave an interesting bound about $t$-normality in terms of degrees of defining equations of $X$: Let $X \subset \mathbb{P}^n$ be the scheme-theoretic intersection of $m$ equations $f_1,f_2,\ldots,f_m$ with $d_1 \geq d_2 \geq \cdots \geq d_m$, $m \geq e = \text{codim}(X)$. Then we have

$$H^i(\mathbb{P}^n, \mathcal{I}_X(j)) = 0, \quad i \geq 1, \quad j \geq d_1 + d_2 + \cdots + d_e - n.$$ 

For a smooth $\text{qci}$ of type $(d_1,d_2,d_3)$, we get the following theorem.

**Theorem 1.2.** Let $X$ be a smooth codimension two subvariety in $\mathbb{P}^n$, $n = 4, 5$. If $X$ is a quasi-complete intersection of type $(d_1,d_2,d_3)$ with a sheaf of ideals $\mathcal{I}_X$, then $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 - 2n + 1$ and $i \geq 1$.

On the other hand, for projective varieties of small codimension, it has also been focused on the vanishing of cohomology groups $H^i(\mathbb{P}^n, \mathcal{I}_X(j))$ for a small positive integer $j$. For example, a famous theorem of Zak's tells us that any smooth threefold is linearly normal, i.e. $H^1(\mathbb{P}^5, \mathcal{I}_X(1)) = 0$. In this sense, it is quite useful to know the following conjecture due to Peskine and Zak: Let $X$ be a nondegenerate, not necessarily smooth, projective variety of codimension $e$ in $\mathbb{P}^n$:

1. $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) = 0$ for $i \geq 1$, $j \geq 0$, $i + j < \frac{\dim(X)}{e - 1}$.
2. For $i \geq 1$, $j \geq 0$, $i + j = \frac{\dim(X)}{e - 1}$, it is possible to describe all varieties for which $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) \neq 0$.

Now we would like to consider this conjecture from the point of defining equations of codimension two $\text{qci}$ subvarieties. Note that all complete intersections and almost complete intersections of codimension two are projectively Cohen-Macaulay. For quasi-complete intersections, we can prove the following theorem using liaison theory and the Arapura vanishing theorem.

**Theorem 1.3.** Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a reduced locally complete intersection of codimension 2. If $X$ is a quasi-complete intersection of type $(d_1,d_2,d_3)$, then $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k < d_3$. Furthermore, if $\dim(X) = 3$ and the singular locus of $X$ is finite, then $H^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k \leq d_3$ and $H^2(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k < d_3$.

A corollary of these two theorems is:

**Corollary 1.4.** If $X$ is a smooth threefold in $\mathbb{P}^5$ that is cut out scheme-theoretically by three equations of degrees $d_1 \geq d_2 \geq d_3$ with $d_1 + d_2 \leq 10$, then $X$ is projectively normal, and if $d_1 + d_2 \leq 9$, then $X$ is projectively Cohen-Macaulay.

In Section 2, we will review some basic backgrounds including the Peskine-Szpiro Theorem ([16], Proposition 4.1) and the Gherardelli Linkage Theorem ([10], Theorem 2.5) needed in this paper. In Section 3, we will prove the above theorems and give some corollaries and examples.

## 2. Preliminaries

The following two theorems are generalizations of the classical Kodaira vanishing theorem for ample vector bundles and for singular subvarieties.

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Theorem 2.1. Let $E$ be a vector bundle of rank $r$ on $\mathbb{P}^n$, and let $L$ be an ample line bundle. Assume that $E$ is generated by its global sections. Then

$$h^q(\mathbb{P}^n, \bigwedge^t E \otimes L \otimes K_{\mathbb{P}^n}) = 0 \quad \text{for} \quad q > r - t.$$  

Proof. See [8], page 54.

