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ALMOST EVERYWHERE CONVERGENCE OF INVERSE FOURIER TRANSFORMS

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ABSTRACT. We show that if $\log(2-\Delta)f \in L^2(\mathbb{R}^d)$, then the inverse Fourier transform of f converges almost everywhere. Here the partial integrals in the Fourier inversion formula come from dilates of a closed bounded neighbourhood of the origin which is star shaped with respect to 0. Our proof is based on a simple application of the Rademacher-Menshov Theorem. In the special case of spherical partial integrals, the theorem was proved by Carbery and Soria. We obtain some partial results when $\sqrt{\log(2-\Delta)}f \in L^2(\mathbb{R}^d)$ and $\log\log(4-\Delta)f \in L^2(\mathbb{R}^d)$. We also consider sequential convergence for general elements of $L^2(\mathbb{R}^d)$.

1. Introduction

We treat the almost everywhere convergence of partial integrals of inverse Fourier transforms on Euclidean space, for functions in L^2 with logarithmic Sobolev properties. The partial integrals are formed by integrating over dilates of a fixed closed bounded region V which is star shaped with respect to the origin and has the origin in its interior. Particular choices of V give rise to the familiar cases of spherical and polyhedral partial integrals. Our results are proved by a very simple application of the Rademacher-Menshov Theorem. In particular, we show that if the Fourier transform satisfies

$$\int_{\mathbb{R}^d} \left(\log(2+|y|^2)\right)^2 \left|\widehat{f}(y)\right|^2 dy < \infty, \tag{1}$$

then the partial integrals

$$S_R f(x) = \int_{RV} \widehat{f}(y) e^{2\pi i x\cdot y}\, dy$$

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converge almost everywhere as $R \to \infty$. If we reduce the power of the logarithmic factor, we have a partial result. We show that if

$$\int_{\mathbb{R}^d} \log(2+|y|^2) \left|\widehat{f}(y)\right|^2 dy < \infty, \tag{2}$$

then $S_{R_n} f(x)$ converges almost everywhere as $R_n = n^{\log(n)} \to \infty$. When the logarithm is replaced by $\log\log$, we find that if

$$\int_{\mathbb{R}^d} \left(\log\log(4+|y|^2)\right)^2 \left|\widehat{f}(y)\right|^2 dy < \infty, \tag{3}$$

then $S_{r_m} f(x)$ converges almost everywhere as $m \to \infty$, for unbounded sequences (r_m) whose terms are in a second order lacunary set, as defined in [2].

The first case, with V a sphere, was done by Carbery and Soria [4, Theorem 3]. The introduction to their paper provides a broad description to the background of this area of Fourier analysis. See also [6] for some weighted norm estimates in the spherical case. The third case is a slight extension of the main result in [2], where they work with integrals over spheres.

Our contribution is the simplicity of the proof and the fact that it is independent of the geometry of V. The method seems to depend only on the Plancherel formula, and follows the same idea as used in [8].

2. The Rademacher-Menshov Theorem

Theorem 1. *Suppose that (X, μ) is a positive measure space. There is a positive constant c with the following property.*

For each orthogonal subset $\{P_n : n \in \mathbb{N}\}$ in $L^2(X, \mu)$ which satisfies

$$\sum_{n=1}^{\infty} (\log(n+1))^2 \|P_n\|_2^2 < \infty, \tag{4}$$

the maximal function $\mathcal{M}(x) = \sup_{N \geq 1} \left|\sum_{n=1}^N P_n(x)\right|$ is in $L^2(X, \mu)$ and

$$\|\mathcal{M}\|_2 \leq c \left(\sum_{n=1}^{\infty} (\log(n+1))^2 \|P_n\|_2^2\right)^{1/2}. \tag{5}$$

In particular, when (4) holds, then the series $\sum_{n=1}^{\infty} P_n(x)$ converges almost everywhere on X.

