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PERTURBATION THEORETIC ENTROPY OF THE BOUNDARY ACTIONS OF FREE GROUPS

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ABSTRACT. We compute the exact value of Voiculescu's perturbation theoretic entropy of the boundary actions of free groups. This result is a partial answer of Voiculescu's question.

1. INTRODUCTION

In the serial papers [Voi1, Voi2, Voi3] on perturbations of Hilbert space operators, D. Voiculescu studied a certain numerical invariant $k_\infty^-(\tau)$ for a tuple τ of operators. This number can be viewed as a kind of dimension of τ with respect to the Macaev ideal (see [Voi1] and [DV]). In [Oka1, Oka2], we computed the exact value of $k_\infty^-(\tau)$ for a family τ of operators in the following two classes: (1) creation operators associated with a subshift, which were introduced by K. Matsumoto [Mat], (2) unitary operators in the left regular representation of a finitely generated group.

In [Voi4, Voi5], Voiculescu defined the perturbation theoretic entropy $H_P(\Omega, \mathfrak{M})$ for a von Neumann algebra $\mathfrak{M}$ and a set Ω of unitary operators normalizing $\mathfrak{M}$. Thanks to the theory of the standard form of von Neumann algebras, the perturbation theoretic entropy can be defined even for a tuple of automorphisms of a von Neumann algebra without assuming the existence of invariant states. In the case where $\mathfrak{M} = L^\infty(X)$ and Ω consists of the unitary operator U_T induced by an ergodic measure preserving transform T on a probability space (X, μ), he obtained that

$$\frac{1}{2}h(T) \leq H_P(U_T, L^\infty(X)) \leq 6h(T),$$

and moreover if T is Bernoulli, then $H_P(U_T, L^\infty(X))$ is proportional to $h(T)$, where $h(T)$ is Kolmogorov-Sinai entropy. These results suggest that the perturbation theoretic entropy has a certain relation to Connes-Størmer entropy [CS]. However for the free shift σ on the free group factor $L(\mathbb{F}_\infty)$, it is quite different. Indeed, Connes-Størmer entropy $H(\sigma) = 0$, but the perturbation theoretic entropy $H_P(\sigma, L(\mathbb{F}_\infty)) = +\infty$. Thus the perturbation theoretic entropy may be a better concept of entropy in some highly non-commutative cases. We also mention that the perturbation theoretic entropy is well defined for transforms with quasi-invariant measures and depends only on the class of the measure. However, it is an open

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problem whether $H_P(U_T, L^\infty(X))$ can be non-zero and finite in the absence of an invariant measure.

We consider in the present paper the boundary actions of free groups. If $\partial\mathbb{F}_N$ is the boundary of the free group $\mathbb{F}_N$ with the canonical symmetric generating set S, then using the similar technique of [Oka1], we obtain the lower bound

$$H_P(\Omega_N, L^\infty(\partial\mathbb{F}_N)) \geq \log(2N-1),$$

where $\Omega_N = \{U_a\}_{a\in S}$, U_a is the corresponding unitary on $L^2(\partial\mathbb{F}_N, \mu)$ and μ is a certain harmonic measure on $\partial\mathbb{F}_N$. Unfortunately we cannot get the upper bound of $H_P(\Omega_N, L^\infty(\partial\mathbb{F}_N))$. However if we replace $L^\infty(\partial\mathbb{F}_N)$ by the smaller algebra $C(\partial\mathbb{F}_N)$, then we obtain the equation

$$H_P(\Omega_N, C(\partial\mathbb{F}_N)) = \log(2N-1).$$

This result is a partial answer of the above-mentioned problem because there is no invariant measure on $\partial\mathbb{F}_N$ under the action of $\mathbb{F}_N$. We also remark that this number coincides with the topological entropy $h(\mathbb{F}_N; \partial\mathbb{F}_N)$ (see e.g. [BW]).

2. Preliminaries

We quickly introduce several definitions and results of [Voi3, Voi4, Voi5], which we will use. The Macaev ideal $(\mathcal{C}_\infty^-(H), \|\cdot\|_\infty^-)$ in the C^*-algebra $\mathbb{B}(H)$ of bounded operators on a separable infinite-dimensional Hilbert space H is defined by the norm

$$\|T\|_\infty^- = \sum_{j=1}^\infty s_j(T) j^{-1},$$

where $s_1(T) \geq s_2(T) \geq \cdots$ are the eigenvalues of $(T^*T)^{1/2}$. Its dual is denoted by $(\mathcal{C}_1^+(H), \|\cdot\|_1^+)$. The set $\mathbb{F}(H)$ of finite rank operators is dense in $\mathcal{C}_\infty^-(H)$ but not in $\mathcal{C}_1^+(H)$.