Theorem 2.2. Let $Y$ be a projective variety, and let $L$ be an ample line bundle on $Y$. Fix $k \geq 1$. Assume that $k < \text{codim}(\text{Sing}(Y))$ and that $Y$ is $S_{k+1}$. Then $H^k(Y, \mathcal{L}^{-1}) = 0$. (If $Y$ is smooth, we define $\text{codim}(\text{Sing}(Y)) = \dim(Y)$.)

Proof. See [10], Theorem 1.1.

Theorem 2.3 (Gherardelli Linkage Theorem). Let $X \subset \mathbb{P}^n$ be a Cohen-Macaulay subvariety of pure codimension 2 and $Z$ a complete intersection of two hypersurfaces $S_1$ and $S_2$. Assume that $Z$ has pure codimension 2. Then $X \subset Z$ is a complete intersection, i.e. $K_X = \mathcal{O}(k)$ for some $k \in \mathbb{Z}$ if and only if its residual scheme $Y \subset Z$ of $X$ is scheme-theoretically defined by the intersection of $S_1$ and $S_2$ with a third hypersurface $S_3$. In addition, if $d_i = \deg(S_i)$ for $i = 1, 2, 3$, then

$$K_X = \mathcal{O}_X(d_1 + d_2 - d_3 - n - 1).$$

Proof. See [16], Proposition 4.1.

Theorem 2.4 (Peskine-Szpiro Theorem). Let $X$ be a closed subscheme without embedded components, that is a generically complete intersection of codimension 2 in $\mathbb{P} = \mathbb{P}_k^n$, defined by a graded ideal $I$ in $k[X_0, \ldots, X_l]$. Let $U_1$ be an open set consisting of points in $X$ where $X$ is complete intersection in $\mathbb{P}$ and $U_2 \subset X$ consisting of regular points in $X$.

Then there exist homogeneous equations $\alpha_1, \alpha_2 \in I$ defining two hypersurfaces $S_{\alpha_1}$ and $S_{\alpha_2}$ which intersect properly such that $S_{\alpha_1} \cap S_{\alpha_2} = X \cap Y$, where

1) $X$ and $Y$ have no common components.
2) $Y \cap U_1$ is a locally complete intersection.
3) $Y$ is a complete intersection at all points of codim $\leq 3$ in $Y \cap U_1$.
4) $Y - X \cap Y$ is non-singular.
5) $Y \cap U_2$ is non-singular in codimension 2 and

$$\text{codim}(Y \cap (U_1 - U_2), (U_1 - U_2)) \geq 1.$$  

6) $Y$ is non-singular at all points $x \in Y \cap U_2$ of codim $\leq 3$ in $Y$. Moreover, if char $k = 0$, one can take the elements $\alpha_i$ of degree $d_i$ provided that

$$d_1 \geq \inf \{ t : I_t k[X_0, X_1, \ldots, X_l] \text{ defines } X \},$$

$$d_2 \geq \inf \{ t : I_t k[X_0, X_1, \ldots, X_l]/\alpha_1 \text{ defines } X \text{ in } k[X_0, X_1, \ldots, X_l]/\alpha_1 \}.$$

Proof. See [10], Proposition 4.1.

We will apply Theorem 2.4 for a locally complete intersection $X$ of codimension two in $\mathbb{P}^n$.

Remark 2.5. If $X$ is a qci of type $(d_1, d_2, d_3)$, then by Theorem 2.4 one can find two hypersurfaces $S_{\alpha_1}$ and $S_{\alpha_2}$ of degrees $d_1$ and $d_2$ satisfying the properties of Theorem 2.4 in such a way that $\{\alpha_1, \alpha_2\}$ belong to a set of minimal generators of $I_X$. By Theorem 2.3, the residual $Y$ of $X$ (in the complete intersection of $S_{\alpha_1}$ and $S_{\alpha_2}$) is subcanonical and holds the properties of Theorem 2.4.
Theorem 2.6. Let $E$ be a locally free sheaf on $\mathbb{P}^n$. If $\operatorname{rank}(E) < n$ and $n$ is even, then $E$ splits if and only if $H^i_+(E) = 0$ for $1 < i < n - 1$. If $\operatorname{rank}(E) < n - 1$ and $n$ is odd, then $E$ splits if and only if $H^i_+(E) = 0$ for $1 < i < n - 1$.