See Theorem XIII.10.21 from [11], Proposition 2.3.1, and Theorem 2.3.2 from [1, Pages 79–80]. Here log means logarithm with base 2. For an application in $L^2(\mathbb{R}^d)$, see part (b) of Lemma 5.1 in [5].

3. Setting up the partial integrals

Suppose that V is a bounded closed subset of $\mathbb{R}^d$ having 0 as an interior point and star shaped with respect to 0. Let $\beta = d(0, \partial V) > 0$. For each $R > 0$ dilate V to get $RV = \{Ry : y \in V\}$, so that the dilated set has measure

$$|RV| = R^d |V| \quad \text{and} \quad d(0, \partial(RV)) = R\beta.$$

Define partial integrals by

$$S_R f(x) = \int_{RV} \widehat{f}(y) e^{2\pi i x \cdot y}\, dy, \qquad \forall f \in L^2(\mathbb{R}^d), \tag{6}$$

which give Fourier inversion in norm, $\lim_{R\to\infty} \|S_R f - f\|_2 = 0$. If $f \in L^2(\mathbb{R}^d)$ and $S_R f(x)$ converges almost everywhere as $R \to \infty$, then its limit equals $f(x)$ almost everywhere.

Now let $(R_n)_{n=1}^{\infty}$ be an unbounded increasing sequence of positive real numbers and fix an element $f \in L^2(\mathbb{R}^d)$. We think of the partial integrals $S_{R_n} f(x)$ as partial sums of the orthogonal expansion

$$S_{R_1} f(x) + \sum_{n=2}^{\infty} \left(S_{R_n} f(x) - S_{R_{n-1}} f(x)\right). \tag{7}$$

Define $P_n f \in L^2(\mathbb{R}^d)$ by setting

$$P_n f(x) = \begin{cases} S_{R_1} f(x) & \text{if } n = 1, \\ S_{R_n} f(x) - S_{R_{n-1}} f(x) & \text{if } n \geq 2. \end{cases} \tag{8}$$

Then the partial sums of (7) are

$$S_{R_n} f(x) = \sum_{k=1}^{n} P_k f(x), \qquad \forall n \geq 1, x \in \mathbb{R}^d,$$

and $m \neq n$ implies that $P_m f \perp P_n f$. The Plancherel formula says that

$$\|P_n f\|_2^2 = \int_{R_n V \setminus R_{n-1} V} \left|\widehat{f}(y)\right|^2 dy. \tag{9}$$

4. Convergent subsequences

Suppose that $f \in L^2(\mathbb{R}^d)$. Since $S_R f$ converges to f in norm, there exists a sequence $(R_n)_{n=0}^{\infty}$ with $\lim_{n\to\infty} S_{R_n} f(x) = f(x)$ almost everywhere. The Rademacher-Menshov Theorem gives a way of describing one such sequence.

Proposition 2. *Suppose $f \in L^2(\mathbb{R}^d)$. If an increasing unbounded sequence $0 = R_0 < R_1 < R_2 < \cdots$ has the property*

$$\sum_{n=1}^{\infty} \left\|S_{R_n} f - S_{R_{n-1}} f\right\|_2^2 (\log(n+1))^2 < \infty, \tag{10}$$

then $\lim_{n\to\infty} S_{R_n} f(x) = f(x)$ almost everywhere. Furthermore, for each $f \in L^2(\mathbb{R}^d)$ there is an increasing unbounded sequence $(R_n)_{n=0}^{\infty}$ with property (10).

