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ON QUASI-COMPLETE INTERSECTIONS OF CODIMENSION 2

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ABSTRACT. In this paper, we prove that if $X \subset \mathbb{P}^n$, $n \geq 4$, is a locally complete intersection of pure codimension 2 and defined scheme-theoretically by three hypersurfaces of degrees $d_1 \geq d_2 \geq d_3$, then $H^1(\mathbb{P}^n, \mathcal{I}_X(j)) = 0$ for $j < d_3$ using liaison theory and the Arapura vanishing theorem for singular varieties. As a corollary, a smooth threefold $X \subset \mathbb{P}^5$ is projectively normal if X is defined by three quintic hypersurfaces.

1. Introduction

Let X be a nondegenerate projective variety of codimension e and degree d in $\mathbb{P}^n$, defined over a complex number field $\mathbb{C}$. X is said to be the scheme-theoretic intersection of $k+1$ homogeneous equations $f_1, f_2, \ldots, f_{k+1}$ if the saturation of an ideal $J = (f_1, \ldots, f_{k+1})$ is equal to the homogeneous ideal $I_X = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^N, \mathcal{I}_X(j))$ of X. X is called a quasi-complete intersection (qci for short) if X is a scheme-theoretic intersection of $e(= \operatorname{codim} X)+1$ equations. In their paper ([3], Proposition 1), Beorchia and Ellia showed that if X is a smooth codimension 2 qci, which is not a complete intersection, of three hypersurfaces of degrees d_1, d_2, d_3, then the integers d_i are determined by X; that is, if X is a qci of three other hypersurfaces of degrees ${d_1}', {d_2}', {d_3}'$, then $d_i = {d_i}'$ for $i = 1, 2, 3$ up to order. This result is generalized by Bresinsky, Schenzel, and Stückrad ([5]) for irreducible and reduced (not necessarily smooth) qci's of codimension 2 which are not complete intersections.

Definition 1.1. A variety $X \subset \mathbb{P}^n$ is called a qci of type (d_1, d_2, d_3) if X is a codimension two, quasi-complete intersection of three hypersurfaces of degrees d_1, d_2, d_3 with $d_1 \geq d_2 \geq d_3$ and general 3-dimensional plane sections of X are smooth. A variety whose homogeneous ideal I_X is generated by $e+1$ equations is called an almost complete intersection.

This paper is concentrated on the cohomological properties of quasi-complete intersections of codimension two. $X \subset \mathbb{P}^n$ is said to be t-normal if the cohomology group $H^1(\mathbb{P}^n, \mathcal{I}_X(t)) = 0$. By the Serre vanishing theorem, X is t-normal for all $t \gg 0$, and it would be very interesting to find an *explicit* lower bound $B(d, e)$ in terms of its degree d and codimension e such that X is t-normal for all $t \geq B(d, e)$. For example, S. Kwak ([13]) showed that any smooth threefold $X \subset \mathbb{P}^5$ of degree

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d is t-normal for all $t \geq d-4$ and that this bound is sharp as the Palatini scroll of degree 7 shows. In their paper ([4], Proposition 1), Bertram-Ein-Lazarsfeld gave an interesting bound about t-normality in terms of degrees of defining equations of X: Let $X \subset \mathbb{P}^n$ be the scheme-theoretic intersection of m equations $f_1, f_2, \dots, f_m$ with degrees $d_1 \geq d_2 \geq \dots \geq d_m,\ m \geq e = \operatorname{codim}(X)$. Then we have

$$H^i(\mathbb{P}^n, \mathcal{I}_X(j)) = 0,\ i \geq 1,\ \ j \geq d_1 + d_2 + \dots + d_e - n.$$

For a smooth qci of type (d_1, d_2, d_3), we get the following theorem.

