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AN ULTRAFILTER WITH PROPERTY σ

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Abstract. Let κ be a λ-supercompact cardinal. We show that $\mathcal{P}_\kappa\lambda$ carries a normal ultrafilter with a property introduced by Menas. With it we give a transparent proof of Kamo's theorem that $\mathcal{P}_\kappa\lambda$ carries a normal ultrafilter with the partition property.

Recently Kamo [1] established the following:

Theorem. *Let κ be a λ-supercompact cardinal and λ a cardinal $\geq \kappa$. Then $\mathcal{P}_\kappa\lambda$ carries a normal ultrafilter with the partition property.*

Throughout the note, U denotes a normal ultrafilter on $\mathcal{P}_\kappa\lambda$, where κ is λ-supercompact, and $j : V \to M$ is the embedding induced by U. Recall from [6] that U has the partition property iff there is $X \in U$ such that $|x| < |y \cap \kappa|$ for every pair of $x \subsetneq y$ from X. Kamo closed the gap in the classical result of Menas [6], in which κ was assumed to be $2^{\lambda^{<\kappa}}$-supercompact. For background of the issue, we refer the reader to §25 of [3].

In fact Menas obtained normal ultrafilters on $\mathcal{P}_\kappa\lambda$ with property χ. Here U is said to have property χ if there is $f : \kappa \to \kappa$ such that $\{x \in \mathcal{P}_\kappa\lambda : f(|x \cap \kappa|) = |x|\} \in U$, or equivalently $j(f)(\kappa) = \lambda$. Property χ implies the partition property [6], but not vice versa [4]. Therefore left open is the following:

Problem. *If κ is λ-supercompact, does $\mathcal{P}_\kappa\lambda$ carry a normal ultrafilter with property χ?*

In [5] Menas introduced a weak form of property χ: U is said to have property σ if there is $g : \kappa \to \kappa$ such that $\{x \in \mathcal{P}_\kappa\lambda : |x| \leq g(|x \cap \kappa|)\} \in U$, or equivalently $\lambda \leq j(g)(\kappa)$. Motivated by the Problem, we prove the following.

Theorem. *Every minimal ultrafilter on $\mathcal{P}_\kappa\lambda$ has property σ.*

Here U is said to be minimal if κ is not λ-supercompact in M. As observed by Menas [5], U is minimal if $j(\kappa)$ is as small as possible.

Proof of the Theorem. Assume that U is minimal. For $\alpha < \kappa$ define $f(\alpha)$ to be the least γ such that $\alpha \leq \gamma < \kappa$ and α is not γ-supercompact if there is one, and 0 otherwise. Since U is minimal, $\kappa \leq j(f)(\kappa) \leq \lambda$.

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Let μ be the largest strong limit cardinal $\leq \lambda$. Then $\kappa \leq \mu$. If $\kappa < \mu$, κ is $< \mu$-supercompact in M, since $\mathcal{PPP}_\kappa\gamma \subset M$ for every $\kappa \leq \gamma < \mu$. Hence $\mu \leq j(f)(\kappa) \leq \lambda$ in either case.

Thus we have $n < \omega$ such that $\beth_n(j(f)(\kappa)) \leq \lambda < \beth_{n+1}(j(f)(\kappa))$. Since M is closed under λ-sequences, $\beth_i^M(j(f)(\kappa)) = \beth_i(j(f)(\kappa))$ by induction on $i \leq n$, and $\beth_{n+1}(j(f)(\kappa)) \leq \beth_{n+1}^M(j(f)(\kappa))$. Define $g : \kappa \to \kappa$ by $g(\alpha) = \beth_{n+1}(f(\alpha))$. Then $\lambda \leq \beth_{n+1}^M(j(f)(\kappa)) = j(g)(\kappa)$, as desired. $\square$

As a corollary to the proof we see that a minimal ultrafilter on $\mathcal{P}_\kappa\lambda$ has property χ if λ is strong limit or GCH holds.

In [2] Kamo simplified his earlier proof. Our Theorem enables us to isolate Kamo's key idea through further simplification. For instance we do not make use of Magidor's lemma. This is possible because in the presence of property σ, the partition property can be reduced to a more tractable property: U is said to have the weak partition property iff there is $Z \in U$ such that $x \cap \kappa \neq y \cap \kappa$ for every pair of $x \subsetneq y$ from Z. To see this, let $g : \kappa \to \kappa$ and $Z \in U$ witness property σ and the weak partition property of U respectively. Then $X = \{x \in Z : |x| \leq g(|x \cap \kappa|) \wedge g``(x \cap \kappa) \subset x \cap \kappa = |x \cap \kappa|\} \in U$. It is easy to see that X witnesses the partition property of U.

