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A NOTE ON NON-EXISTENCE RESULTS FOR SEMI-LINEAR COOPERATIVE ELLIPTIC SYSTEMS VIA MOVING SPHERES

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ABSTRACT. In this note, we extend some earlier non-existence, monotonicity and one-dimensionality results of W. Reichel and the author, by replacing the local Lipschitz continuity hypothesis on the non-linearities by a so-called boundedly uniform Lipschitz condition in the *magnitude* of $\mathbf{u}$.

1. INTRODUCTION

For an integer $k \geq 1$, denote by Π^k the first (open) octant of $\mathbb{R}^k$,

$$\Pi^k = \{\mathbf{u} = (u_1, u_2, \cdots, u_k) \in \mathbb{R}^k \,|\, u_i > 0\},$$

and by $\Xi^k = \overline{\Pi^k}$ the closure of Π^k.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a connected domain and let

$$\mathbf{f} = (f^1, f^2, \cdots, f^k) : \Omega \times \Xi^k \to \mathbb{R}^k$$

be a continuous (vector-valued)[1] function. Consider the semi-linear elliptic problem

$$\begin{aligned} \Delta \mathbf{u} + \mathbf{f}(x, \mathbf{u}) &= 0 \text{ in } \Omega, \\ \mathbf{u} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Note that the boundary data is prescribed only if Ω has a non-empty boundary.

The system (1.1) was studied in [2] and several results on monotonicity, one-dimensionality and non-existence on non-negative solutions were obtained. The following notations were used in [2].

(A) Let $\Gamma \subseteq \Xi^k$. A function $\mathbf{f}$ defined on $\Omega \times \Xi^k$ is locally Lipschitz in $\mathbf{u}$ on Γ, provided that for $\mathbf{u}_0 \in \Gamma$ and $\omega \subseteq \Omega$ bounded, there exists a (relatively) open neighborhood $U(\mathbf{u}_0) \subset \Gamma$ such that $\mathbf{f}$ is Lipschitz continuous in every u_i on $\omega \times U(\mathbf{u}_0)$.

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[1] All relations here involving vectors are understood in the component-wise sense.

(B) Suppose that either Ω or $\mathbb{R}^n \setminus \overline{\Omega}$ is star-shaped with respect to the origin $\mathcal{O}$. We say that $\mathbf{f}$ has subcritical (supercritical) growth provided that for each fixed $(x, \mathbf{u}) \in \Omega \times \Xi^k$ the function

$$\lambda^{-(n+2)/(n-2)} \mathbf{f}(\lambda^{-2/(n-2)} x, \lambda \mathbf{u})$$

is non-increasing (non-decreasing) in $\lambda > 0$ (wherever it is defined).

(C) The system (1.1) or the function $\mathbf{f}$ is called cooperative, provided that

$$f^i(x, \mathbf{u}) \le f^i(x, \mathbf{v}), \qquad i = 1, \cdots, k,$$

for all $(x, \mathbf{u}), (x, \mathbf{v}) \in \Omega \times \Xi^k$ with $\mathbf{u} \le \mathbf{v}$ and $u_i = v_i$.

In this note, we want to replace the local Lipschitz continuity (A) of $\mathbf{f}$ in $\mathbf{u}$ on Ξ^k, which was a key assumption in [2] (see also (L1)-(L2) on p. 237), by the following conditions.

(F1) There exists a function $\mathbf{g} : \Omega \times \Pi^k \to \Xi^k$ such that

$$f^i(x, \mathbf{u}) + g_i(x, \mathbf{u}) u_i \ge 0, \qquad i = 1, \cdots, k,$$

for all $(x, \mathbf{u}) \in \Omega \times \Pi^k$ where $\mathbf{g}$ is bounded on bounded subsets $\omega \times \Gamma \subset \Omega \times \Pi^k$.

(F2) The functions $L_i : \Omega \times \Pi^k \times \Pi^k \to \mathbb{R}$

$$L_i(x, \mathbf{u}, \mathbf{v}) := \frac{f^i(x, \mathbf{u}) - f^i(x, \mathbf{v})}{u_i - v_i} \qquad i = 1, \cdots, k,$$

are bounded from below on every compact subset $\omega \times \Gamma \subset \Omega \times \Pi^k$, where $(x, \mathbf{u}), (x, \mathbf{v}) \in \Omega \times \Pi^k$ with $u_i \neq v_i$ and $u_l = v_l$ for $l \neq i$.

(F3) For any bounded subset $\omega \times \Gamma \subset \Omega \times \Xi^k$, there exist multi-indexes $\mathbf{p}_i = \mathbf{p}_i(\omega, \Gamma) \in \Xi^k$, with $|\mathbf{p}_i| \ge 1$, and $C_0 = C_0(\omega, \Gamma, \mathbf{p}_i) > 0$ such that for $i, j = 1, \cdots, k$

$$\frac{f^i(x, \mathbf{u}) - f^i(x, \mathbf{v})}{u_j - v_j} \le C_0 + C_0 \min\{u_j^{p_{ij}-1}, v_j^{p_{ij}-1}\} \cdot \prod_{l \neq j} u_l^{p_{il}}$$

for all $(x, \mathbf{u}), (x, \mathbf{v}) \in \omega \times \Gamma$ with $u_j > v_j$ and $u_l = v_l$ for $l \neq j$ ($0^{-\sigma} = \infty$ for any $\sigma > 0$).

Clearly the condition (F3) is weaker than the local Lipschitz continuity in $\mathbf{u}$ on Ξ^k. The example in section 3 further illustrates this point. We shall refer to (F3) as boundedly uniform Lipschitz in the *magnitude* of $\mathbf{u}$ on Ξ^k.

Throughout this paper, we shall assume that the function $\mathbf{f}$ is cooperative and satisfies (F1)-(F3) and, for simplicity, that the domain Ω has a non-empty and smooth (say C^2) boundary $\partial\Omega$.