Theorem 1.2. *Let X be a smooth codimension two subvariety in $\mathbb{P}^n$, $n = 4, 5$. If X is a quasi-complete intersection of type (d_1, d_2, d_3) with a sheaf of ideals $\mathcal{I}_X$, then $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 - 2n + 1$ and $i \geq 1$.*

On the other hand, for projective varieties of small codimension, it has also been focused on the vanishing of cohomology groups $H^i(\mathbb{P}^n, \mathcal{I}_X(j))$ for a small positive integer j. For example, a famous theorem of Zak's tells us that any smooth threefold is linearly normal, i.e. $H^1(\mathbb{P}^5, \mathcal{I}_X(1)) = 0$. In this sense, it is quite useful to know the following conjecture due to Peskine and Zak: Let X be a nondegenerate, not necessarily smooth, projective variety of codimension e in $\mathbb{P}^n$:

(1) $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) = 0$ for $i \geq 1$, $j \geq 0$, $i + j < \frac{\dim(X)}{e-1}$.
(2) For $i \geq 1$, $j \geq 0$, $i + j = \frac{\dim(X)}{e-1}$, it is possible to describe all varieties for which $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) \neq 0$.

Now we would like to consider this conjecture from the point of defining equations of codimension two qci subvarieties. Note that all complete intersections and almost complete intersections of codimension two are projectively Cohen-Macaulay. For quasi-complete intersections, we can prove the following theorem using liaison theory and the Arapura vanishing theorem.

Theorem 1.3. *Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a reduced locally complete intersection of codimension 2. If X is a quasi-complete intersection of type (d_1, d_2, d_3), then $H^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k < d_3$. Furthermore, if $\dim(X) = 3$ and the singular locus of X is finite, then $H^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k \leq d_3$ and $H^2(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k < d_3$.*

A corollary of these two theorems is:

Corollary 1.4. *If X is a smooth threefold in $\mathbb{P}^5$ that is cut out scheme-theoretically by three equations of degrees $d_1 \geq d_2 \geq d_3$ with $d_1 + d_2 \leq 10$, then X is projectively normal, and if $d_1 + d_2 \leq 9$, then X is projectively Cohen-Macaulay.*

In Section 2, we will review some basic backgrounds including the Peskine-Szpiro Theorem ([16], Proposition 4.1) and the Gherardelli Linkage Theorem ([10], Theorem 2.5) needed in this paper. In Section 3, we will prove the above theorems and give some corollaries and examples.

2. Preliminaries

The following two theorems are generalizations of the classical Kodaira vanishing theorem for ample vector bundles and for singular subvarieties.

Theorem 2.1. *Let $\mathcal{E}$ be a vector bundle of rank r on $\mathbb{P}^n$, and let $\mathcal{L}$ be an ample line bundle. Assume that $\mathcal{E}$ is generated by its global sections. Then*

$$h^q(\mathbb{P}^n, \bigwedge^t \mathcal{E} \otimes \mathcal{L} \otimes K_{\mathbb{P}^n}) = 0 \quad \text{for} \quad q > r - t.$$

Proof. See [8], page 54. □

Theorem 2.2. *Let Y be a projective variety, and let $\mathcal{L}$ be an ample line bundle on Y. Fix $k \geq 1$. Assume that $k < \operatorname{codim}(Sing(Y))$ and that Y is S_{k+1}. Then $H^k(Y, \mathcal{L}^{-1}) = 0$. (If Y is smooth, we define $\operatorname{codim}(Sing(Y)) = \dim(Y)$.)*

Proof. See [1], Proposition 1.1. □

Theorem 2.3 (Gherardelli Linkage Theorem)**.** *Let $X \subset \mathbb{P}^n$ be a Cohen-Macaulay subvariety of pure codimension 2 and Z a complete intersection of two hypersurfaces S_1 and S_2. Assume that Z has pure codimension 2. Then $X \subset Z$ is subcanonical, i.e. $K_X = \mathcal{O}(k)$ for some $k \in \mathbb{Z}$ if and only if its residual scheme $Y \subset Z$ of X is scheme-theoretically defined by the intersection of S_1 and S_2 with a third hypersurface S_3. In addition, if $d_i = \deg(S_i)$ for $i = 1, 2, 3$, then*

$$K_X = \mathcal{O}_X(d_1 + d_2 - d_3 - n - 1).$$

Proof. See [10], Theorem 2.5. □

Theorem 2.4 (Peskine-Szpiro Theorem)**.** *Let X be a closed subscheme without embedded components, that is a generically complete intersection of codimension 2 in $\mathbb{P} = \mathbb{P}^l_k$, defined by a graded ideal I in $k[X_0, \ldots, X_l]$. Let U_1 be an open set consisting of points in X where X is complete intersection in $\mathbb{P}$ and $U_2 \subset X$ consisting of regular points in X.*