Proof. See [13], Theorem 1.

Corollary 2.7. Let $X$ be a Cohen-Macaulay surface in $\mathbb{P}^4$ defined by three hypersurfaces. Then $X$ is projectively normal if and only if $X$ is projectively Cohen-Macaulay.

Proof. If $X$ is defined by three hypersurfaces of degrees $d_1, d_2, d_3$, then $\mathcal{I}_X$ has the following resolution:

$$0 \rightarrow E \rightarrow \mathcal{O}_{\mathbb{P}^4}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^4}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^4}(-d_3) \rightarrow \mathcal{I}_X \rightarrow 0,$$

where $E$ is a kernel of a canonical map $\varphi$. Since $X$ is Cohen-Macaulay, $E$ is a vector bundle of rank two on $\mathbb{P}^4$. If $X$ is projectively normal, then $H^1(\mathbb{P}^4, \mathcal{I}(k)) = H^2(\mathbb{P}^4, E(k)) = 0$ for all $k \in \mathbb{Z}$. By Theorem 2.6, $E$ splits into line bundles, which implies that $X$ is projectively Cohen-Macaulay.

3. Main results and applications

As Mumford defined, for a coherent sheaf $F$ on $\mathbb{P}^n$ it is said to be $m$-regular if $H^i(\mathbb{P}^n, F(m-i)) = 0$ for all $i > 0$, and the regularity of $F$ is defined by the formula

$$\operatorname{reg} F = \min \{ m \in \mathbb{Z} : F \text{ is } m\text{-regular} \}.$$

Lemma 3.1. Let $X$ be a smooth non-degenerate subvariety of codimension two in $\mathbb{P}^n$, $n = 4, 5$. Assume that $X$ is defined by $(k+1)$ hypersurfaces of degrees $d_1, \ldots, d_{k+1}$, i.e. there is a locally free resolution

$$0 \rightarrow E \rightarrow \bigoplus_{i=1}^{k+1} \mathcal{O}_{\mathbb{P}^n}(-d_i) \rightarrow \mathcal{I}_X \rightarrow 0.$$

Then

(a) If $\dim X = 2$, then $\operatorname{reg}(E^*) \leq -3$.

(b) If $\dim X = 3$ and $X$ is not in a cubic hypersurface, then $\operatorname{reg}(E^*) \leq -4$.

Proof. (a) First note that $E$ is a locally free sheaf of rank $k$ because $X$ is smooth of codimension two. By definition, $E^*$ is $(-3)$-regular if and only if

$$H^i(\mathbb{P}^4, E^*(-3-i)) = 0$$

for all $i \geq 1$. By Serre’s duality, this is equivalent to $H^j(\mathbb{P}^4, E(2-j)) = 0$ for $0 \leq j \leq 3$. From an exact sequence (14), when $j = 3$, we have by the Kodaira vanishing theorem

$$H^3(\mathbb{P}^4, E(-1)) = H^2(\mathcal{I}_X(-1)) = H^1(\mathcal{O}_X(-1)) = 0.$$

Note also that for $j = 0, 1, 2$

$$H^2(\mathbb{P}^4, E) = H^1(\mathcal{I}_X) = 0 \ (X \text{ is smooth}),$$

$$H^1(\mathbb{P}^4, E(1)) = 0 \ (X \text{ is non-degenerate}),$$

$$H^0(\mathbb{P}^4, E(2)) = 0.$$

Note that the last vanishing comes from the exact sequence (1) and the non-degeneracy of $X$.

(b) See [6], Lemma 3.7. □
Note that by Faltings theorem ([9], §4), all smooth quasi-complete intersections $X \subset \mathbb{P}^n$ of codimension 2 are complete intersections if $n \geq 6$.