Under the assumptions (F1)-(F3), all results in [2] remain valid. Below are statements of the main results. The first is a Pohožaev-type non-existence result for non-variational systems.

Theorem 1.1 (Pohožaev)**.** *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ be bounded and star-shaped with respect to the origin $\mathcal{O}$ and suppose that $\mathbf{f}(x, \mathbf{u})$ is supercritical. Then there holds*

(I) (1.1) *has no positive $C^2(\Omega) \cap C(\overline{\Omega})$-solution, and*

(II) *if $\mathbf{f}(x, 0) = 0$ for all $x \in \Omega$, then* (1.1) *has no non-negative and non-trivial $C^2(\Omega) \cap C(\overline{\Omega})$-solution.*

The second is a Liouville-type result on unbounded domains. For $\mathbf{p}$, $\mathbf{u} \in \Pi^k$, we write

$$|\mathbf{p}| = \sum_{i=1}^{k} p_i, \qquad \mathbf{u}^{\mathbf{p}} = \prod_{i=1}^{k} u_i^{p_i} \ (0^0 = 1).$$

Theorem 1.2. *Let $\mathbb{R}^n \setminus \overline{\Omega}$ $(n \geq 2)$ be star-shaped with respect to the origin $\mathcal{O}$ and suppose that $\mathbf{f}(x, \mathbf{u})$ is subcritical. Moreover, we assume that there exist an integer $l \in \{1, \cdots, k\}$, constants $C > 0$, $\sigma_i > -2$ and multi-indices $\mathbf{p}_i \in \Xi^l$ such that*

$$f^i(x, \mathbf{u}) \geq C|x|^{\sigma_i} \bar{\mathbf{u}}^{\mathbf{p}_i}, \qquad i = 1, \cdots, l,$$

in $\Omega \times \Pi^k$ with $\bar{\mathbf{u}} = (u_1, \cdots, u_l)$ and

$$|\mathbf{p}_i| \in (1, \frac{n + 2 + 2\sigma_i}{n - 2}), \qquad i = 1, \cdots, l.$$

Then (1.1) *has no non-negative and non-trivial $C^2(\Omega) \cap C(\overline{\Omega})$-solution.*

The third is a one-dimensionality result on the half-space.

Theorem 1.3. *Let $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$ and suppose that $\mathbf{f}(x, \mathbf{u})$ is subcritical. Then the non-negative solutions $\mathbf{u}$ of* (1.1) *depend only on x_n and are non-decreasing* (*the positive components of $\mathbf{u}$ are in fact strictly increasing*).

We finally state the following monotonicity result, which is of independent interest. Moreover, it plays the key role for proving the one-dimensionality result on the half-space and non-existence theorem on unbounded domains.

Theorem 1.4. *Let $\mathbf{u}$ be a non-negative $C^2(\Omega) \cap C(\overline{\Omega})$-solution of* (1.1) *and suppose that $\mathbb{R}^n \setminus \overline{\Omega}$ is star-shaped with respect to the origin $\mathcal{O}$. Assume that $\mathbf{f}(x, \mathbf{u})$ is subcritical. Then $|x|^{(n-2)/2}\mathbf{u}(x)$ is non-decreasing, while its positive components are strictly increasing, in the radius $|x|$.*

Theorems 1.1, 1.2, 1.3 and 1.4 were obtained in [2], but under the assumption that $\mathbf{f}$ is locally Lipschitz continuous in $\mathbf{u}$ on Ξ^k. The proofs of Theorems 1.2 and 1.3 are exactly the same as those given in [2] and the reader is referred to [2] for details. We shall prove Theorems 1.1 and 1.4 in section 2. Due to $\mathbf{f}$'s lack of the local Lipschitz continuity in $\mathbf{u}$ on Ξ^k, however, supremum *a posteriori* estimates (from above) on the coefficient-matrices $A(x)$ and $B(t, \theta)$ are needed. By a posteriori we mean that the estimates depend not only on the structural quantities but also on the solution itself; see section 2 for details. These estimates, based on Lemma 2.1 and the assumption (F3), are new and we would like to point out that similar ideas have been used in a recent article of the author [3]. On the other hand, in section 3, a prototype example is given to demonstrate that the super-linearity (see (3.4) for definition) of $\mathbf{f}$, which is substantially weaker than the local Lipschitz continuity of $\mathbf{f}$ in $\mathbf{u}$ on Ξ^k, implies the (F3) property.

2. Proof of Theorems 1.1 and 1.4

In this section, we use the same idea introduced in [2] to prove Theorems 1.1 and 1.4. Since the proofs are essentially the same as those developed in [2], we shall only sketch the proofs and refer the reader to [2] for further details.

Let $\mathbf{u}$ be a positive solution of (1.1). For $1 \leq i \leq k$ and $1 \leq j \leq k$, denote

$$m_{ij}(x) := \frac{u_i(x)}{u_j(x)} > 0, \qquad x \in \Omega.$$

We can bound the quantities $m_{ij}(x)$ as follows.

Lemma 2.1. *Let* $\Gamma \subset \Omega$ *be a bounded subset of* Ω *and let* $\mathbf{u}$ *be a positive solution of* (1.1)*. Then there exists* $C_{ij} = C_{ij}(\mathbf{u}, \partial\Omega, \Gamma, \mathbf{g}) > 0$ *such that*

$$\sup_{x\in\Gamma} m_{ij}(x) \le C_{ij} < \infty \tag{2.1}$$

for every $1 \le i \le k$ *and* $1 \le j \le k$.

Remark. The a posteriori estimate (2.1) continues to hold for positive components of a non-negative solution and the proof remains essentially unchanged.