Then there exist homogeneous equations $\alpha_1, \alpha_2 \in I$ defining two hypersurfaces S_{α_1} and S_{α_2} which intersect properly such that $S_{\alpha_1} \cap S_{\alpha_2} = X \cup Y$, where

1) *X and Y have no common components.*
2) *$Y \cap U_1$ is a locally complete intersection.*
3) *Y is a complete intersection at all points of $\operatorname{codim} \leq 3$ in $Y \cap U_1$.*
4) *$Y - X \cap Y$ is non-singular.*
5) *$Y \cap U_2$ is non-singular in codimension 2 and*

$$\operatorname{codim}(Y \cap (U_1 - U_2), (U_1 - U_2)) \geq 1.$$

6) *Y is non-singular at all points $x \in Y \cap U_2$ of codimension ≤ 3 in Y.*

Moreover, if char $k = 0$, one can take the elements α_i of degree d_i provided that

$$d_1 \geq \inf\{s : I_s k[X_0, X_1, \ldots, X_l] \text{ defines } X\},$$
$$d_2 \geq \inf\{t : I_t k[X_0, X_1, \ldots, X_l]/\alpha_1 \text{ defines } X \text{ in } k[X_0, X_1, \ldots, X_l]/\alpha_1\}.$$

Proof. See [16], Proposition 4.1. □

We will apply Theorem 2.4 for a locally complete intersection X of codimension two in $\mathbb{P}^n$.

Remark 2.5. If X is a qci of type (d_1, d_2, d_3), then by Theorem 2.4 one can find two hypersurfaces S_{α_1} and S_{α_2} of degrees d_1 and d_2 satisfying the properties of Theorem 2.4 in such a way that $\{\alpha_1, \alpha_2\}$ belong to a set of minimal generators of I_X. By Theorem 2.3, the residual Y of X (in the complete intersection of S_{α_1} and S_{α_2}) is subcanonical and holds the properties of Theorem 2.4.

Theorem 2.6. *Let $\mathcal{E}$ be a locally free sheaf on $\mathbb{P}^n$. If $\operatorname{rank}(\mathcal{E}) < n$ and n is even, then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$. If $\operatorname{rank}(\mathcal{E}) < n-1$ and n is odd, then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$.*

Proof. See [14], Theorem 1. □

Corollary 2.7. *Let X be a Cohen-Macaulay surface in $\mathbb{P}^4$ defined by three hypersurfaces. Then X is projectively normal if and only if X is projectively Cohen-Macaulay.*

Proof. If X is defined by three hypersurfaces of degrees d_1, d_2, d_3, then $\mathcal{I}_X$ has the following resolution:

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^4}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^4}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^4}(-d_3) \xrightarrow{\varphi} \mathcal{I}_X \to 0,$$

where $\mathcal{E}$ is a kernel of a canonical map φ. Since X is Cohen-Macaulay, $\mathcal{E}$ is a vector bundle of rank two on $\mathbb{P}^4$. If X is projectively normal, then $H^1(\mathbb{P}^4, \mathcal{I}(k)) = H^2(\mathbb{P}^4, \mathcal{E}(k)) = 0$ for all $k \in \mathbb{Z}$. By Theorem 2.6, $\mathcal{E}$ splits into line bundles, which implies that X is projectively Cohen-Macaulay. □

3. Main results and applications

As Mumford defined, for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ it is said to be m-regular if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for all $i > 0$, and the regularity of $\mathcal{F}$ is defined by the formula

$$\operatorname{reg} \mathcal{F} = \min\{m \in \mathbb{Z} \ : \mathcal{F} \text{ is } m\text{-regular}\}.$$

