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SHORT-TIME EXISTENCE OF SOLUTIONS TO THE CROSS CURVATURE FLOW ON 3-MANIFOLDS

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Abstract. Given a compact 3-manifold with an initial Riemannian metric of positive (or negative) sectional curvature, we prove the short-time existence of a solution to the cross curvature flow. This is achieved using an idea first introduced by DeTurck (1983) in his work establishing the short-time existence of solutions to the Ricci flow.

Recently, B. Chow and R. Hamilton [3] introduced the notion of the cross curvature tensor—a kind of dual to the Ricci tensor—and considered the evolution equation which deforms metrics on 3-manifolds in the direction of their cross curvature tensor. They conjecture that, for an initial metric with negative sectional curvature, the cross curvature flow will exist for all time and converge to a hyperbolic metric after an appropriate normalization, and have obtained several monotonicity formulae to support this conjecture.

In contrast to the Ricci flow, which is a quasi-linear flow first introduced by Hamilton in [7], the cross curvature flow is fully nonlinear. Both flows, however, are only weakly parabolic, and so special care needs to be taken to prove the short-time existence of solutions. In the case of the Ricci flow, Hamilton was able to affirm the existence of a solution for a short time by drawing recourse to the Nash-Moser implicit function theorem. He first established a very general result concerning the short-time existence of solutions to weakly parabolic equations satisfying a certain first-order integrability condition, then showed for the Ricci flow this integrability condition is fulfilled by the contracted second Bianchi identity.

Hamilton's general short-time existence theorem has also been applied extensively to the case of hypersurfaces evolving by weakly parabolic curvature-driven flows, where the integrability is met using the orthogonal projection map onto the tangent space; see e.g. [1], [2] and [6].

Shortly after the publication of Hamilton's pioneering work on the Ricci flow, DeTurck [4], [5] discovered a much-simplified proof of the short-time existence of solutions to the Ricci flow using only the classical existence and uniqueness theorems for quasi-linear parabolic systems. He shows that the Ricci flow is equivalent

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to a strictly parabolic system (for which the short-time existence follows by standard theory), modulo the action of the diffeomorphism group on the manifold, and establishes the correspondence between the solutions of the two systems.

In an attempt to establish the short-time existence of solutions to the cross curvature flow when the sectional curvature has a sign, Chow and Hamilton appealed to Hamilton's general existence theorem. However, the integrability condition, as stipulated in [3], is not valid as it is second-order in the metric. Although the question remains of whether a general short-time existence theorem for weakly parabolic equations with second-order integrability conditions holds, the aim of this paper is to establish the short-time existence of solutions to the cross curvature flow using the more elementary, classical method of DeTurck.

We shall adopt notation similar to that used in [3] and let μ_{ijk} denote the volume form, raise indices by $\mu^{ijk} = g^{ip}g^{jq}g^{kr}\mu_{pqr}$ and normalize such that $\mu_{123} = \mu^{123} = 1$. This implies

$$\mu_{ijk}\mu^{klm} = \delta_i^l\delta_j^m - \delta_i^m\delta_j^l. \tag{1}$$

We write

$$\Lambda^{ab} = \frac{1}{4}\mu^{apq}\mu^{brs}R_{pqrs} \tag{2}$$

for the Einstein tensor, and then define the cross curvature tensor by

$$X_{ij} = \frac{1}{2}\mu_{ipq}\mu_{jrs}\Lambda^{pr}\Lambda^{qs} = \frac{1}{2}\Lambda^{uv}R_{iujv}. \tag{3}$$

Theorem 1. *Let (M, g) be a closed 3-manifold with positive sectional curvature. Then the evolution equation*

$$\frac{\partial}{\partial t}g_{ij} = -2X_{ij}, \qquad g(0,x) = g_0(x), \tag{XCF}$$

has a unique solution for a short time, for any smooth initial metric g_0.

Remark 2. In the case of negative sectional curvature, the same result is true if we consider the flow given by

$$\frac{\partial}{\partial t}g_{ij} = 2X_{ij}.$$

Proof. As in [5], we show that, modulo the action of the diffeomorphism group of M, (XCF) is equivalent to a strictly parabolic initial-value problem, for which the existence of a unique solution for a short time follows by standard parabolic theory. The correspondence between solutions of the parabolic system and the original system is then established via a one-parameter family of diffeomorphisms.