Now we present Kamo's key idea as we understand it.

Lemma. *Let U be a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ and $\kappa \leq \nu \leq \lambda$ a cardinal. Then there is $Y \in U$ such that $x \cap 2^\nu \neq y \cap 2^\nu$ implies $x \cap \nu \neq y \cap \nu$ for every pair of $x \subsetneq y$ from Y.*

Proof. First note that $\min\{\lambda, 2^\nu\} \leq 2^\nu \leq (2^\nu)^M$, since $\mathcal{P}(\nu) \subset M$. Hence $X = \{x \in \mathcal{P}_\kappa\lambda : \omega \subset x \wedge |x \cap 2^\nu| \leq 2^{|x \cap \nu|}\} \in U$. By Solovay's argument [7] using an ω-Jónsson function for $\min\{\lambda, 2^\nu\}$, we can assume that $x \cap 2^\nu \neq y \cap 2^\nu$ implies $|x \cap 2^\nu| \neq |y \cap 2^\nu|$ for every pair of $x \subsetneq y$ from X. For each infinite $z \in \mathcal{P}_\kappa\nu$ fix an injection $\pi_z : 2^{|z|} + 1 \to \mathcal{P}(z)$. For $x \in X$ set $h(x) = \pi_{x\cap\nu}(|x \cap 2^\nu|) \subset x \cap \nu$. Since $[h] \subset j``\lambda \cap j(\nu) = j``\nu$, we have $A \subset \nu$ with $[h] = j``A = j``\lambda \cap j(A)$. Then $Y = \{x \in X : h(x) = x \cap A\} \in U$. We claim that Y works.

Fix $x \subsetneq y$ both from Y with $x \cap 2^\nu \neq y \cap 2^\nu$. Then $|x \cap 2^\nu| \neq |y \cap 2^\nu|$ by $x, y \in X$. To see that $x \cap \nu \neq y \cap \nu$, suppose otherwise. Since $\pi_{x\cap\nu} = \pi_{y\cap\nu}$ is injective, $h(x) = \pi_{x\cap\nu}(|x \cap 2^\nu|) \neq \pi_{y\cap\nu}(|y \cap 2^\nu|) = h(y)$. On the contrary $h(x) = x \cap A = y \cap A = h(y)$, since $x, y \in Y$ and $A \subset \nu$. This is the desired contradiction. $\square$

Kamo proved in fact that a minimal ultrafilter has the partition property. This follows from our Theorem and the following.

Proposition. *Every minimal ultrafilter on $\mathcal{P}_\kappa\lambda$ has the weak partition property.*

Proof. Assume that U is minimal. Let μ be the largest strong limit cardinal $\leq \lambda$. As in the proof of our Theorem, $\kappa \leq \mu \leq \beth_n(\mu) \leq \lambda < \beth_{n+1}(\mu)$ for some $n < \omega$. By applying Kamo's Lemma several times (at most $n + 1$), we have $X \in U$ such that $x \cap \mu \neq y \cap \mu$ for every pair of $x \subsetneq y$ from X. Hence we are done if $\kappa = \mu$.

Now assume $\kappa < \mu$. By the Solovay argument we can assume that $|x \cap \mu| \neq |y \cap \mu|$ for every pair of $x \subsetneq y$ from X. As in the proof of our Theorem, M satisfies the following: κ is $< \mu$-supercompact but not λ-supercompact, μ is strong limit $> \kappa$ and $\lambda < \beth_{n+1}(\mu)$. Hence $Z = \{x \in X : |x \cap \kappa|$ is $< |x \cap \mu|$-supercompact but not

$|x|$-supercompact $\wedge$ $|x \cap \mu|$ is strong limit $> |x \cap \kappa| \wedge |x| < \beth_{n+1}(|x \cap \mu|)\} \in U$. We claim that Z works.

Fix $x \subsetneq y$ both from Z. Then $|x \cap \mu| < |y \cap \mu|$ by $x, y \in X$. Hence $|x| < \beth_{n+1}(|x \cap \mu|) < |y \cap \mu|$ by $x, y \in Z$. To see that $x \cap \kappa \neq y \cap \kappa$, suppose otherwise. Then $|x \cap \kappa| = |y \cap \kappa|$ is $< |y \cap \mu|$-supercompact by $y \in Z$, hence $|x|$-supercompact. This contradicts $x \in Z$, as desired. □

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