Lemma 3.1. *Let X be a smooth non-degenerate subvariety of codimension two in $\mathbb{P}^n$, $n = 4, 5$. Assume that X is defined by $(k+1)$ hypersurfaces of degrees $d_1, \ldots, d_{k+1}$, i.e. there is a locally free resolution*

$$0 \to \mathcal{E} \to \bigoplus_{i=1}^{k+1} \mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{I}_X \to 0. \tag{1}$$

Then

(a) *If $\dim X = 2$, then $\operatorname{reg}(\mathcal{E}^*) \le -3$.*
(b) *If $\dim X = 3$ and X is not in a cubic hypersurface, then $\operatorname{reg}(\mathcal{E}^*) \le -4$.*

Proof. (a) First note that $\mathcal{E}$ is a locally free sheaf of rank k because X is smooth of codimension two. By definition, $\mathcal{E}^*$ is (-3)-regular if and only if

$$H^i(\mathbb{P}^4, \mathcal{E}^*(-3-i)) = 0$$

for all $i \ge 1$. By Serre's duality, this is equivalent to $H^j(\mathbb{P}^4, \mathcal{E}(2-j)) = 0$ for $0 \le j \le 3$. From an exact sequence (1), when $j = 3$, we have by the Kodaira vanishing theorem

$$H^3(\mathbb{P}^4, \mathcal{E}(-1)) = H^2(\mathcal{I}_X(-1)) = H^1(\mathcal{O}_X(-1)) = 0.$$

Note also that for $j = 0, 1, 2$

$$H^2(\mathbb{P}^4, \mathcal{E}) = H^1(\mathcal{I}_X) = 0 \ (X \text{ is smooth}),$$
$$H^1(\mathbb{P}^4, \mathcal{E}(1)) = 0 \ (\text{X is non-degenerate}),$$
$$H^0(\mathbb{P}^4, \mathcal{E}(2)) = 0.$$

Note that the last vanishing comes from the exact sequence (1) and the non-degeneracy of X.

(b) See [6], Lemma 3.7. □

Note that by Faltings theorem ([9], §4), all smooth quasi-complete intersections $X \subset \mathbb{P}^n$ of codimension 2 are complete intersections if $n \geq 6$.

Theorem 3.2. *Let X be a smooth codimension two subvariety in $\mathbb{P}^n$, $n = 4, 5$. If X is a quasi-complete intersection of type (d_1, d_2, d_3) with an ideal sheaf $\mathcal{I}_X$, then:*

(a) $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ *for* $k \leq \dim X$ *and* $1 \leq i < \dim X$.
(b) $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ *for* $k \geq d_1 + d_2 + d_3 - 2n + 1$ *and* $i \geq 1$.

Proof. (a) Assume that $\dim X = 3$. If X is contained in a cubic hypersurface, then it is projectively Cohen-Macaulay ([7], Proposition 5.1), so we assume that X is not contained in a cubic hypersurface. By Lemma 3.1, $\mathcal{E}^*$ is (-4)-regular, so $\mathcal{E}^*(-4)$ is generated by its global sections. Applying Theorem 2.1, one can get

$$H^i(\mathbb{P}^5, K_{\mathbb{P}^5} \otimes \mathcal{E}^*(-4) \otimes \mathcal{O}(k)) = 0 \quad \text{for } k \geq 1,\ i > 1. \tag{2}$$

By Serre duality, $H^i(\mathbb{P}^5, \mathcal{E}(4-k)) = 0$ for $k \geq 1$ and $0 \leq i \leq 3$. This gives $H^i(\mathcal{I}_X(k)) = 0$ for $k \leq 3$ and $1 \leq i < 3$. For $\dim X = 2$, by Lemma 3.1, $\mathcal{E}^*$ is (-3)-regular and, in the same way, we get $H^1(\mathcal{I}_X(k)) = 0$ for $k \leq 2$.