**Theorem 3.2.** Let $X$ be a smooth codimension two subvariety in $\mathbb{P}^n$, $n = 4, 5$. If $X$ is a quasi-complete intersection of type $(d_1, d_2, d_3)$ with an ideal sheaf $\mathcal{I}_X$, then:

(a) $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k \leq \dim X$ and $1 \leq i < \dim X$.

(b) $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3$ and $i \geq 1$.

**Proof.** (a) Assume that $\dim X = 3$. If $X$ is contained in a cubic hypersurface, then it is projectively Cohen-Macaulay ([7], Proposition 5.1), so we assume that $X$ is not contained in a cubic hypersurface. By Lemma 3.1, $\mathcal{E}^*$ is $(-4)$-regular, so $\mathcal{E}^*(-4)$ is generated by its global sections. Applying Theorem 2.1 one can get

$$H^i(\mathbb{P}^5, K_{\mathbb{P}^5} \otimes \mathcal{E}^*(-4) \otimes \mathcal{O}(k)) = 0 \quad \text{for} \quad k \geq 1, \ i > 1.$$  

By Serre duality, $H^i(\mathbb{P}^5, \Omega(4 - k)) = 0$ for $k \geq 1$ and $0 \leq i \leq 3$. This gives $H^i(\mathcal{I}_X(k)) = 0$ for $k \leq 3$ and $1 \leq i < 3$. For $\dim X = 2$, by Lemma 3.1, $\mathcal{E}^*$ is $(-3)$-regular and, in the same way, we get $H^i(\mathcal{I}_X(k)) = 0$ for $k \leq 2$.

(b) Assume that $\dim X = 3$. Note that $\mathcal{E}$ in the exact sequence (1) is locally free of rank 2 and $c_1(\mathcal{E}) = -(d_1 + d_2 + d_3)$. So $\mathcal{E}^*(-4) = \mathcal{E}(d_1 + d_2 + d_3 - 4)$, and for $k \geq 1$ and $i > 1$,

$$H^i(\mathbb{P}^n, \mathcal{E}(d_1 + d_2 + d_3 - 10 + k)) = H^i(\mathbb{P}^5, K_{\mathbb{P}^5} \otimes \mathcal{E}^*(-4) \otimes \mathcal{O}(k)) = 0.$$  

This implies $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 - 9$ and $i \geq 1$. Similar proof goes for $\dim X = 2$. \hfill $\Box$

This method can be applied to a smooth threefold in $\mathbb{P}^5$ which is cut out by four equations.

**Theorem 3.3.** Let $X$ be a smooth threefold in $\mathbb{P}^5$. If $X$ is cut out scheme-theoretically by four hypersurfaces of degrees $d_1, d_2, d_3, d_4$, then:

(a) $H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for $k \leq 3$.

(b) $H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 + d_4 - 13$ and $i \geq 1$.

**Proof.** (a) The proof is similar to the proof of Theorem 3.2(a).

(b) By Lemma 3.1, $\mathcal{E}^*(-4)$ is generated by its global sections, and by Theorem 2.1

$$H^i(\mathbb{P}^5, K_{\mathbb{P}^5} \otimes \bigwedge^2 \mathcal{E}^*(-4) \otimes \mathcal{O}(k)) = 0 \quad \text{for} \quad k \geq 1, \ i > 1.$$  

Since $X$ is cut out by 4 hypersurfaces, $\mathcal{E}^*$ is locally free of rank 3. So $\bigwedge^2 \mathcal{E}^*(-4) = \mathcal{E}(d_1 + d_2 + d_3 + d_4 - 8)$ and $H^i(\mathbb{P}^5, \mathcal{E}(d_1 + d_2 + d_3 + d_4 - 14 + k)) = 0$ for $i > 1$ and $k \geq 1$. Therefore, $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 + d_4 - 13$ and $i \geq 1$. \hfill $\Box$

**Corollary 3.4.** In the situation of Theorem 3.3 above, if $d_1 + d_2 + d_3 + d_4 \leq 17$, then $X$ is projectively normal.