Let us begin by analyzing the linearization of (XCF),

$$\frac{\partial}{\partial t}\tilde{g}_{ij} = DE(g)\,\tilde{g}_{ij},$$

where $E(g_{ij}) = -2X_{ij}$ is the second-order nonlinear operator defined above, $DE(g_{ij})\,\tilde{g}_{ij} \equiv \frac{d}{d\varepsilon}\big|_{\varepsilon=0} E(g_{ij} + \varepsilon\tilde{g}_{ij})$, and $\tilde{g}_{ij}$ denotes a variation in the metric. The evolution equation $\frac{\partial}{\partial t}g_{ij} = E(g_{ij})$ is said to be parabolic if the linearized operator $DE(g)\,\tilde{g}_{ij}$ is elliptic, for any (symmetric) tensor $\tilde{g}$. This is equivalent to requiring all the eigenvalues of the symbol $\sigma DE(g)(\xi)$ have strictly positive real parts when $\xi \neq 0$. Unfortunately, due to the invariance of the Riemann curvature tensor under the action of diffeomorphisms, the symbol of the linearization of the cross curvature tensor turns out to have degeneracies ($\lambda = 0$ is a repeated eigenvalue), and so is only weakly elliptic. These degeneracies, however, are of the same nature as those present in the symbol of the linearized Ricci operator and so can be dealt with using the idea of DeTurck [5].

The variation $\tilde{g}_{ij}$ in the metric produces a variation in all the associated curvature tensors. Denoting these with tildes, we have

$$\tilde{R}_{ipjr} = \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{pj}}{\partial x^i \partial x^r} - \frac{\partial^2 \tilde{g}_{pr}}{\partial x^i \partial x^j} + \frac{\partial^2 \tilde{g}_{ir}}{\partial x^p \partial x^j} - \frac{\partial^2 \tilde{g}_{ij}}{\partial x^p \partial x^r}\right) + \cdots,$$

where the dots denote terms involving at most one derivative of the metric, and

$$\tilde{\Lambda}^{ab} = \frac{1}{4}\mu^{apq}\mu^{brs}\tilde{R}_{pqrs} + \cdots.$$

Hence, on combining these results and using (1), we obtain

$$\begin{aligned}\tilde{X}_{ij} &= \Lambda^{pr}\widetilde{R}_{ipjr} + \cdots \\ &= \frac{1}{2}\Lambda^{pr}\left(\frac{\partial^2 \tilde{g}_{pj}}{\partial x^i \partial x^r} - \frac{\partial^2 \tilde{g}_{pr}}{\partial x^i \partial x^j} + \frac{\partial^2 \tilde{g}_{ir}}{\partial x^p \partial x^j} - \frac{\partial^2 \tilde{g}_{ij}}{\partial x^p \partial x^r}\right) + \cdots,\end{aligned}$$

and so

$$DE\left(g_{ij}\right)\tilde{g}_{ij} = \Lambda^{pr}\left(\frac{\partial^2 \tilde{g}_{pr}}{\partial x^i \partial x^j} - \frac{\partial^2 \tilde{g}_{pj}}{\partial x^i \partial x^r} - \frac{\partial^2 \tilde{g}_{ir}}{\partial x^p \partial x^j} + \frac{\partial^2 \tilde{g}_{ij}}{\partial x^p \partial x^r}\right) + \cdots.$$

The symbol of the linear differential operator $DE\left(g_{ij}\right)\tilde{g}_{ij} = -2\widetilde{X}_{ij}$ in the direction of an arbitrary cotangent vector ξ is then obtained by replacing each derivative $\partial/\partial x^k$ by ξ_k in the highest (second) order terms above:

$$\sigma DE\left(g\right)\left(\xi\right)\tilde{g}_{ij} = \Lambda^{pr}\left(\xi_i\xi_j\tilde{g}_{pr} - \xi_i\xi_r\tilde{g}_{pj} - \xi_p\xi_j\tilde{g}_{ir} + \xi_p\xi_r\tilde{g}_{ij}\right).$$

Since this is homogenous, we may assume ξ has length one and rotate the coordinates so that $\xi_1 = 1$ and $\xi_i = 0$ for all $i \neq 1$. In these coordinates, we then have

$$\sigma DE\left(g\right)\left(\xi\right)\tilde{g}_{ij} = \Lambda^{11}\tilde{g}_{ij} + \Lambda^{pr}\left(\delta_{i1}\delta_{j1}\tilde{g}_{pr} - \delta_{i1}\delta_{r1}\tilde{g}_{pj} - \delta_{p1}\delta_{j1}\tilde{g}_{ir}\right).$$