(b) Assume that $\dim X = 3$. Note that $\mathcal{E}$ in the exact sequence (1) is locally free of rank 2 and $c_1(\mathcal{E}) = -(d_1 + d_2 + d_3)$. So $\mathcal{E}^*(-4) = \mathcal{E}(d_1 + d_2 + d_3 - 4)$, and for $k \geq 1$ and $i > 1$,

$$H^i(\mathbb{P}^5, \mathcal{E}(d_1 + d_2 + d_3 - 10 + k)) = H^i(\mathbb{P}^5, K_{\mathbb{P}^5} \otimes \mathcal{E}^*(-4) \otimes \mathcal{O}(k)) = 0.$$

This implies $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 - 9$ and $i \geq 1$. Similar proof goes for $\dim X = 2$. $\square$

This method can be applied to a smooth threefold in $\mathbb{P}^5$ which is cut out by four equations.

Theorem 3.3. *Let X be a smooth threefold in $\mathbb{P}^5$. If X is cut out scheme-theoretically by four hypersurfaces of degrees d_1, d_2, d_3, d_4, then:*

(a) $H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ *for* $k \leq 3$.
(b) $H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ *for* $k \geq d_1 + d_2 + d_3 + d_4 - 13$ *and* $i \geq 1$.

Proof. (a) The proof is similar to the proof of Theorem 3.2(a).

(b) By Lemma 3.1, $\mathcal{E}^*(-4)$ is generated by its global sections, and by Theorem 2.1,

$$H^i(\mathbb{P}^5, K_{\mathbb{P}^5} \otimes \bigwedge^2 \mathcal{E}^*(-4) \otimes \mathcal{O}(k)) = 0 \quad \text{for } k \geq 1,\ i > 1.$$

Since X is cut out by 4 hypersurfaces, $\mathcal{E}^*$ is locally free of rank 3. So $\bigwedge^2 \mathcal{E}^*(-4) = \mathcal{E}(d_1 + d_2 + d_3 + d_4 - 8)$ and $H^i(\mathbb{P}^5, \mathcal{E}(d_1 + d_2 + d_3 + d_4 - 14 + k)) = 0$ for $i > 1$ and $k \geq 1$. Therefore, $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 + d_4 - 13$ and $i \geq 1$. $\square$

Corollary 3.4. *In the situation of Theorem 3.3 above, if $d_1 + d_2 + d_3 + d_4 \leq 17$, then X is projectively normal.*

Theorem 3.5. *Let X be a smooth surface in $\mathbb{P}^4$. If X is cut out scheme-theoretically by four hypersurfaces of degrees d_1, d_2, d_3, d_4, then $H^i(\mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 + d_4 - 10$, and $i \geq 1$.*

Proof. The proof is similar to the proof of Theorem 3.3(b). $\square$

Remark 3.6. Assume that X is a projected Veronese surface in $\mathbb{P}^4$. Clearly, X is not a linearly normal, i.e. $H^1(\mathcal{I}_X(1)) \neq 0$. By the way, X has the following exact sequence:

$$0 \to \mathcal{F}^*(-4) \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 4} \to \mathcal{I}_X \to 0,$$

where $\mathcal{F}$ is the Tango bundle on $\mathbb{P}^4$. So, X is cut out by four cubic hypersurfaces. This shows that Theorem 3.5 is sharp.

From now on, we do not assume that X is smooth. As usual, liaison theory is a tool for studying a qci of codimension two.

Theorem 3.7. *Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a Cohen-Macaulay and quasi-complete intersection of type (d_1, d_2, d_3) and Y a residual of X in the complete intersection of hypersurfaces S_1 and S_2 of degrees d_1 and d_2, i.e. $X \cup Y = S_1 \cap S_2$. If* $\operatorname{codim}(Sing(Y)) = t$ *for some $t \in \mathbb{Z}$, then:*

(a) $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ *for $0 \leq i < t$ and $k < d_3$, and*
(b) $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ *for $i > n-2-t, k \geq d_1 + d_2 - n$.*