If $\tau = (T_1, \ldots, T_N)$ with $T_k \in \mathbb{B}(H)$ $(1 \leq k \leq N)$, the invariant $k_\infty^-(\tau)$ is defined by

$$k_\infty^-(\tau) = \liminf_{A\in\mathbb{F}(H)_1^+} \max_{1\leq i\leq N} \|[A, T_i]\|_\infty^-,$$

where $\mathbb{F}(H)_1^+ = \{A \in \mathbb{F}(H) \mid 0 \leq A \leq I\}$ with the natural order. We will evaluate $k_\infty^-(\tau)$ using the following facts.

Fact 1. If $A_m \in \mathbb{F}(H)_1^+$ with $A_m \uparrow I$, then we have

$$k_\infty^-(\tau) \leq \liminf_{m\to\infty} \max_{1\leq i\leq N} \|[A_m, T_i]\|_\infty^-.$$

Fact 2. Let $\{X_i\}_{1\leq i\leq N} \subset \mathcal{C}_1^+(H)$. If $\sum_{i=1}^N [T_i, X_i] \in \mathcal{C}_1(H) + \mathbb{B}(H)_+$, then we have

$$\left|\mathrm{Tr}\left(\sum_{i=1}^N [T_i, X_i]\right)\right| \leq k_\infty^-(\tau) \sum_{i=1}^N \|X_i\|_1^{\widetilde{+}},$$

where $\mathcal{C}_1(H)$ is the trace class ideal and

$$\|X_i\|_1^{\widetilde{+}} = \inf_{Y\in\mathbb{F}(H)} \|X_i + Y\|_1^+.$$

Remark 1. Note that for $X \in \mathcal{C}_1^+(H)$, we have

$$\|X\|_1^{\widetilde{+}} = \limsup_{n\to\infty} \frac{\sum_{j=1}^n s_j(X)}{\sum_{j=1}^n j^{-1}}.$$

Moreover if an operator X has singular values $c\alpha^{-n}$ with multiplicities $d\alpha^n$ $(n \geq 1)$, where some constant $c > 0$ and positive integers d, α, then one can obtain that

$$\|X\|_1^{\widetilde{+}} = \frac{cd}{\log \alpha}.$$

Definition 2 (cf. [Voi4], [Voi6])**.** Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space H and let Ω be a set of unitaries on H normalizing $\mathfrak{A}$, i.e., $U\mathfrak{A}U^* = \mathfrak{A}$ for $U \in \Omega$. The perturbation theoretic entropy $H_P(\Omega, \mathfrak{A})$ is defined by

$$H_P(\Omega, \mathfrak{A}) = \sup\{k_\infty^-(\omega \cup \tau) \mid \omega \in \mathcal{F}(\Omega), \tau \in \mathcal{F}(\mathfrak{A}), \dim C^*(\tau) < +\infty\},$$

where $\mathcal{F}(\Omega)$ and $\mathcal{F}(\mathfrak{A})$ are sets of all finite subsets of Ω and $\mathfrak{A}$, respectively.

Remark 3. The above definition is slightly different from the original one in [Voi4]. Our definition is introduced in [Voi6]. One can easily check that all results in [Voi4] and [Voi5] can be also obtained if the above definition is adopted. We believe that our definition is better, because we do not need to deal with the invariant $k_\infty^-(\tau)$ for a family τ of infinitely many operators. We also remark that if we consider the original definition, then our main result can be obtained by the same proof.

We note that $H_P(\Omega, \mathfrak{A}) = +\infty$ unless $k_\infty^-(\tau) = 0$ for all $\tau \in \mathcal{F}(\mathfrak{A})$ with $\dim C^*(\tau) < +\infty$. It is clear that if $\mathfrak{M} = \mathfrak{A}''$, then we have that

$$H_P(\Omega, \mathfrak{M}) \geq H_P(\Omega, \mathfrak{A}).$$

3. Results

Let $\mathbb{F}_N = \langle a_1, \ldots, a_N \rangle$ be the free group with the canonical generating set $S = \{a_1^{\pm 1}, \ldots, a_N^{\pm 1}\}$ and $\partial\mathbb{F}_N$ its boundary with the product topology,

$$\partial\mathbb{F}_N = \{(x_n)_{n=1}^\infty \mid x_n \in S, x_{n+1} \neq x_n^{-1}\}.$$