Proof. The first statement is a direct consequence of Theorem 1. It remains to prove the second statement. If $\widehat{f}$ has bounded support, then it is integrable and the statement is immediate. Now suppose that $\widehat{f}$ is not compactly supported. The function $R \mapsto F(R) = \|S_R f\|_2$ is continuous and its values are non-negative. If $R' < R''$, then $F(R') \leq F(R'')$, and $\lim_{R\to\infty} F(R) = \|f\|_2$. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers with

$$\sum_{n=1}^{\infty} a_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (\log(n+1))^2 a_n < \infty.$$

There is an increasing unbounded sequence $(R_n)_{n=1}^{\infty}$ with the property

$$F(R_n)^2 = \|f\|_2^2 \sum_{m=1}^{n} a_m, \quad \forall n \geq 1.$$

In particular, $\|S_{R_{n+1}}f\|_2^2 - \|S_{R_n}f\|_2^2 = a_{n+1}\|f\|_2^2$, for all $n \geq 1$. Define the projections as in (8). Then we have that

$$\sum_{n=1}^{\infty}(\log(n+1))^2\|P_n f\|_2^2 < \infty$$

and we can apply Theorem 1. □

The Cauchy-Schwarz inequality and the Plancherel formula imply that when a sequence of partial integrals converges, then the sequence can be perturbed slightly and still preserve convergence.

Lemma 3. *Suppose $(R_n)_{n=1}^{\infty}$ is an increasing unbounded sequence. For each $\rho > 0$ and $n \geq 1$ define the set*

$$E_\rho(n) = \left\{r > 0 \;:\; \left|r^d - R_n^d\right| \leq \rho\right\}.$$

For these sets and $f \in L^2(\mathbb{R}^d)$ there is the inequality,

$$\sup_{n\geq 1}\left(\sup_{r\in E_\rho(n)}|S_r f(x) - S_{R_n}f(x)|\right) \leq \|f\|_2\sqrt{\rho|V|}, \qquad \forall x \in \mathbb{R}^d.$$

Now fix $f \in L^2(\mathbb{R}^d)$ and suppose $(R_n)_{n=1}^{\infty}$ is an increasing unbounded sequence for which $S_{R_n}f(x)$ converges almost everywhere. Furthermore, let $E_\rho = \bigcup_{n=1}^{\infty}E_\rho(n)$. If $(r_m)_{m=1}^{\infty}$ is an increasing unbounded sequence whose terms belong to a set E_ρ, then $\lim_{m\to\infty} S_{r_m}f(x) = f(x)$, almost everywhere.

Proof. If $0 \leq R_n^d - r^d \leq \rho$, then $rV \subset R_nV$ and $|R_nV \setminus rV| \leq \rho|V|$, so that

$$|S_{R_n}f(x) - S_r f(x)| \leq \left(\int_{R_nV\setminus rV}\left|\widehat{f}(y)\right|^2 dy\right)^{1/2}\sqrt{\rho|V|}. \tag{11}$$

Since $f \in L^2(\mathbb{R}^d)$, the right-hand side tends to zero as $R_n \to \infty$. A similar argument applies to the case $0 \leq r^d - R_n^d \leq \rho$. □

We can apply the Rademacher-Menshov Theorem again to give a minor extension of Proposition 2.

Lemma 4. *Suppose that $f \in L^2(\mathbb{R}^d)$ and $(R_n)_{n=0}^{\infty}$ satisfy (10) of Proposition 2. If $(r_m)_{m=1}^{\infty}$ is an unbounded increasing sequence with the property that*

$$|\{m : R_n \leq r_m \leq R_{n+1}\}| \leq cn^\gamma, \qquad \forall n \geq 1,$$

for some positive constants c and γ, then

$$\lim_{m\to\infty} S_{r_m}f(x) = f(x), \qquad \textit{almost everywhere.}$$

Proof. For each $n \geq 1$, suppose that there is a finite set of M_n real numbers arranged in the interval (R_n, R_{n+1}), say

$$R_n = r_1(n) < \cdots < r_{M_n}(n) = R_{n+1}$$

and define functions

$$Q_{k,n}(x) = S_{r_{k+1}(n)}f(x) - S_{r_k(n)}f(x), \quad 1 \leq k < M_n.$$

These functions form an orthogonal subset of $L^2(\mathbb{R}^d)$ and so the Rademacher-Menshov Theorem says that

$$\max_{1\le m<M_n}\left|S_{r_m(n)}f(x)-S_{R_n}f(x)\right| = \max_{1\le m<M_n}\left|\sum_{k=1}^{m} Q_{k,n}(x)\right|$$

has L^2 norm bounded by

$$c\left(\log M_n\right)\left\|S_{R_{n+1}}f-S_{R_n}f\right\|_2 = c\left(\log M_n\right)\left\|P_{n+1}f\right\|_2.$$