Proof. (a) First of all, since d_3 is a minimal degree of a hypersurface containing X ([11]), $H^0(\mathcal{I}_X(k)) = 0$ for $k < d_3$. Since X is a qci of codimension 2, by Theorem 2.3, Y is a subcanonical with $K_Y = \mathcal{O}_Y(d_1+d_2-d_3-n-1)$. Since $X \cup Y = S_1 \cap S_2$, the liaison theory gives the following exact sequence ([15], page 117):

$$0 \to \mathcal{I}_{X\cup Y} \to \mathcal{I}_X \to K_Y(n+1-d_1-d_2) \to 0. \tag{3}$$

Therefore we have

$$0 \to \mathcal{I}_{X\cup Y}(d_3) \to \mathcal{I}_X(d_3) \to \mathcal{O}_Y \to 0.$$

As $X \cup Y$ is a complete intersection, $H^i(\mathcal{I}_{X\cup Y}(k)) = 0$ for $1 \leq i \leq n-2$ and for all $k \in \mathbb{Z}$. Note that the residual Y is Cohen-Macaulay if X is Cohen-Macaulay ([10], Proposition 2.1). So by Theorem 2.2, we have $H^i(\mathcal{O}_Y(k)) = 0$ for $0 \leq i < t$ and $k < 0$. Therefore $H^i(\mathcal{I}_X(k)) = 0$ for $k < d_3$ and $0 \leq i < t$.

(b) By Serre duality and Theorem 2.2, $H^i(K_Y(k)) = 0$ for $i > n-2-t$ and $k \geq 1$. Since $X \cup Y$ is a complete intersection of degrees d_1 and d_2, $H^i(\mathcal{I}_{X\cup Y}(k)) = 0$ for $i \geq 1$ and for all $k \geq d_1 + d_2 - n$. So we get, from the exact sequence (3), $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $i > n-2-t$, $k \geq d_1 + d_2 - n$. $\square$

In their paper ([4]), Bertram-Ein-Lazarsfeld show that if X is a smooth qci of type (d_1, d_2, d_3), then $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 - n$ and $i \geq 1$. So if Y is smooth, the Kodaira vanishing theorem holds for Y, i.e. $H^i(\mathbb{P}^n, K_Y(k)) = 0$ for $k \geq 1$, $i \geq 1$. So one may consider (b) as a generalization of the Bertram-Ein-Lazarsfeld theorem for singular varieties.

If X is a locally complete intersection, then one can control singular locus $Sing(Y)$ of residual Y of X via the Peskine-Szpiro theorem (Theorem 2.4). Using this theorem, one can state the theorem without mentioning a residual Y of X.

Theorem 3.8. *Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a reduced locally complete intersection of codimension 2. If X is a quasi-complete intersection of type (d_1, d_2, d_3), then:*

(a) $H^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ *for $k < d_3$ and*
(b) $H^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ *for $i > n-4$, $k \geq d_1 + d_2 - n$.*

If $\dim(X) = 3$ *and the singular locus of* X *is finite, then*

(c) $H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ *for* $k \leq d_3$ *and*
(d) $H^2(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ *for* $k < d_3$ *and*
(e) $H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ *for* $i \geq 1,\ k \geq d_1 + d_2 - n$.

Proof. (a) As in Remark 2.5, one can find two hypersurfaces S_1 and S_2 of degree d_1, d_2 containing X such that the residual Y of X (in the complete intersection of S_1 and S_2) is subcanonical and holds the properties of Theorem 2.4. Properties (4) and (6) in Theorem 2.4 say Y is regular in codimension 3 at all points in $Y - Sing(X) \cap Y$. From property (5), $\operatorname{codim}(Sing(X) \cap Y, Sing(X)) \geq 1$. Since X is reduced, $\operatorname{codim}(Sing(X), X) \geq 1$. So we can conclude that $\operatorname{codim}(Sing(Y), Y) \geq 2$. By Theorem 2.2, $H^1(\mathcal{O}_Y(k)) = 0$ for $k < 0$. From the above exact sequence (3), $H^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $k < d_3$.

(b) As in the proof of (a), one can find a residual Y of X satisfying $H^1(\mathcal{O}_Y(k)) = 0$ for $k < 0$. Using Serre duality and an exact sequence (3), $H^i(\mathcal{I}_X(k)) = 0$ for $i > n - 4,\ k \geq d_1 + d_2 - n$.

