A CHARACTERISATION OF $0^\#$ IN TERMS OF FORCING

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Abstract. We show that “saturation” of the universe with respect to forcing over $L$ with partial orders on $\omega_1$ is equivalent to the existence of $0^\#$. 

If $P$ is a constructible forcing notion, then we say that $G \subseteq P$ is $P$-generic iff $G$ is $P$-generic over $L$. The statement that all countable constructible forcings have generics is rather weak, and holds for example in $L[R]$, where $R$ is a Cohen real over $L$. But it is not possible that all constructible forcings have generics: consider the forcing that collapses $\omega_1$ to $\omega$ with finite conditions.

Definition. $V$ is $L$-saturated for $\omega_1$-forcings iff whenever $P$ is a constructible forcing of $L$-cardinality $\omega_1$ such that for any $p \in P$ there is a $P$-generic containing $p$ in some $\omega_1$-preserving extension of $V$, then there is a $P$-generic in $V$.

Theorem 1. The following are equivalent:

(a) $V$ is $L$-saturated for $\omega_1$-forcings.

(b) $0^\#$ exists.

Proof. (a) $\rightarrow$ (b) The existence of $0^\#$ is equivalent to the statement that every stationary constructible subset of $\omega_1$ contains a CUB subset (see [2]). Now use the following:

Fact (Baumgartner; see [1]). If $X$ is a stationary constructible subset of $\omega_1$, then there is a forcing $P \in L$ of $L$-cardinality $\omega_1$ which preserves cardinals over $V$ and adds a CUB subset to $X$. ($P$ adds a CUB subset of $X$ using “finite conditions”.)

(b) $\rightarrow$ (a) Assume that $0^\#$ exists, and suppose that $P$ is a constructible forcing of $L$-cardinality $\omega_1$ such that every condition in $P$ belongs to a generic in an $\omega_1$-preserving extension of $V$. We will show that there is a $P$-generic in $V$. Assume that the universe of $P$ is exactly $\omega_1$. Let $P$ be of the form $t(\vec{i}, \omega_1, \infty)$, where $\vec{i} < \omega_1 < \infty$ is a finite increasing sequence of indiscernibles and $t$ is an $L$-term. We claim that if $\vec{i} < k_0 < k_1$ are countable indiscernibles and $G_{k_0}$ is $P_{k_0}$-generic over $L$, then there is $G_{k_1}$ containing $G_{k_0}$ which is $P_{k_1}$-generic over $L$, where $P_k = t(\vec{i}, k, \infty)$. If not, then player $I$ wins the open game $G(k_0, k_1, G_{k_0})$, where $I$ chooses constructible dense subsets of $P_{k_1}$ and $II$ responds with increasingly strong conditions meeting...
these dense sets which are compatible with all conditions in $G_{k_0}$. The latter is a property of the model $L[G_{k_0}]$. Let $p \in P_{k_0}$ be a condition forcing that $I$ wins $G(k_0, k_1, G_{k_0})$. Then $p$ forces that $I$ wins $G(k_2, k_3, G_{k_2})$, where $k_2 < k_3$ are any indiscernibles $\geq k_0$ and $G_{k_2}$ denotes the $P_{k_2}$-generic. But now let $G$ be a $P$-generic containing $p$ in an $\omega_1$-preserving extension of $V$. As $G$ preserves $\omega_1$ over $V$, there are indiscernibles $k_2 < k_3$ with $k_0 \leq k_2$ such that $G \cap k_2$ is $P_{k_2}$-generic and $G \cap P_{k_3}$ is $P_{k_3}$-generic, so clearly player $II$ has a winning strategy in the game $G(k_2, k_3, G \cap P_{k_3})$, in contradiction to the choice of $p$.

Now it is easy to build a $P$-generic: List the countable indiscernibles greater than $i$ as $j_0 < j_1 < j_2 < \cdots$ and inductively choose $P_{j_\alpha}$-generic $G_\alpha$ such that $\alpha < \beta$ implies $G_\alpha \subseteq G_\beta$. At the first step, $G_{j_0}$ is an arbitrary $P_{j_0}$-generic. By the previous paragraph there is no difficulty at the successor steps, where one extends $G_{j_\alpha}$ to $G_{j_{\alpha+1}}$. At limit stages $\lambda$, the $P_{j_\lambda}$-genericity of the union $G_{j_\lambda}$ of the $G_{j_\alpha}$, $\alpha < \lambda$, follows by indiscernibility. The desired $P$-generic is the union of the $G_{j_\alpha}$, $\alpha < \omega_1$.

Remark. The proof of (a) implies (b) shows that the theorem still holds if “$\omega_1$-preserving extension” is taken to be “$\omega_1$-preserving set-generic extension” in the definition of $L$-saturation for $\omega_1$- forcings.

Question. Suppose that $0^\#$ exists. Then does part (a) of the theorem hold (in the obvious sense) for constructible $\omega_1^+L$-forcings, i.e. constructible $P$ of $L$-cardinality $\omega_1^+L$-forcing?