Suppose that

$$\log M_n \le \gamma \log n = \log\left(n^{\gamma}\right), \quad \forall n\ge 2.$$

Because of (10) we see that

$$\sum_{n=1}^{\infty}\max_{1\le m\le M_n}\left|S_{r_m(n)}f(x)-S_{R_n}f(x)\right|^2$$

is in $L^1(\mathbb{R}^d)$. We then have that as $n\to\infty$,

$$\max_{1\le m\le M_n}\left|S_{r_m(n)}f(x)-S_{R_n}f(x)\right|\to 0, \text{ almost everywhere.}$$

□

5. The main result

Proposition 5. *Suppose that $f\in L^2(\mathbb{R}^d)$ satisfies the condition (1). Then $\lim_{R\to\infty} S_R f(x)=f(x)$, almost everywhere on $\mathbb{R}^d$. Furthermore, there is a constant $c>0$ so that for all $w\in\mathbb{R}^d$,*

$$\int_{|x-w|\le 1}\left|\sup_{R>0}\left|S_Rf(x)\right|\right|^2 dx \le c\int_{\mathbb{R}^d}\left(\log\left(2+|y|^2\right)\right)^2\left|\widehat{f}(y)\right|^2 dy. \tag{12}$$

Proof. Take the sequence $R_n=n^{1/d}$ in setting up (8) and let

$$\mathcal{M}f(x)=\sup_{n\ge 1}\left|S_{R_n}f(x)\right|, \qquad \forall x\in\mathbb{R}^d.$$

When y is in the shell $R_nV\setminus R_{n-1}V$ it satisfies $|y|\ge (n-1)^{1/d}\beta$ and for large n there is a constant $c>0$ for which

$$\log(n+1)\le c\log\left(2+|y|^2\right), \qquad \forall y\in R_nV\setminus R_{n-1}V.$$

Combine this with (9) to see that

$$(\log(n+1))^2\|P_n\|_2^2 \le c\int_{R_nV\setminus R_{n-1}V}\left(\log\left(2+|y|^2\right)\right)^2\left|\widehat{f}(y)\right|^2 dy.$$

Since f satisfies inequality (1), the sum of the terms on the right-hand side is finite. This verifies the hypothesis (4) in Theorem 1 and so $S_{R_n}f(x)$ converges almost everywhere as $n\to\infty$. Furthermore, we see that since (1) holds, then inequality (5) says that

$$\|\mathcal{M}f\|_2 \le c\left(\int_{\mathbb{R}^d}\left(\log(2+|y|^2)\right)^2\left|\widehat{f}(y)\right|^2 dy\right)^{1/2}. \tag{13}$$

We can dominate the maximal function over $R \geq 1$ by the maximal function over the sequence $(R_n)_{n=1}^{\infty}$ plus a remainder,

$$\sup_{R\geq 1} |S_R f(x)| \leq \mathcal{M}f(x) + \sup_{n>0}\left(\sup_{R_n\leq r<R_{n+1}} |S_r f(x) - S_{R_n} f(x)|\right).$$

We chose the sequence $R_n = n^{1/d}$ so that the increments in the measure of the dilates of V are constant,

$$|R_n V \setminus R_{n-1} V| = n|V| - (n-1)|V| = |V|.$$

If $R_n \leq r < R_{n+1}$, then $n \leq r^d < n+1$ and $|r^d - n| = |r^d - R_n^d| \leq 1$, so that we can apply Lemma 3 with $\rho = 1$. Hence

$$\left\|\sup_{n>0}\left(\sup_{R_n\leq r<R_{n+1}} |S_r f - S_{R_n} f|\right)\right\|_{\infty} \leq c\|f\|_2. \tag{14}$$

Combine inequalities (13) and (14) to prove (12). □

See [6, Chapter 2] for more sophisticated methods for estimating $S_{R_n} f(x) - S_r f(x)$.