For the second statement, if $\dim(X) = 3$ and X is a local complete intersection with a zero-dimensional singular locus, then by Theorem 2.4 one can find a smooth residual Y of X which has no component in common with X. So, by Theorem 2.2, $H^i(\mathcal{O}_Y(k)) = 0$ for $k < 0$ and $i = 1, 2$. Using Serre duality and an exact sequence (3), one gets (d) and (e). For (c), since $\dim Y = 3$ and Y is smooth, by Barth's theorem, $H^1(\mathcal{O}_Y) = 0$. So we get $H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for $k \leq d_3$. □

Corollary 3.9. *If* X *is a smooth threefold in* $\mathbb{P}^5$ *cut out scheme-theoretically by three equations of degrees* $d_1 \geq d_2 \geq d_3$ *with* $d_1 + d_2 \leq 10$, *then* X *is projectively normal, and if* $d_1 + d_2 \leq 9$, *then* X *is projectively Cohen-Macaulay.*

Proof. By Theorem 3.8, $H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for $k < d_3,\ i = 1, 2$, and by Theorem 3.2, $H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for $k \geq d_1 + d_2 + d_3 - 9$ and $i \geq 1$. So if $d_1 + d_2 \leq 9$, then $H^i(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for all $k \in \mathbb{Z}$ and $i = 1, 2$. Then $H^i(\mathbb{P}^5, \mathcal{E}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i = 2, 3$, where $\mathcal{E}$ is is a rank two vector bundle in the exact sequence (1) of Lemma 3.1. By Theorem 2.6, $\mathcal{E}$ splits into line bundles. So X is projectively Cohen-Macaulay.

Since $H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for $k \leq d_3$ by Theorem 3.8 (c), it is clear that if $d_1 + d_2 \leq 10$, then X is projectively normal, i.e. $H^1(\mathbb{P}^5, \mathcal{I}_X(k)) = 0$ for all $k \in \mathbb{Z}$. □

Corollary 3.10. *If* X *is a smooth threefold in* $\mathbb{P}^5$ *of degree* $d \leq 6$ *and* X *is cut out scheme-theoretically by three equations of degrees* $d_1 \geq d_2 \geq d_3$, *then* X *is projectively normal.*

Proof. Since $d_3 \leq d_2 \leq d_1 \leq d - 1$ ([3], Corollary 4), $d_1 + d_2 \leq 10$. By Corollary 3.9, X is projectively normal. □

Example 3.11. Let $\mathcal{E}$ be a Horrocks-Mumford vector bundle of rank 2 with $c_1(\mathcal{E}) = 5$ and $c_2(\mathcal{E}) = 10$. If X is a non-minimal abelian surface of degree 15 in $\mathbb{P}^4$, then X is cut out by three quintics, and we have the following locally free resolution ([2], §7):

$$0 \to \mathcal{E}(-10) \to \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^4}(-5) \to \mathcal{I}_X \to 0.$$

Then, from the table of $\dim H^i(\mathcal{E}(q))$ in [12], page 74, one can find that $H^1(\mathbb{P}^4, \mathcal{I}_X(k)) = 0$ for $k \neq 5$ and $\dim H^1(\mathbb{P}^4, \mathcal{I}_X(5)) = 2$. Note that $H^2(\mathbb{P}^4, \mathcal{I}_X(k)) \neq 0$ for $0 \leq k \leq 3$. This gives the sharpness of Theorem 3.8.

Remark 3.12. Hartshorne conjectures that every holomorphic rank 2 vector bundle over $\mathbb{P}^n$, $n \geq 7$, splits. This would be equivalent to the statement that every smooth subvariety $X \subset \mathbb{P}^n$ is a global complete intersection for $n \geq 7$. In fact until now no indecomposable rank 2 vector bundles over $\mathbb{P}^n$, $n \geq 5$, are known. If all the rank 2 vector bundles over $\mathbb{P}^5$ split, then all the Cohen-Macaulay quasi-complete intersections are projectively Cohen-Macaulay and almost complete intersections.

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