Proof. Suppose for contradiction that (2.1) is false. Then there exist $1 \le i \le k$, $1 \le j \le k$ and a sequence $\{x^l \in \Gamma\}$ such that

$$\lim_{l\to\infty} m_{ij}(x^l) = \infty. \tag{2.2}$$

Denote (up to a subsequence)

$$\lim_{l\to\infty} x^l = x^0 \in \overline{\Gamma} \subset \mathbb{R}^n.$$

The point x^0 must be at $\partial\Omega$ since $\Gamma\subset\Omega$ is bounded and $\mathbf{u}$ is strictly positive in Ω.

Fix j and rewrite $(1.1)_j$ as

$$\begin{aligned} \Delta u_j - c_j(x)u_j = -(g_j(x,\mathbf{u})u_j + f^j(x,\mathbf{u})) &\le 0 \text{ in } \Omega, \\ u_j &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $c_j(x) = g_j(x,\mathbf{u})$ is non-negative and bounded on $B\cap\Omega$, and B is the unit ball centered at x^0. By the strong maximum principle (i.e., Theorem 8, p. 67, [1]), for every $x \in (\partial\Omega)\cap B$ there exists $C = C(x, \mathbf{u}, \partial\Omega, \mathbf{g}) > 0$ such that

$$\frac{\partial u_j}{\partial \nu_x}(x) \le -C < 0$$

where ν_x is the (unit) outer-normal to $\partial\Omega$ at x.

For $x \in \Omega$, let $x_{\partial\Omega} \in \partial\Omega$ be such that $\operatorname{dist}(x,\partial\Omega) = \operatorname{dist}(x, x_{\partial\Omega})$ (taking x sufficiently close to $\partial\Omega$ to avoid ambiguity) and denote by ν_x the (unit) outer-normal of $\partial\Omega$ at $x_{\partial\Omega}$. One thus readily infers that there exist $\delta = \delta(\mathbf{u}, \partial\Omega, x^0, \mathbf{g}) > 0$ and $C_j = C_j(\mathbf{u}, \partial\Omega, x^0, \mathbf{g}) > 0$ such that

$$\sup_{x\in\Omega_\delta\cap B'} \frac{\partial u_j}{\partial \nu_x}(x) \le -C_j < 0,$$

since $u_j \in C^1(\overline{\Omega})$, where B' is the ball centered at x^0 with radius $\delta/2$ and

$$\Omega_\delta = \{x \in \Omega \,|\, \operatorname{dist}(x,\partial\Omega) < \delta\}.$$

By the mean value theorem, it follows that there exists a point $\xi \in \Omega$ such that

$$u_j(x) = u_j(x) - u_j(x_{\partial\Omega}) = -\frac{\partial u_j}{\partial \nu_x}(\xi)\cdot \operatorname{dist}(x, x_{\partial\Omega}) \ge C_j \operatorname{dist}(x, x_{\partial\Omega}) \tag{2.3}$$

for every $x \in \Omega_\delta \cap B'$, since $u_j = 0$ on $\partial\Omega$.

On the other hand, there exists $C_i = C_i(\mathbf{u}, \Gamma) > 0$ such that

$$u_i(x) = u_i(x) - u_i(x_{\partial\Omega}) \le C_i \operatorname{dist}(x, x_{\partial\Omega}), \qquad x \in \Omega_\delta \cap B', \tag{2.4}$$

since u_i is in $C^1(\overline{\Omega})$.

For l sufficiently large, we have $x^l \in (\Omega_\delta \cap B')$. Therefore, by (2.3)-(2.4), one has

$$m_{ij}(x^l) = \frac{u_i(x^l)}{u_j(x^l)} \le \frac{C_i \mathrm{dist}(x^l, (x^l)_{\partial\Omega})}{C_j \mathrm{dist}(x^l, (x^l)_{\partial\Omega})} = \frac{C_i}{C_j} < \infty.$$

This contradicts (2.2) and completes the proof. □

Now we are ready to prove Theorems 1.1 and 1.4. For convenience, we write

$$\mathbf{u}^j = (u_1, \cdots, u_j, 0, \cdots, 0) \in \Xi^k \text{ for } \mathbf{u} = (u_1, u_2, \cdots, u_k) \in \Xi^k,$$

this being the j-truncation of $\mathbf{u}$, where $\mathbf{u}^0 = (0, \cdots, 0)$.

Proof of Theorem 1.1. Let $B_R(\mathcal{O})$ be the smallest ball centered at $\mathcal{O}$ that contains Ω. For any $\rho \in (0, R)$ we define the cap $\Sigma_\rho = \Omega \setminus \overline{B}_\rho(\mathcal{O})$. Let $\mathbf{u}$ be a positive solution of (1.1). For $x \in \Sigma_\rho$, we make the following Kelvin-transform:

$$\mathbf{u}_\rho(x) = \left(\frac{\rho}{|x|}\right)^{n-2} \mathbf{u}(x^\rho), \qquad x^\rho = \frac{\rho^2 x}{|x|^2}.$$

Since Ω is star-shaped with respect to $\mathcal{O}$, the function $\mathbf{u}_\rho$ is well defined in Σ_ρ and satisfies

$$\Delta \mathbf{u}_\rho + \left(\frac{\rho}{|x|}\right)^{n+2} \mathbf{f}\left(\frac{\rho^2 x}{|x|^2}, \left(\frac{|x|}{\rho}\right)^{n-2} \mathbf{u}_\rho\right) = 0.$$

By the super-criticality of $\mathbf{f}$ (taking $\lambda = (|x|/\rho)^{n-2} > 1$ in (B)), we get

$$\Delta \mathbf{u}_\rho + \mathbf{f}(x, \mathbf{u}_\rho) \le 0 \text{ in } \Sigma_\rho.$$