**Theorem 3.5.** Let $X$ be a smooth surface in $\mathbb{P}^4$. If $X$ is cut out scheme-theoretically by four hypersurfaces of degrees $d_1, d_2, d_3, d_4$, then $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 + d_4 - 10$, and $i \geq 1$.

**Proof.** The proof is similar to the proof of Theorem 3.3(b). \hfill $\Box$
Proposition 2.1). So by Theorem 2.2, we have
\[ 0 \to \mathcal{F}^*(-4) \to \mathcal{O}_{\mathbb{P}^4}(-3) \oplus^4 \to \mathcal{I}_X \to 0, \]
where \( \mathcal{F} \) is the Tango bundle on \( \mathbb{P}^4 \). So, \( X \) is cut out by four cubic hypersurfaces. This shows that Theorem 3.5 is sharp.

From now on, we do not assume that \( X \) is smooth. As usual, liaison theory is a tool for studying a qci of codimension two.

**Theorem 3.7.** Let \( X \subset \mathbb{P}^n \), \( n \geq 4 \), be a Cohen-Macaulay and quasi-complete intersection of type \( (d_1, d_2, d_3) \) and \( Y \) a residual of \( X \) in the complete intersection of hypersurfaces \( S_1 \) and \( S_2 \) of degrees \( d_1 \) and \( d_2 \), i.e. \( X \cup Y = S_1 \cap S_2 \). If \( \text{codim}(\text{Sing}(Y)) = t \) for some \( t \in \mathbb{Z} \), then:

1. \( H^i(\mathcal{I}_X(k)) = 0 \) for \( 0 \leq i < t \) and \( k < d_3 \), and
2. \( H^i(\mathcal{I}_X(k)) = 0 \) for \( i > n-2-t \), \( k \geq d_1 + d_2 - n \).

**Proof.** (a) First of all, since \( d_3 \) is a minimal degree of a hypersurface containing \( X \) \((11)\), \( H^0(\mathcal{I}_X(k)) = 0 \) for \( k < d_3 \). Since \( X \) is a qci of codimension 2, by Theorem 2.2 \( Y \) is a subcanonical with \( K_Y = \mathcal{O}_Y(d_1 + d_2 - d_3 - n - 1) \). Since \( X \cup Y = S_1 \cap S_2 \), the liaison theory gives the following exact sequence \((13)\), page 117:

\[ 0 \to \mathcal{I}_{X \cup Y} \to \mathcal{I}_X \to K_Y(n + 1 - d_1 - d_2) \to 0. \]

Therefore we have
\[ 0 \to \mathcal{I}_{X \cup Y}(d_3) \to \mathcal{I}_X(d_3) \to \mathcal{O}_Y \to 0. \]

As \( X \cup Y \) is a complete intersection, \( H^i(\mathcal{I}_{X \cup Y}(k)) = 0 \) for \( 1 \leq i \leq n-2 \) and for all \( k \in \mathbb{Z} \). Note that the residual \( Y \) is Cohen-Macaulay if \( X \) is Cohen-Macaulay \((10)\), Proposition 2.1). So by Theorem 2.2 we have \( H^i(\mathcal{O}_Y(k)) = 0 \) for \( 0 \leq i < t \) and \( k < 0 \). Therefore \( H^i(\mathcal{I}_X(k)) = 0 \) for \( k < d_3 \) and \( 0 \leq i < t \).