6. The case of one power of logarithm

The first part of the method used above can be applied to other sequences.

Proposition 6. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisifies the condition (2) and that $R_n = n^{\log n}$, for $n \geq 1$. Then $\lim_{n\to\infty} S_{R_n} f(x) = f(x)$, almost everywhere on $\mathbb{R}^d$.*

Proof. We have that $\log(R_n) = (\log n)^2$ and for large n there is a constant c for which

$$(\log(n+1))^2 \|P_n f\|_2^2 \leq c \int_{R_n V\setminus R_{n-1}V} \log\left(2+|y|^2\right) \left|\widehat{f}(y)\right|^2 dy. \tag{15}$$

Inequality (2) means that the sum of the terms on the right-hand side is finite and so Theorem 1 applies. □

Note that $n^{\log n} = 2^{(\log n)^2}$ grows slower than any unbounded geometric progression but faster than n^k, for each $k \in \mathbb{N}$. The measure of the shell $R_n V \setminus R_{n-1} V$ grows too rapidly to use the estimate from Lemma 3. However, Lemma 4 gives convergence for some other sequences.

Corollary 7. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies (2) and $(r_m)_{m=1}^{\infty}$ is an unbounded increasing sequence with the property that*

$$\left|\left\{m : n^{\log n} \leq r_m \leq (n+1)^{\log(n+1)}\right\}\right| \leq cn^{\gamma}, \qquad \forall n \geq 1,$$

for some positive constants c and γ. Then $\lim_{m\to\infty} S_{r_m} f(x) = f(x)$, almost everywhere.

7. Iterated logarithm

Fix $a > 1$ and define the geometric progression $R_n = a^n$, for all $n \geq 1$. For $y \in R_nV \setminus R_{n-1}V$ we have $|y| \geq a^{n-1}\beta$ and for large n there is a constant $\kappa > 0$ with

$$\log\log(4 + |y|^2) \geq \kappa \log(n+1).$$

This means that for large n we have

$$\kappa^2 \left(\log(n+1)\right)^2 \int_{R_nV\setminus R_{n-1}V} \left|\widehat{f}(y)\right|^2 \, dy \leq \int_{R_nV\setminus R_{n-1}V} \left(\log\log(4+|y|^2)\right)^2 \left|\widehat{f}(y)\right|^2 \, dy$$

and we can again apply Theorem 1.

Corollary 8. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisifies (3) and that $a > 1$ is fixed. Then $\lim_{n\to\infty} S_{a^n} f(x) = f(x)$, almost everywhere on $\mathbb{R}^d$.*

Remark 7.1. For lacunary spherical partial integrals there is a much stronger result in [3, Theorem B] and in [7].

Lemma 4 can be applied to the case of $R_n = a^n$.

Corollary 9. *Fix $a > 1$ and let $(r_m)_{m=1}^{\infty}$ be an unbounded increasing sequence with the property that*

$$\left|\left\{m : a^n \leq r_m \leq a^{n+1}\right\}\right| \leq cn^{\gamma}, \qquad \forall n \geq 1,$$

for some positive constants c and γ. If $f \in L^2(\mathbb{R}^d)$ satisfies (3), then

$$\lim_{m\to\infty} S_{r_m} f(x) = f(x), \qquad \textit{almost everywhere.}$$

We can combine Corollary 8 with Lemma 3 and Corollary 9 to extend the result of [2] to the case of general V.