Note that $\mathbb{F}_N$ acts on $\partial\mathbb{F}_N$ by left multiplications. We always assume that the expression such as $x_1 \cdots x_n$ is reduced throughout this paper. We define the probability measure μ on $\partial\mathbb{F}_N$ by

$$\mu(C(x_1 \cdots x_n)) = \frac{1}{2N(2N-1)^{n-1}},$$

where $C(x_1 \cdots x_n)$ is the cylinder set. We consider the C^*-algebra $C(\partial\mathbb{F}_N)$ acting on the Hilbert space $L^2(\partial\mathbb{F}_N, \mu)$ by left multiplications and the unitary operator U_g on $L^2(\partial\mathbb{F}_N, \mu)$ for $g \in \mathbb{F}_N$ given by

$$(U_g f)(\boldsymbol{x}) = f(g^{-1}\boldsymbol{x})\sqrt{\frac{dg_*\mu}{d\mu}}(\boldsymbol{x}),$$

for $f \in L^2(\partial\mathbb{F}_N, \mu)$. Note that if $\boldsymbol{x} = (x_n)_{n=1}^\infty \in \partial\mathbb{F}_N$, we have

$$\frac{dg_*\mu}{d\mu}(\boldsymbol{x}) = (2N-1)^{|g_n| - |g^{-1}g_n|}$$

for sufficiently large n, where $g_n = x_1 \cdots x_n$ and $|\cdot|$ is the word-length on $\mathbb{F}_N$. We denote by $\chi_{x_1\cdots x_n}$ the characteristic function of $C(x_1\cdots x_n)$. We remark that for $a \in S$ and $n \geq 2$,

$$U_a\chi_{x_1\cdots x_n} = \begin{cases} c_N\chi_{ax_1\cdots x_n} & \text{if } x_1 \neq a^{-1}, \\ c_N^{-1}\chi_{x_2\cdots x_n} & \text{if } x_1 = a^{-1}, \end{cases}$$

where $c_N = \sqrt{2N-1}$. Let $\Omega_N = \{U_a\}_{a\in S}$.

Proposition 4.

$$H_P(\Omega_N, C(\partial\mathbb{F}_N)) \geq \log(2N-1).$$

Hence,

$$H_P(\Omega_N, L^\infty(\partial\mathbb{F}_N)) \geq \log(2N-1).$$

Proof. We set $H_0 = \mathbb{C}\chi$ where $\chi(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \partial\mathbb{F}_N$ and for $n \geq 1$,

$$H_n = \operatorname{span}\{\chi_{x_1\cdots x_n}\}.$$

Note that $H_0 \subset H_1 \subset \cdots \subset H_n \subset \cdots \subset H = L^2(\partial\mathbb{F}_N, \mu)$ and $\dim H_n = 2N(2N-1)^{n-1}$ for $n \geq 1$. We also define the subspace $K_n = H_n \cap H_{n-1}^\perp$ for $n \geq 1$. One can easily check that

$$K_n = \operatorname{span}\{\chi_{x_1\cdots x_{n-1}x} - \chi_{x_1\cdots x_{n-1}y}\}$$

with $\dim K_1 = 2N-1$ and $\dim K_n = \dim H_n - \dim H_{n-1} = 4N(N-1)(2N-1)^{n-2}$ for $n \geq 2$.

We denote by P_n the projection onto K_n. For $a \in S$ we define the operator $X_a \in \mathbb{B}(H)$ by

$$X_a = \sum_{n\geq 1}\sum_{x_1,\ldots,x_n} \frac{1}{2N(2N-1)^{n-1}} P_n U_a^* P_{n+1}.$$

For $a \in S$ and $n \geq 2$, let us denote by $P_n(a)$ the projection onto

$$K_n(a) = \operatorname{span}\{\chi_{x_1\cdots x_{n-1}x} - \chi_{x_1\cdots x_{n-1}y} \mid x_1 = a\}.$$

Then it is easy to check that

$$\sum_{a\in S} P_n(a) = P_n$$

and $\dim K_n(a) = 2(N-1)(2N-1)^{n-2}$ for $a \in S$. Moreover $P_nU_a^*P_{n+1} = (P_n - P_n(a^{-1}))U_a^*$ for $n \geq 2$ and $P_nU_a^*P_{n+1} = U_a^*P_{n+1}(a)$ for $n \geq 1$. It follows that

$$\sum_{a\in S} U_aX_a = \sum_{n\geq 1}\frac{1}{2N(2N-1)^{n-1}}P_{n+1},$$

and

$$\sum_{a\in S} X_aU_a = \frac{1}{2N}\sum_{a\in S}U_a^*P_2(a)U_a + \sum_{n\geq 2}\frac{1}{2N(2N-1)^{n-2}}P_n.$$