Thus,

$$\sigma DE\left(g\right)\left(\xi\right)\begin{pmatrix}\tilde{g}_{11}\\ \tilde{g}_{12}\\ \tilde{g}_{13}\\ \tilde{g}_{22}\\ \tilde{g}_{33}\\ \tilde{g}_{23}\end{pmatrix} = \Lambda^{11}\begin{pmatrix}\tilde{g}_{11}\\ \tilde{g}_{12}\\ \tilde{g}_{13}\\ \tilde{g}_{22}\\ \tilde{g}_{33}\\ \tilde{g}_{23}\end{pmatrix} + \begin{pmatrix}-\Lambda^{11}\tilde{g}_{11} + \Lambda^{22}\tilde{g}_{22} + \Lambda^{33}\tilde{g}_{33} + 2\Lambda^{23}\tilde{g}_{23}\\ -\Lambda^{11}\tilde{g}_{12} - \Lambda^{12}\tilde{g}_{22} - \Lambda^{13}\tilde{g}_{23}\\ -\Lambda^{11}\tilde{g}_{13} - \Lambda^{12}\tilde{g}_{23} - \Lambda^{13}\tilde{g}_{33}\\ 0\\ 0\\ 0\end{pmatrix}$$

from which we deduce

$$\sigma DE\left(g\right)\left(\xi\right) = \begin{pmatrix}0 & 0 & 0 & \Lambda^{22} & \Lambda^{33} & 2\Lambda^{23}\\ 0 & 0 & 0 & -\Lambda^{12} & 0 & -\Lambda^{13}\\ 0 & 0 & 0 & 0 & -\Lambda^{13} & -\Lambda^{12}\\ 0 & 0 & 0 & \Lambda^{11} & 0 & 0\\ 0 & 0 & 0 & 0 & \Lambda^{11} & 0\\ 0 & 0 & 0 & 0 & 0 & \Lambda^{11}\end{pmatrix}. \tag{4}$$

The eignenvalues of this matrix then coincide with the diagonal entries, 0 and Λ^{11}, which, under the assumption of positive sectional curvature, indicate that (XCF) is weakly parabolic. In order to eradicate the zero eigenvalues, we shall introduce a one-parameter family of diffeomorphisms into our equation. To this end, we recall from [5] the following two elementary results.

Lemma 3 ([5]). *Let $V(x,t)$ be a time-varying vector field on M. Then, for small t, there exists a unique one-parameter family of diffeomorphisms $\phi_t : M \longrightarrow M$ such that*

$$\frac{\partial \phi_t(x)}{\partial t} = V(\phi_t(x), t)$$

for all $x \in M$, with ϕ_0 equal to the identity diffeomorphism.

Lemma 4 ([5]). *Let $g_{ij}(x,t)$ be a time-varying Riemannian metric on M, and let ϕ_t be the family of diffeomorphisms from the above lemma. Then*

$$\frac{\partial \phi_t^*(g)}{\partial t}(x) = \phi_t^*\left(\frac{\partial g}{\partial t}(\phi_t(x))\right) + \mathcal{L}_{\widehat{V}}\phi_t^*(g) \tag{5}$$

where $\widehat{V}$ is the pull-back of the covariant one-tensor $V(x,t) = g^{ij}V_j(x)\frac{\partial}{\partial x^i}$, and $\mathcal{L}_W g$ is the Lie derivative of g in the direction W.

Following the work of DeTurck [4], we let T denote any fixed, invertible, symmetric 2-tensor (e.g. $T = g_0$) and consider the vector given by

$$V_i = g^{pq}\left(T^{-1}\right)_i^r \nabla_q\left(\frac{1}{2}\mathrm{tr}_g(T)\, g_{pr} - T_{pr}\right),$$

where $\mathrm{tr}_g(T) = g^{ab}T_{ab}$. One computes

$$\begin{aligned}(\mathcal{L}_V g)_{ij} &= \nabla_j V_i + \nabla_i V_j \\ &= g^{pq}\left(\frac{\partial^2 g_{pi}}{\partial x^q \partial x^j} - \frac{\partial^2 g_{pq}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{pj}}{\partial x^q \partial x^i}\right) + \cdots,\end{aligned}$$

and thus

$$\sigma D(\mathcal{L}_V g)(\xi)\,\widetilde{g}_{ij} = g^{pq}\left(\xi_q\xi_j\widetilde{g}_{pi} - \xi_i\xi_j\widetilde{g}_{pq} + \xi_q\xi_i\widetilde{g}_{pj}\right). \tag{6}$$

In our orthonormal basis we have $g_{pq} = \delta_{pq}$ at a point and so, as before, taking $\xi_1 = 1$ and $\xi_i = 0$ for all $i \neq 1$ we obtain

$$\sigma D(\mathcal{L}_V g)(\xi) = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

It then follows from (4) and (7) that, by choosing V as above, the initial-value problem

$$\frac{\partial}{\partial t}\widehat{g}_{ij} = -2\widehat{X}_{ij} + (\mathcal{L}_V\widehat{g})_{ij}, \qquad \widehat{g}(0,x) = \widehat{g}_0(x), \tag{8}$$

is strictly parabolic, and hence a unique solution $\widehat{g}$ exists for a short time by standard parabolic theory. The solution to the original problem (XCF) can then be recovered from $\widehat{g}$ by introducing the diffeomorphism given by

$$\frac{\partial \phi_t(x)}{\partial t} = -V(\phi_t(x), t).$$

See [4] for further details. $\square$

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