Corollary 10. *Fix $a > 1$ and suppose $f \in L^2(\mathbb{R}^d)$ satisfies (3). Let*

$$A = \left\{a^n(1 - a^{-k}) \;:\; n, k \in \mathbb{N}\right\}$$

and suppose that $(r_m)_{m=1}^{\infty}$ is an increasing unbounded sequence whose terms belong to A. Then $\lim_{m\to\infty} S_{r_m} f(x) = f(x)$, almost everywhere.

Proof. Let $R_n = a^n$ and consider the set E_1, as defined in Lemma 3. We need to count how many elements are in $(A \setminus E_1) \cap \left[a^{n-1}, a^n\right]$, for each $n \geq 1$. That is, we count how many k satisfy

$$a^{nd} - a^{nd}(1 - a^{-k})^d > 1. \tag{16}$$

This is equivalent to the inequality

$$1 - (1 - a^{-k})^d > a^{-dn}$$

and the left-hand side is equal to $da^{-k}y^{d-1}$ for some $1 - a^{-k} \leq y \leq 1$. Taking logarithms, we see that if k satisfies the inequality (16), then we must have $k \leq cn$, for some constants c. This shows that $A \setminus E_1$ satisfies the criterion of Corollary 9. If a sequence has its values in A, then it is made up of subsequences in $A \cap E_1$ and $A \setminus E_1$. Apply Lemma 3 for $A \cap E_1$ and Corollary 9 for $A \setminus E_1$. □

8. Capacity

We conclude with an extension of Theorem 1.3 of [9] to summation based on the set V. Following Definition 2 in [9], for each $0 < \alpha < d$ the $(\alpha, 2)$-capacity of a subset $X \subset \mathbb{R}^d$ is

$$C_\alpha(X) = \inf\left\{\|f\|_2^2 : f \in L^2_+(\mathbb{R}^d), \quad G_\alpha * f(x) \geq 1, \forall x \in X\right\}.$$

Here G_α is the Bessel kernel, with $\widehat{G_\alpha}(y) = \left(1 + |y|^2\right)^{-\alpha/2}$. Its properties are cataloged in [10, Section V.3]. Most importantly, $G_\alpha(x) \geq 0$ for all $x \neq 0$. Notice that if $f \in L^2_+(\mathbb{R}^d)$ and $X \subseteq \{x : G_\alpha * f(x) \geq \lambda\}$, then $C_\alpha(X) \leq \lambda^{-2}\|f\|_2^2$. Capacity is subadditive and sets of capacity zero have Lebesgue measure zero.

Let $R_n = n^{1/d}$ for each $n \geq 1$, as in the proof of Proposition 5, and define $\mathcal{M}f(x) = \sup_{n \geq 1} |S_{R_n} f(x)|$. Recall that this satisfies inequality (13).

Lemma 11. *Suppose that $\varphi \in L^2(\mathbb{R}^d)$ satisfies*

$$N(\varphi, \alpha) := \int_{\mathbb{R}^d} |\widehat{\varphi}(y)|^2 \left(1 + |y|^2\right)^\alpha (\log(2 + |y|))^2 \, dy < \infty, \tag{17}$$

for some $0 < \alpha < d$. There is a positive constant c so that

$$C_\alpha(\{x : \mathcal{M}\varphi(x) \geq \lambda\}) \leq c\lambda^{-2} N(\varphi, \alpha), \quad \forall \lambda > 0.$$

Proof. Since φ satisfies (17), there is a $\psi \in L^2(\mathbb{R}^d)$ with $\varphi = G_\alpha * \psi$ and

$$N(\varphi, \alpha) = N(\psi, 0) = \int_{\mathbb{R}^d} \left|\widehat{\psi}(y)\right|^2 (\log(2 + |y|))^2 \, dy < \infty.$$