Therefore,

$$\left|\operatorname{Tr}\left(\sum_{a\in S}[U_a, X_a]\right)\right| = \frac{1}{2N}\operatorname{Tr}\left(\sum_{a\in S}U_a^*P_2(a)U_a\right) = \frac{1}{2N}\operatorname{Tr}(P_2) = 2(N-1).$$

For $a \in S$, we next estimate the norm $\|X_a\|_\infty^{\widetilde{+}}$. By using the fact in Remark 1 as

$$c = \frac{1}{2N}, d = 2(N-1), \alpha = 2N-1,$$

we obtain that

$$\|X_a\|_1^{\widetilde{+}} = \frac{2(N-1)}{2N\log(2N-1)}.$$

Hence it follows from Fact 2 that

$$H_P(\Omega_N, C(\partial\mathbb{F}_N)) \geq k_\infty^-(\Omega_N) \geq \log(2N-1).$$

□

Theorem 5.

$$H_P(\Omega_N, C(\partial\mathbb{F}_N)) = \log(2N-1).$$

Proof. We define the finite-dimensional C^*-subalgebra $\mathfrak{A}_n$ of $\mathfrak{A} = C(\partial\mathbb{F}_N)$ by $\mathfrak{A}_0 = \mathbb{C}\chi$ and

$$\mathfrak{A}_n = \overline{\mathrm{span}}\{\chi_{x_1\cdots x_n}\}$$

for $n \geq 1$. Note that $\dim\mathfrak{A}_0 = 1, \dim\mathfrak{A}_n = 2N(2N-1)^{n-1}$ and

$$\mathfrak{A} = \overline{\bigcup_{n\geq 0}\mathfrak{A}_n}.$$

We denote by P_n the projection onto the subspace $H_n = \mathfrak{A}_n\chi \subset L^2(\partial\mathbb{F}_N, \mu)$. Note that for $a \in S$ and $n \geq 2$,

$$P_{n-1} \leq U_a P_n U_a^* \leq P_{n+1}.$$

Let $\tau \in \mathcal{F}(\mathfrak{A})$ with $\dim C^*(\tau) < +\infty$. Note that $C^*(\tau)$ is contained in $\mathfrak{A}_K$ for some $K > 1$. Then we define

$$Q_m = \frac{1}{m}\sum_{n=K}^{K+m-1} P_n.$$

Since

$$U_a Q_m U_a^* - Q_m = \frac{1}{m}\sum_{n=K}^{K+m-1}(U_a P_n U_a^* - P_n) \leq \frac{1}{m}\sum_{n=K}^{K+m-1}(P_{n+1} - P_n) \leq \frac{1}{m}P_{K+m}$$

and

$$U_a Q_m U_a^* - Q_m \geq \frac{1}{m}\sum_{n=K}^{K+m-1}(P_{n-1} - P_n) \geq -\frac{1}{m}P_{K+m},$$

we obtain that

$$U_a Q_m U_a^* - Q_m = \frac{1}{m}P_{K+m}C_m P_{K+m}$$

for some contraction C_m. Since $P_n \uparrow I$, we also have that $Q_m \uparrow I$. Moreover each Q_m commutes with τ. Therefore we have

$$\liminf_{m\to\infty}\|U_a Q_m U_a^* - Q_m\|_\infty^- \leq \lim_{m\to\infty}\frac{1}{m}\log(2N(2N-1)^{K+m-1}) = \log(2N-1),$$

and our assertion follows from Fact 1 and Proposition 2. □

Remark 6. We also prove that $k_\infty^-(\Omega_N) = \log(2N-1)$, actually. We wonder how this is related to our previous result $k_\infty^-(\lambda_N) = \log(2N-1)$ in [Oka1] for $\lambda_N = \{\lambda_a\}_{a\in S}$, where λ is the left regular representation of $\mathbb{F}_N$, because Ω_N and λ_N generate the same C^*-algebra, although they are not unitarily equivalent. (See, e.g., [FTP] for the representation U of $\mathbb{F}_N$ arising from the Poisson boundary.)

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