(b) By Serre duality and Theorem 2.2 \( H^i(K_Y(k)) = 0 \) for \( i > n-2-t \) and \( k \geq 1 \). Since \( X \cup Y \) is a complete intersection of degrees \( d_1 \) and \( d_2 \), \( H^i(\mathcal{I}_{X \cup Y}(k)) = 0 \) for \( i \geq 1 \) and for all \( k \geq d_1 + d_2 - n \). So we get, from the exact sequence \((3)\),
\[ H^i(\mathcal{I}_X(k)) = 0 \text{ for } i > n - 2 - t, \quad k \geq d_1 + d_2 - n. \]

In their paper \((4)\), Bertram-Ein-Lazarsfeld show that if \( X \) is a smooth qci of type \( (d_1, d_2, d_3) \), then \( H^i(\mathcal{I}_X(k)) = 0 \) for \( k \geq d_1 + d_2 - n \) and \( i \geq 1 \). So if \( Y \) is smooth, the Kodaira vanishing theorem holds for \( Y \), i.e. \( H^i(\mathcal{I}_X(k)) = 0 \) for \( k \geq 1, \quad i \geq 1 \). So one may consider (b) as a generalization of the Bertram-Ein-Lazarsfeld theorem for singular varieties.

If \( X \) is a locally complete intersection, then one can control singular locus \( \text{Sing}(Y) \) of residual \( Y \) of \( X \) via the Peskine-Szpiro theorem (Theorem 2.4). Using this theorem, one can state the theorem without mentioning a residual \( Y \) of \( X \).

**Theorem 3.8.** Let \( X \subset \mathbb{P}^n \), \( n \geq 4 \), be a reduced locally complete intersection of codimension 2. If \( X \) is a quasi-complete intersection of type \( (d_1, d_2, d_3) \), then:

1. \( H^i(\mathcal{I}_X(k)) = 0 \) for \( k < d_3 \) and
2. \( H^i(\mathcal{I}_X(k)) = 0 \) for \( i > n-4 \), \( k \geq d_1 + d_2 - n \).
If \( \dim(X) = 3 \) and the singular locus of \( X \) is finite, then

(c) \( H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( k \leq d_3 \) and
(d) \( H^2(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( k < d_3 \) and
(e) \( H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( i \geq 1, k \geq d_1 + d_2 - n \).

Proof. (a) As in Remark 2.5, one can find two hypersurfaces \( S_1 \) and \( S_2 \) of degree \( d_1, d_2 \) containing \( X \) such that the residual \( Y \) of \( X \) (in the complete intersection of \( S_1 \) and \( S_2 \)) is subcanonical and holds the properties of Theorem 2.4. Properties (4) and (6) in Theorem 2.4 say \( Y \) is regular in codimension 3 at all points in \( Y - \text{Sing}(X) \cap Y \). From property (5), \( \text{codim}(\text{Sing}(X) \cap Y, \text{Sing}(X)) \geq 1 \). Since \( X \) is reduced, \( \text{codim}(\text{Sing}(X), X) \geq 1 \). So we can conclude that \( \text{codim}(\text{Sing}(Y), Y) \geq 2 \).

By Theorem 2.4, \( H^1(\mathcal{O}_Y(k)) = 0 \) for \( k < 0 \). From the above exact sequence (3), \( H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( k < d_3 \).

(b) As in the proof of (a), one can find a residual \( Y \) of \( X \) satisfying \( H^1(\mathcal{O}_Y(k)) = 0 \) for \( k < 0 \). Using Serre duality and an exact sequence (3), \( H^1(\mathcal{I}_X(k)) = 0 \) for \( i > n - 4, k \geq d_1 + d_2 - n \).

For the second statement, if \( \dim(X) = 3 \) and \( X \) is a local complete intersection with a zero-dimensional singular locus, then by Theorem 2.4, one can find a smooth residual \( Y \) of \( X \) which has no component in common with \( X \). So, by Theorem 2.4, \( H^i(\mathcal{O}_Y(k)) = 0 \) for \( k < 0 \) and \( i = 1, 2 \). Using Serre duality and an exact sequence (3), one gets (d) and (e). For (e), since \( \dim(Y) = 3 \) and \( Y \) is smooth, by Barth’s theorem, \( H^1(\mathcal{O}_Y) = 0 \). So we get \( H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( k \leq d_3 \).