By taking the difference, one sees that the function $\mathbf{w} = \mathbf{w}_\rho = \mathbf{u}_\rho - \mathbf{u}$ satisfies

$$\begin{aligned} \Delta \mathbf{w} + A(x)\mathbf{w} &\le 0 \text{ in } \Sigma_\rho, \\ \mathbf{w} &\ge 0 \text{ on } \partial\Sigma_\rho, \end{aligned} \tag{2.5}$$

where $A(x)$ is a $k \times k$ matrix in Σ_ρ with entries

$$a_{ij}(x, \mathbf{u}) = \frac{f^i(x, \mathbf{u}_\rho^j + \mathbf{u} - \mathbf{u}^j) - f^i(x, \mathbf{u}_\rho^{j-1} + \mathbf{u} - \mathbf{u}^{j-1})}{w_j}$$

for $w_j \ne 0$ and $a_{ij}(x) = 0$ if $w_j = 0$. Evidently, $a_{ij}(x) \ge 0$ for $i \ne j$ since $\mathbf{f}$ is cooperative and so is (2.5). By Lemma 2.1 and the fact $\mathbf{u} \in C(\overline{\Omega})$, there exists $C = C(\Omega, \mathbf{u}, \mathbf{f}) > 0$ such that

$$\max_{1 \le i,j \le k} \sup_{x \in \Omega} \sup_{\rho \in (0,R)} \{m_{i,j}(x^\rho), m_{i,j}(x), u_j(x), u_j(x^\rho)\} \le C.$$

It follows that, by taking $\Gamma = \{\mathbf{v} \in \Xi^k \,|\, |\mathbf{v}| \le 2|\mathbf{u}|\}$ in (F3), there exist multi-indices $\mathbf{p}_i = \mathbf{p}_i(\mathbf{u}) \in \Xi^k$ and constants $C_0 = C_0(\mathbf{p}_i, \mathbf{u}) > 0$ and $C = C(\Omega, \mathbf{u}, \mathbf{p}_i, \mathbf{f}) > 0$ such that for $i, j = 1, \cdots, k$ and for $\rho \in (0, R)$, there holds $|\mathbf{p}_i| \geq 1$ and

$$\begin{aligned} a_{ij}(x) &\leq C_0 + C_0 \min\{u_j^{p_{ij}-1}(x^\rho), u_j^{p_{ij}-1}(x)\} \cdot \prod_{l=1}^{j-1} u_l^{p_{il}}(x^\rho) \cdot \prod_{l=j+1}^{k} u_l^{p_{il}}(x) \\ &= C_0 + C_0 \min\{u_j^{p_{ij}-1}(x^\rho), u_j^{p_{ij}-1}(x)\} \cdot u_j^{p_{i1}+\cdots+p_{i,j-1}}(x^\rho) \\ &\quad \cdot u_j^{p_{i,j+1}+\cdots+p_{i,k}}(x) \cdot \prod_{l=1}^{j-1} \Big(\frac{u_l(x^\rho)}{u_j(x^\rho)}\Big)^{p_{il}} \cdot \prod_{l=j+1}^{k} \Big(\frac{u_l(x)}{u_j(x)}\Big)^{p_{il}} \\ &\leq C_0 + C_0 \max\{u_j^{|\mathbf{p}_i|-1}(x^\rho), u_j^{|\mathbf{p}_i|-1}(x)\} \cdot \prod_{l=1}^{j-1} m_{l,j}^{p_{il}}(x^\rho) \cdot \prod_{l=j+1}^{k} m_{l,j}^{p_{i,l}}(x) \leq C, \end{aligned}$$

since $|\mathbf{p}_i| \geq 1$ and $p_{ij} \geq 0$. Moreover, the entries $a_{ii}(x) > -\infty$ are bounded from below on any compact subset $\omega \subset \Sigma_\rho$ by (F2). In conclusion, there exists $C = C(\mathbf{u}, \mathbf{f}, \Omega) > 0$ such that

$$0 \leq a_{ij}(x) \leq C \ \ (i \neq j), \qquad -\infty < a_{ii}(x) \leq C$$

in Σ_ρ uniformly for all $\rho \in (0, R)$ and for all $i, j = 1, \cdots, k$. Now the rest of the proof is exactly the same as that of Theorem 1 in [2] and is thus omitted. □

Proof of Theorem 1.4. In spherical coordinates, we have

$$\Omega = \{(r, \theta) : r > r_0(\theta), \theta \in S^{n-1}\}, \qquad \partial\Omega = \{(r_0(\theta), \theta) : \theta \in S^{n-1}\},$$

where $r_0 : S^{n-1} \to (0, \infty]$ is continuous in the extended sense. Clearly we have

$$\min_{\theta \in S^{n-1}} r_0(\theta) > 0,$$

since $\mathcal{O} \in (\mathbb{R}^n \setminus \overline{\Omega}) \neq \emptyset$.

Let $\mathbf{u}$ be a non-negative solution of (1.1) and introduce the following transform:

$$\mathbf{v}(t, \theta) = r^{(n-2)/2}\mathbf{u}(r, \theta), \qquad t = \ln r. \tag{2.6}$$

Put $t_0(\theta) = \ln r_0(\theta)$ and

$$\Sigma = \{(t, \theta) : t > t_0(\theta), \theta \in S^{n-1}\}, \qquad \tau = \min_{\theta \in S^{n-1}} t_0(\theta) > -\infty.$$

By direct calculation, the function $\mathbf{v} = \mathbf{v}(t, \theta)$ in (2.6) is well defined in Σ and satisfies the equation

$$\partial_t^2 \mathbf{v} + \Delta_\theta \mathbf{v} + e^{\frac{n+2}{2}t}\mathbf{f}(e^t, \theta, e^{-\frac{n-2}{2}t}\mathbf{v}) - \frac{1}{4}(n-2)^2\mathbf{v} = 0 \tag{2.7}$$

in Σ, where we write $\mathbf{f}(r, \theta, \mathbf{u})$ for $\mathbf{f}(x, \mathbf{u})$.