Inequality (13) can be applied to both ψ and to $\varphi = G_\alpha * \psi$, so that the maximal functions satisfy $\mathcal{M}\psi \in L^2(\mathbb{R}^d)$ and $\mathcal{M}(G_\alpha * \psi) \in L^2(\mathbb{R}^d)$. Since G_α is positive, the observation on page 1419 of [9] can be adapted to our sequential maximal function so that

$$\mathcal{M}(G_\alpha * \psi)(x) \leq G_\alpha * (\mathcal{M}\psi)(x), \qquad \forall x \in \mathbb{R}^d.$$

For each $\lambda > 0$ let

$$X_\lambda = \{x : \mathcal{M}(G_\alpha * \psi)(x) \geq \lambda\} \subseteq \{x : G_\alpha * (\mathcal{M}\psi)(x) \geq \lambda\}.$$

From the definition of capacity, $C_\alpha(X_\lambda) \leq \lambda^{-2}\|\mathcal{M}\psi\|_2^2 \leq c\lambda^{-2}N(\psi, 0)$. □

Proposition 12. *Suppose that $\varphi \in L^2(\mathbb{R}^d)$ satisfies (17) for some $0 < \alpha < d$. The set on which $S_R\varphi(x)$ does not converge to $\varphi(x)$, as $R \to \infty$, has $(\alpha, 2)$-capacity zero.*

Proof. The argument based on Lemma 3 shows that it is enough to consider the convergence of $S_{R_n}\varphi(x)$ as $n \to \infty$. Let ψ be the function in the previous proof, so that $\varphi = G_\alpha * \psi$. For $\delta > 0$ let $H \in C_c^\infty(\mathbb{R}^d)$ satisfy

$$N(\psi - H, 0) = \int_{\mathbb{R}^d} \left|\widehat{\psi}(y) - \widehat{H}(y)\right|^2 (\log(2 + |y|))^2 \, dy < \delta.$$

We know that $\lim_{R \to \infty} S_R(G_\alpha * H)(x) = G_\alpha * H(x)$, for all x. For each $\eta > 0$,

$$\left\{x : \limsup_{n \to \infty} |S_{R_n}\varphi(x) - \varphi(x)| > \eta\right\}$$

$$\subseteq \left\{x : \sup_{n \geq 1} |S_{R_n}(\varphi - G_\alpha * H)(x)| > \frac{\eta}{2}\right\} \cup \left\{x : |\varphi(x) - G_\alpha * H(x)| > \frac{\eta}{2}\right\}.$$

Lemma 11 shows that

$$C_\alpha\left(\left\{x:\sup_{n\geq 1}\left|S_{R_n}\left(G_\alpha*\psi-G_\alpha*H\right)(x)\right|>\frac{\eta}{2}\right\}\right)\leq 4c\eta^{-2}\delta.$$

Observe that $|G_\alpha*\psi-G_\alpha*H|\leq G_\alpha*|\psi-H|$. The definition of capacity shows that

$$C_\alpha\left(\left\{x:|G_\alpha*\psi(x)-G_\alpha*H(x)|>\frac{\eta}{2}\right\}\right)\leq 4\eta^{-2}\|\psi-H\|_2^2<4\eta^{-2}c_d^2\delta.$$

Letting $\delta\to 0$, we find that

$$C_\alpha\left(\left\{x:\limsup_{n\to\infty}|S_{R_n}\varphi(x)-\varphi(x)|>\eta\right\}\right)=0,$$

for every $\eta>0$. The set of divergence is

$$\bigcup_{k\geq 1}\left\{x:\limsup_{n\to\infty}|S_{R_n}\varphi(x)-\varphi(x)|>\frac{1}{k}\right\},$$

which is a countable union of sets of $(\alpha,2)$-capacity zero and so it also has $(\alpha,2)$-capacity zero. □

One consequence of this proposition is that the partial inverse Fourier integrals of functions in Sobolev classes $L^2_\alpha(\mathbb{R}^d)$ converge pointwise, with the possible exception of sets with zero $(\alpha-\varepsilon,2)$-capacity, for every $\varepsilon>0$.

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