Corollary 3.9. If \( X \) is a smooth threefold in \( \mathbb{P}^5 \) cut out scheme-theoretically by three equations of degrees \( d_1 \geq d_2 \geq d_3 \) with \( d_1 + d_2 \leq 10 \), then \( X \) is projectively normal, and if \( d_1 + d_2 \leq 9 \), then \( X \) is projectively Cohen-Macaulay.

Proof. By Theorem 3.8, \( H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( k < d_3 \), \( i = 1, 2 \), and by Theorem 3.2, \( H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( k \geq d_1 + d_2 + d_3 - 9 \) and \( i \geq 1 \). So if \( d_1 + d_2 \leq 9 \), then \( H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for all \( k \in \mathbb{Z} \) and \( i = 1, 2 \). Then \( H^1(\mathbb{P}^5, \mathcal{E}(k)) = 0 \) for all \( k \in \mathbb{Z} \) and \( i = 2, 3 \), where \( \mathcal{E} \) is a rank two vector bundle in the exact sequence (1) of Lemma 3.1. By Theorem 2.6, \( \mathcal{E} \) splits into line bundles. So \( X \) is projectively Cohen-Macaulay.

Since \( H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for \( k \leq d_3 \) by Theorem 3.8 (c), it is clear that if \( d_1 + d_2 \leq 10 \), then \( X \) is projectively normal, i.e. \( H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0 \) for all \( k \in \mathbb{Z} \).

Corollary 3.10. If \( X \) is a smooth threefold in \( \mathbb{P}^5 \) of degree \( d \leq 6 \) and \( X \) is cut out scheme-theoretically by three equations of degrees \( d_1 \geq d_2 \geq d_3 \), then \( X \) is projectively normal.

Proof. Since \( d_3 \leq d_2 \leq d_1 \leq d - 1 \) (3, Corollary 4), \( d_1 + d_2 \leq 10 \). By Corollary 3.9, \( X \) is projectively normal.

Example 3.11. Let \( \mathcal{E} \) be a Horrocks-Mumford vector bundle of rank 2 with \( c_1(\mathcal{E}) = 5 \) and \( c_2(\mathcal{E}) = 10 \). If \( X \) is a non-minimal abelian surface of degree 15 in \( \mathbb{P}^4 \), then \( X \) is cut out by three quintics, and we have the following locally free resolution (2, §7):

\[
0 \rightarrow \mathcal{E}(-10) \rightarrow \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-5) \rightarrow \mathcal{I}_X \rightarrow 0.
\]
Then, from the table of \( \dim H^i(E(q)) \) in [12], page 74, one can find that \( \dim H^1(\mathbb{P}^4, I_X(k)) = 0 \) for \( k \neq 5 \) and \( \dim H^1(\mathbb{P}^4, I_X(5)) = 2 \). Note that \( \dim H^2(\mathbb{P}^4, I_X(k)) \neq 0 \) for \( 0 \leq k \leq 3 \). This gives the sharpness of Theorem 3.8.

**Remark 3.12.** Hartshorne conjectures that every holomorphic rank 2 vector bundle over \( \mathbb{P}^n \), \( n \geq 7 \), splits. This would be equivalent to the statement that every smooth subvariety \( X \subset \mathbb{P}^n \) is a global complete intersection for \( n \geq 7 \). In fact until now no indecomposable rank 2 vector bundles over \( \mathbb{P}^n \), \( n \geq 5 \), are known. If all the rank 2 vector bundles over \( \mathbb{P}^5 \) split, then all the Cohen-Macaulay quasi-complete intersections are projectively Cohen-Macaulay and almost complete intersections.

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**References**


Department of Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Yusung-dong Yusung-gu, Daejeon, Korea

E-mail address: ychoi@math.kaist.ac.kr