For $T \in \mathbb{R}$ we define

$$\Sigma_T = \Sigma \cap (-\infty, T) \times S^{n-1}, \qquad S_T = \Sigma \cap \{T\} \times S^{n-1}.$$

Obviously, $\Sigma_T \neq \emptyset$ for $T > \tau$. Furthermore, we let $(t, \theta)_T$ be the reflection of (t, θ) with respect to $\{T\} \times S^{n-1}$, namely,

$$(t, \theta)_T = (2T - t, \theta).$$

Then the functions

$$\mathbf{v}_T(t, \theta) = \mathbf{v}((t, \theta)_T) = \mathbf{v}(2T - t, \theta), \qquad \mathbf{w}_T = \mathbf{v} - \mathbf{v}_T$$

are well defined in Σ_T for $T > \tau$.

As a consequence of (2.7) the function $\mathbf{w}_T$ satisfies the following equation in Σ_T:

$$\partial_t^2 \mathbf{w}_T + \Delta_\theta \mathbf{w}_T + e^{\frac{n+2}{2}t}\mathbf{f}(e^t, \theta, e^{-\frac{n-2}{2}t}\mathbf{v}) \\ -e^{\frac{n+2}{2}(2T-t)}\mathbf{f}(e^{2T-t}, \theta, e^{-\frac{n-2}{2}(2T-t)}\mathbf{v}_T) - \frac{1}{4}(n-2)^2\mathbf{w}_T = 0.$$

For fixed $\theta \in S^{n-1}$ and $\mathbf{v}_T \in \Pi^k$, we have

$$e^{\frac{n+2}{2}(2T-t)}\mathbf{f}(e^{2T-t}, \theta, e^{-\frac{n-2}{2}(2T-t)}\mathbf{v}_T) \geq e^{\frac{n+2}{2}t}\mathbf{f}(e^t, \theta, e^{-\frac{n-2}{2}t}\mathbf{v}_T)$$

for $t \leq T$, since $\mathbf{f}$ is subcritical (taking $\lambda = e^{-(n-2)(T-t)} > 0$ in (B)). In turn, the function $\mathbf{w}_T$ satisfies

$$\begin{aligned} \partial_t^2 \mathbf{w}_T + \Delta_\theta \mathbf{w}_T + (e^{\frac{n+2}{2}t}B - \tfrac{1}{4}(n-2)^2 I)\mathbf{w}_T &\geq 0 \text{ in } \Sigma_T, \\ \mathbf{w}_T &\leq 0 \text{ on } \partial\Sigma_T, \end{aligned} \tag{2.8}$$

where $B = B(t, \theta, \mathbf{v})$ is a $k \times k$ matrix in Σ_T with entries

$$b_{ij}(t,\theta,\mathbf{v}) = \frac{f^i(e^t, \theta, e^{-\frac{n-2}{2}t}(\mathbf{v}_T^j + \mathbf{v} - \mathbf{v}^j)) - f^i(e^t, \theta, e^{-\frac{n-2}{2}t}(\mathbf{v}_T^{j-1} + \mathbf{v} - \mathbf{v}^{j-1}))}{(w_T)_j}$$

for $(w_T)_j \neq 0$ and $b_{ij}(t,\theta) = 0$ if $(w_T)_j = 0$. As before, $b_{ij}(t,\theta) \geq 0$ for $i \neq j$ since $\mathbf{f}$ is cooperative and so is (2.8) ($e^{\frac{n+2}{2}t} > 0$). Similarly as in the proof of Theorem 1.1, we shall bound the matrix B (from above) in Σ_T uniformly for $T \in [\tau, T_0]$ for every fixed $T_0 > \tau$.

By the strong maximum principle and (F1), every component u_i of $\mathbf{u}$ is either strictly positive or identically zero in Ω. Without loss of generality, there exists $0 < K < k$ such that

$$u_l > 0 \quad \text{for } l = 1, \cdots, K; \qquad u_l \equiv 0 \quad \text{for } l = K+1, \cdots, k.$$

(If $K = 0$, there is nothing to prove; if $K = k$, then $\mathbf{u} > 0$ and early arguments apply with little change.) It follows that

$$v_l > 0 \quad \text{for } l = 1, \cdots, K; \qquad v_l = (w_T)_l \equiv 0 \quad \text{for } l = K+1, \cdots, k. \tag{2.9}$$

In turn,

$$b_{il}(t, \theta, \mathbf{v}) \equiv 0 \qquad \text{for } l = K+1, \cdots, k.$$

By (F3), for each fixed $T_0 > 0$ there exist $\mathbf{p}_i = \mathbf{p}_i(\mathbf{u}, T_0) \in \Xi^k$, $C_0 = C_0(\mathbf{p}_i, T_0, \mathbf{u}) > 0$ such that for $i = 1, \cdots, k$, $j = 1, \cdots, K$ and for $\tau \leq t \leq T \leq T_0$, there hold $|\mathbf{p}_i| \geq 1$ and

$$b_{ij}(t,\theta) \leq C_0 + C_0 \min\{(v_T)_j^{p_{ij}-1}(t,\theta), v_j^{p_{ij}-1}(t,\theta)\} \cdot \prod_{l=1}^{j-1}(v_T)_l^{p_{il}}(t,\theta) \cdot \prod_{l=j+1}^{k} v_l^{p_{i,l}}(t,\theta). \tag{2.10}$$

Next we fix $1 \leq i \leq k$, $1 \leq j \leq K$ and consider two cases.

Case I). There exists $l > K$ with $p_{il} > 0$. Then (2.9) and (2.10) imply $b_{ij}(t,\theta) \leq C_0$ since $v_l(t,\theta) \equiv 0$ and $p_{il} > 0$.

Case II). $p_{il} = 0$ for $l = K+1, \cdots, k$. By Lemma 2.1 (see the remark after it), by the fact $\mathbf{u} \in C(\overline{\Omega})$ and by virtue of the transform (2.6), for each fixed $T_0 > \tau$ there exists $C = C(\Omega, \mathbf{g}, \mathbf{u}, T_0) > 0$ such that for $\tau \leq t \leq T \leq T_0$, there holds

$$\max_{1 \leq i,j \leq K} \sup_{(t,\theta)\in\Sigma_T} \sup_{T \in (\tau, T_0)} \{n_{ij}(2T-t,\theta), n_{ij}(t,\theta), v_j(2T-t,\theta), v_j(t,\theta)\} \leq C,$$

where $n_{ij}(t,\theta) = v_i(t,\theta)/v_j(t,\theta)$. By (2.10), using the fact $p_{il} = 0$ for $l = K+1,\cdots,k$, we infer that (with $0^0 = 1$)

$$
\begin{aligned}
b_{ij}(t,\theta) &\leq C_0 + C_0 \min\{(v_T)_j^{p_{ij}-1}(t,\theta), v_j^{p_{ij}-1}(t,\theta)\} \\
&\quad \cdot \prod_{l=1}^{j-1} (v_T)_l^{p_{il}}(t,\theta) \cdot \prod_{l=j+1}^{K} v_l^{p_{il}}(t,\theta) \\
&= C_0 + C_0 \min\{(v_T)_j^{p_{ij}-1}(t,\theta), v_j^{p_{ij}-1}(t,\theta)\} \cdot (v_T)_j^{p_{i1}+\cdots+p_{i,j-1}}(t,\theta) \\
&\quad \cdot v_j^{p_{i,j+1}+\cdots+p_{iK}}(t,\theta) \cdot \prod_{l=1}^{j-1} \Big(\frac{(v_T)_l(t,\theta)}{(v_T)_j(t,\theta)}\Big)^{p_{il}} \cdot \prod_{l=j+1}^{K} \Big(\frac{v_l(t,\theta)}{v_j(t,\theta)}\Big)^{p_{il}} \\
&\leq C_0 + C_0 \max\{v_j^{|\mathbf{p}_i|-1}(2T-t,\theta), v_j^{|\mathbf{p}_i|-1}(t,\theta)\} \\
&\quad \cdot \prod_{l=1}^{j-1} n_{lj}^{p_{i1}}(2T-t,\theta) \cdot \prod_{l=j+1}^{K} n_{lj}^{p_{il}}(t,\theta) \leq C,
\end{aligned}
$$

since $|\mathbf{p}_i| \geq 1$ and $p_{ij} \geq 0$, where we have omitted the factor $e^{-\frac{n-2}{2}|\mathbf{p}_i|t}$ which is clearly bounded for $\tau \leq t \leq T \leq T_0$. Again, the entries $a_{ii}(t,\theta) > -\infty$ are bounded from below on any compact subset $\sigma \subset \Sigma_T$ by (F2). In conclusion, there exists $C = C(\Omega, \mathbf{u}, \mathbf{f}, T_0) > 0$ such that

$$0 \leq b_{ij}(t,\theta) \leq C \ \ (i \neq j), \qquad -\infty < b_{ii}(t,\theta) \leq C$$

in Σ_T uniformly for all $T \in [\tau, T_0]$ for each fixed T_0 and for all $i,j = 1,\cdots,k$. Now the rest of the proof is (almost) exactly the same as that of Theorem 7 in [2].[2] □

3. An example

In this section, we discuss the following prototype model in which

$$f^i(x,\mathbf{u}) = c_i \mathbf{u}^{\mathbf{p}_i}, \qquad i = 1,\cdots,k, \tag{3.1}$$

where $k \geq 1$ is an integer, $\mathbf{p}_i \in \Xi^k$ and $c_i > 0$. For convenience, we simply write $c_i \equiv 1$. It is straightforward to verify that $\mathbf{f}$ is cooperative and satisfies (F1) with $\mathbf{g} \equiv 0$ and (F2) with $L_i \geq 0$. Moreover, the *sub-criticality* of $\mathbf{f}$ is equivalent to

$$\max_i |\mathbf{p}_i| \leq \frac{n+2}{n-2}, \tag{3.2}$$

while the *super-criticality* is equivalent to

$$\min_i |\mathbf{p}_i| \geq \frac{n+2}{n-2}. \tag{3.3}$$

It is understood that $(n+2)/(n-2) = \infty$ when $n = 2$. In particular, every $\mathbf{f}$ is sub-critical if $n = 2$.

This example was included in [2] (Theorem 4) for the case $k = 2$.

Theorem 3.1 ([2]). *Let $k = 2$ and let $\mathbf{f}$ be given by* (3.1) *with $p_{ij} \geq 1$ for $i,j = 1,2$. Then the following conclusions hold.*

---

[2]The inequality $\partial_t \mathbf{w}_T(t,\theta) > 0$ on S_T in (13) on p. 228 need no longer hold here (even for positive components) since $\mathbf{f}$ is not assumed locally Lipschitz on Π^k. However, this claim is non-essential to the rest of the proof which remains valid and carries over exactly in the same way, in view of our assumptions (F1)-(F3) and the strong maximum principle; see also the proof of Theorem 1 in [2].

I) *Assume that* $\mathbb{R}^n \setminus \overline{\Omega}$ $(n \geq 2)$ *is star-shaped with respect to the origin* $\mathcal{O}$. *Suppose*

$$\max_i |\mathbf{p}_i| < \frac{n+2}{n-2}.$$

Then (1.1) *has no non-trivial and non-negative solution.*

II) *Assume that* Ω $(n \geq 3)$ *is bounded and star-shaped with respect to the origin* $\mathcal{O}$ *and that* (3.3) *holds. Then* (1.1) *has no non-trivial and non-negative solution.*

The restriction $p_{ij} \geq 1$ for $i, j = 1, 2$ (either $p_{ij} \geq 1$ or $p_{ij} = 0$ should suffice) in Theorem 3.1 is required by the local Lipschitz continuity of $\mathbf{f}$ in $\mathbf{u}$ on Ξ^k, which was a key assumption in [2] (see also (L1)-(L2) on p. 237 there). However, as shown in Theorems 1.1, 1.2, 1.3 and 1.4, the condition (F3) is sufficient and the local Lipschitz continuity of $\mathbf{f}$ in $\mathbf{u}$ on Ξ^k is superfluous. In particular, one can show that (F3) holds if $\mathbf{f}$ is assumed merely super-linear (see Lemma 3.2 below). As a result, all conclusions of Theorem 3.1 continue to hold for $\mathbf{f}$'s given by (3.1) with all non-negative exponents $p_{ij} \geq 0$ satisfying $|\mathbf{p}_i| \geq 1$; see Theorem 3.2 below.

We next show that a super-linear function $\mathbf{f}$ indeed possesses the (F3) property, by beginning with an elementary lemma.

Lemma 3.1. *Suppose that* $a, b \geq 0$ *and* $\alpha \in (0, 1)$ *are non-negative constants. Then*

$$\alpha \int_0^1 [ta + (1-t)b]^{\alpha-1} dt \leq \min\{a^{\alpha-1}, b^{\alpha-1}\},$$

where it is understood that $0^{-\beta} = \infty$ *for* $\beta > 0$, $\int_0^1 \infty dt = \infty$ *and* $\infty = \infty$.

This is straightforward.

Lemma 3.2. *Let* $\mathbf{f}$ *be given by* (3.1) *and suppose that* $\mathbf{f}$ *is super-linear in the sense that*

$$\min_i\{|\mathbf{p}_i|\} \geq 1. \tag{3.4}$$

Then $\mathbf{f}$ *satisfies* (F3).

Proof. For $\Gamma \subset \Pi^k$ bounded, put

$$\gamma = \gamma(\Gamma) = \sup_{\mathbf{u} \in \Gamma}(|\mathbf{u}| + 1) \in [1, \infty), \qquad P = P(\mathbf{p}_i) = \sum_{i=1}^k |\mathbf{p}_i|.$$

For $(x, \mathbf{u}), (x, \mathbf{v}) \in \Omega \times \Gamma$ with $u_l = v_l$ for $l \neq j$ and $u_j > v_j > 0$, we have

$$\frac{f^i(x, \mathbf{u}) - f^i(x, \mathbf{v})}{u_j - v_j} = \prod_{l \neq j} u_l^{p_{il}} \cdot \frac{u_j^{p_{ij}} - v_j^{p_{ij}}}{u_j - v_j}, \qquad i, j = 1, \cdots, k.$$

We are going to estimate the right-hand side product from above. We consider three cases.

Case I). $p_{ij} = 0$. It becomes zero.

Case II). $p_{ij} \geq 1$. By the mean value theorem, there exists $\xi \in (v_j, u_j)$ such that

$$\frac{u_j^{p_{ij}} - u_j^{p_{ij}}}{u_j - v_j} = p_{ij}\xi^{p_{ij}-1} \leq p_{ij} u_j^{p_{ij}-1} \leq P\gamma^{p_{ij}-1} \leq P\gamma^{p_{ij}}$$

since $p_{ij}-1\ge 0$ and $\gamma\ge\max\{u_j,1\}$. Therefore,

$$\frac{f^i(x,\mathbf{u})-f^i(x,\mathbf{v})}{u_j-v_j}\le\prod_{l\neq j}\gamma^{p_{il}}\cdot(P\gamma^{p_{ij}})=P\gamma^{|\mathbf{p}_i|}\le P\gamma^P,$$

since $p_{ij}\ge 0$ and $\gamma\ge\max_l\{u_l,1\}$.

Case III). $p_{ij}\in(0,1)$. Using integration, we have

$$\frac{u_j^{p_{ij}}-u_j^{p_{ij}}}{u_j-v_j}=p_{ij}\int_0^1[tu_j+(1-t)v_j]^{p_{ij}-1}dt.$$

By Lemma 3.1,

$$p_{ij}\int_0^1[tu_j+(1-t)v_j]^{p_{ij}-1}dt\le\min\{u_j^{p_{ij}-1},v_j^{p_{ij}-1}\}.$$

It follows that

$$\frac{f^i(x,\mathbf{u})-f^i(x,\mathbf{v})}{u_j-v_j}\le\prod_{l\neq j}u_l^{p_{il}}\cdot\min\{u_j^{p_{ij}-1},v_j^{p_{ij}-1}\}.$$

Combining I)-III), one immediately infers that

$$\frac{f^i(x,\mathbf{u})-f^i(x,\mathbf{v})}{u_j-v_j}\le P\gamma^P+\min\{u_j^{p_{ij}-1},v_j^{p_{ij}-1}\}\cdot\prod_{l\neq j}u_l^{p_{il}}.$$

This completes the proof of the lemma by taking $C_0(\mathbf{p}_i,\Gamma)=P\gamma^P\ge 1$ and $\mathbf{p}_i(\Gamma)=\mathbf{p}_i$ since $\min_i\{|\mathbf{p}_i|\}\ge 1$. ∎

As a consequence of Lemma 3.2, we can extend Theorem 3.1 to include all non-negative exponents $p_{ij}\ge 0$. Below is the statement.

Theorem 3.2. *Let* $\mathbf{f}$ *be given by* (3.1) *and suppose that* (3.4) *holds. Then the following conclusions hold.*

I) *Assume that* $\Omega\subset\mathbb{R}^n$ $(n\ge 3)$ *is bounded and star-shaped with respect to the origin* $\mathcal{O}$ *and that* (3.3) *holds. Then the only non-negative* $C^2(\Omega)\cap C(\overline{\Omega})$*-solution of* (1.1) *is the trivial solution* $\mathbf{u}\equiv 0$.

II) *Assume* $\mathbb{R}^n\setminus\overline{\Omega}$ $(n\ge 2)$ *is star-shaped with respect to the origin* $\mathcal{O}$ *and that* $\mathbf{f}(\mathbf{u})$ *is strictly subcritical,*

$$1<\min_i\{|\mathbf{p}_i|\}\le\max_i\{|\mathbf{p}_i|\}<\frac{n+2}{n-2}.$$

Then (1.1) *has no non-negative non-trivial* $C^2(\Omega)\cap C(\overline{\Omega})$*-solution.*

III) *Assume* $\Omega=\mathbb{R}^n_+=\{x\in\mathbb{R}^n\mid x_n>0\}$ *and that* (3.2) *holds. Then* (1.1) *has no non-negative non-trivial* $C^2(\Omega)\cap C(\overline{\Omega})$*-solution.*

IV) *Assume that* $\mathbb{R}^n\setminus\overline{\Omega}$ *is star-shaped with respect to the origin* $\mathcal{O}$ *and that* (3.2) *holds. Let* $\mathbf{u}$ *be a non-negative* $C^2(\Omega)\cap C(\overline{\Omega})$*-solution of* (1.1)*. Then* $|x|^{(n-2)/2}\mathbf{u}(x)$ *is non-decreasing, while its positive components are strictly increasing, in the radius* $|x|$.

Proof. Since $\mathbf{f}$ is cooperative that the conditions (F1)-(F3) are satisfied, it is easy to see with the aid of Lemma 3.2 (recall $\mathbf{f}$ is super-linear) that Theorems 1.1, 1.2, 1.3 and 1.4 hold. In particular, II) and IV) follow directly from Theorems 1.2 and 1.4 respectively.

We shall reduce I) to the case of Theorem 1.1. Suppose that (1.1) has a non-negative $C^2(\Omega) \cap C(\overline{\Omega})$-solution $\mathbf{u}$ which is not the trivial solution. By the strong maximum principle, every component u_i of $\mathbf{u}$ is either strictly positive or identically zero in Ω. Without loss of generality, there exists $1 \leq k' \leq k$ such that (note $k' > 0$ by our assumption)

$$u_i > 0 \quad \text{for } i = 1, \cdots, k'; \qquad u_i \equiv 0 \quad \text{for } i = k'+1, \cdots, k.$$

If $k' = k$, one readily applies Theorem 1.1 (i.e., (1.1) has a positive solution) to deduce a contradiction. So $k' < k$ and we claim that

$$\mathbf{p}_i = (p_{i1}, \cdots, p_{ik'}, 0, \cdots, 0) \quad \text{for } i = 1, \cdots, k'.$$

Otherwise there exist, say, $i_0 = 1$ and $j_0 > k'$ with $p_{1j_0} > 0$ and so $\mathbf{u}^{\mathbf{p}_1}(x) \equiv 0$ in Ω. Thus $\Delta u_1 = -\mathbf{u}^{\mathbf{p}_1}(x) = 0$ in Ω and $u_1 = 0$ on $\partial\Omega$, which implies $u_1(x) \equiv 0$ in Ω, a contradiction. It follows that for $i = 1, \cdots, k'$

$$\Delta u_i + \bar{\mathbf{u}}^{\bar{\mathbf{p}}_i} = 0, \qquad \text{in } \Omega, \tag{3.5}$$

where

$$\bar{\mathbf{p}}_i = (p_{i1}, \cdots, p_{ik'}), \quad \bar{\mathbf{u}} = (u_1, \cdots, u_{k'}), \qquad i = 1, \cdots, k'.$$

Clearly, the exponents $\bar{\mathbf{p}}_i$ satisfy (3.3) since $|\bar{\mathbf{p}}_i| = |\mathbf{p}_i|$. In particular, the new system (3.5) fulfills the requirements of Theorem 1.1. Therefore, Theorem 1.1 applies to (3.5) and one again sees a contradiction to the fact that (3.5) has a positive solution $\bar{\mathbf{u}} > 0$.

To prove III), we first apply Theorem 1.3 to deduce that all non-negative $C^2(\Omega) \cap C(\overline{\Omega})$-solutions of (1.1) depend only on x_n and are non-decreasing. Let $\mathbf{u}$ be a non-negative non-trivial $C^2(\Omega) \cap C(\overline{\Omega})$-solution of (1.1). Then $\mathbf{u} = \mathbf{u}(x_n)$ is strictly positive by the strong maximum principle and is non-decreasing in x_n. Now a simple integration of (1.1) yields a contradiction since $\mathbf{u} = \mathbf{u}(x_n) > 0$ is non-decreasing in x_n, $\mathbf{u}(0) = 0$ and $\min_i |\mathbf{p}_i| \geq 1$. □

To conclude, we would like to point out that one may consider

$$f^i(x, \mathbf{u}) = \sum_{j=1}^{l(i)} c_{ij}(x) \mathbf{u}^{\mathbf{p}_{ij}}, \qquad i = 1, \cdots, k,$$

where $\mathbf{p}_{ij} \in \Xi^k$, $l(i) \geq 0$ and $c_{ij}(x)$ is continuous on Ω. The case $l(i) = 0$ simply means $f^i(x, \mathbf{u}) \equiv 0$. It goes without saying that certain care is needed to have appropriate assumptions on the functions c_{ij} and the exponents $\mathbf{p}_{ij} \in \Xi^k$ so that Theorems 1.1, 1.2, 1.3 and 1.4 are applicable. For instance, proper relations between the sign of $c_{ij}(x)$ and the exponents $\mathbf{p}_{ij}$ are needed to make $\mathbf{f}$ cooperative. In order for $\mathbf{f}$ to satisfy (F3), one would require either $|\mathbf{p}_{ij}| = 0$ or $|\mathbf{p}_{ij}| \geq 1$ for the super-linearity. We leave the details to the reader.

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