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ON HOMEOMORPHISMS AND QUASI-ISOMETRIES OF THE REAL LINE

PARAMESWARAN SANKARAN

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Abstract. We show that the group of piecewise-linear homeomorphisms of $\mathbb{R}$ having bounded slopes surjects onto the group $QI(\mathbb{R})$ of all quasi-isometries of $\mathbb{R}$. We prove that the following groups can be imbedded in $QI(\mathbb{R})$: the group of compactly supported piecewise-linear homeomorphisms of $\mathbb{R}$, the Richard Thompson group F, and the free group of continuous rank.

1. Introduction

We begin by recalling the notion of quasi-isometry. Let $f : X \longrightarrow X'$ be a map (which is not assumed to be continuous) between metric spaces. We say that f is a C-quasi-isometric embedding if there exists a $C > 1$ such that

$$(*) \qquad C^{-1}d(x,y) - C \leq d'(f(x), f(y)) \leq Cd(x,y) + C$$

for all $x, y \in X$. Here d, d' denote the metrics on X, X', respectively. If, further, every $x' \in X'$ is within distance C from the image of f, we say that f is a C-quasi isometry. If f is a quasi-isometry (for some C), then there exists a quasi-isometry $f' : X' \longrightarrow X$ (for a possibly different constant C') such that $f' \circ f$ (resp. $f \circ f'$) is quasi-isometry equivalent to the identity map of X (resp. X'). (Two maps $f, g : X \longrightarrow X$ are said to be quasi-isometrically equivalent if there exists a constant M such that $d(f(x), g(x)) \leq M$ for all $x \in X$.) Let $[f]$ denote the equivalence class of a quasi-isometry $f : X \longrightarrow X$. The set $QI(X)$ of all equivalence classes of quasi-isometries of X is a group under composition: $[f].[g] = [f \circ g]$ for $[f], [g] \in QI(X)$. If X' is quasi-isometry equivalent to X, then $QI(X')$ is isomorphic to $QI(X)$. We refer the reader to [1] for basic facts concerning quasi-isometry. For example $t \mapsto [t]$ is a quasi-isometry from $\mathbb{R}$ to $\mathbb{Z}$.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any homeomorphism of $\mathbb{R}$. Denote by $B(f)$ the set of break points of f, i.e., points where f fails to have derivative and by $\Lambda(f)$ the set of slopes of f, i.e., $\Lambda(f) = \{f'(t) \mid t \in \mathbb{R} \setminus B(f)\}$. Note that $B(f) \subset \mathbb{R}$ is discrete if f is piecewise differentiable.

Definition 1.1. We say that a subset Λ of $\mathbb{R}^*$, the set of non-zero real numbers, is *bounded* if there exists an $M > 1$ such that $M^{-1} < |\lambda| < M$ for all $\lambda \in \Lambda$. We say that a homeomorphism f of $\mathbb{R}$ which is piecewise differentiable has *bounded slopes* if $\Lambda(f)$ is bounded.

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We denote by $PL_\delta(\mathbb{R})$ the set of all those piecewise-linear homeomorphisms f of $\mathbb{R}$ such that $\Lambda(f)$ is bounded. It is clear that $PL_\delta(\mathbb{R})$ is a subgroup of the group $PL(\mathbb{R})$ of all piecewise-linear homeomorphisms of $\mathbb{R}$.

It is easy to see that each $f \in PL_\delta(\mathbb{R})$ is a quasi-isometry. (See Lemma 2.1 below.) One has a natural homomorphism $\varphi : PL_\delta(\mathbb{R}) \longrightarrow QI(\mathbb{R})$, where $\varphi(f) = [f]$ for all $f \in PL_\delta(\mathbb{R})$.

Theorem 1.2. *The natural homomorphism $\varphi : PL_\delta(\mathbb{R}) \longrightarrow QI(\mathbb{R})$, defined as $f \mapsto [f]$, is surjective.*

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a homeomorphism, recall that $\mathrm{Supp}(f)$, the support of f, is the closure of the set $\{x \in \mathbb{R} \mid f(x) \neq x\}$ of all points moved by f. Denote by $PL_\kappa(\mathbb{R})$ the group of all piecewise-linear homeomorphisms of $\mathbb{R}$ which have compact support. It is obvious that $PL_\kappa(\mathbb{R}) \subset \ker(\varphi)$.

Let Γ be a group of homeomorphisms of $\mathbb{S}^1$. Any $f \in \Gamma$ can be lifted to obtain a homeomorphism $\widetilde{f}$ of $\mathbb{R}$ over the covering projection $p : \mathbb{R} \longrightarrow \mathbb{S}^1$, $t \mapsto \exp(2\pi\sqrt{-1}t)$. The set $\widetilde{\Gamma}$ of all homeomorphisms of $\mathbb{R}$ which are lifts of elements of Γ is a subgroup of the group $Homeo(\mathbb{R})$ of all homeomorphisms of $\mathbb{R}$. Indeed $\widetilde{\Gamma}$ is a central extension of Γ by the infinite cyclic group generated by translation by 1: $x \mapsto x + 1$. Denote by $Diff(\mathbb{S}^1)$ the group of all C^∞ diffeomorphisms of the circle. When Γ is one of the groups $PL(\mathbb{S}^1)$, $Diff(\mathbb{S}^1)$, any element of $\widetilde{f} \in \widetilde{\Gamma}$ has bounded slope and is quasi-isometrically equivalent to the identity map of $\mathbb{R}$ (since $\widetilde{f}(x + n) = \widetilde{f}(x) + n$ for $n \in \mathbb{Z}$).

Recall that Richard Thompson discovered the group

$$F = \langle x_0, x_1, \cdots \mid x_i x_j x_i^{-1} = x_{j+1},\ i < j \rangle$$

and used it in some constructions in logic related to word problems. The group F is finitely presentable with two generators x_0, x_1 and two relations. This group and a closely related larger group G have since then appeared in several contexts including homotopy theory [6], homological group theory [3], Teichmüller theory [9], etc. The group F is isomorphic to the subgroup of piecewise-linear homeomorphisms of $\mathbb{R}$ which are the identity outside the unit interval I such that $B(f)$ is contained in dyadic rationals and $\Lambda(f)$ is contained in the subgroup of $\mathbb{R}^*$ generated by 2. Although F satisfies no (non-trivial) group law, it contains no non-abelian free group. The group G is the group of piecewise-linear homeomorphisms f of the circle $\mathbb{S}^1 = I/\{0, 1\}$ with $B(\widetilde{f})$ contained in dyadic rationals and $\Lambda(\widetilde{f})$ contained in the multiplicative subgroup of $\mathbb{R}^*$ generated by 2 for some lift $\widetilde{f}$ of f. It is the first known example of a finitely presented infinite simple group. We recommend the beautiful survey article [4] for further information about Richard Thompson's groups.

We shall prove the following theorem:

Theorem 1.3. *The following groups can be imbedded in $QI(\mathbb{R})$:*

(i) *the groups $\widetilde{Diff}(\mathbb{S}^1)$ and $\widetilde{PL}(\mathbb{S}^1)$,*
(ii) *the group $PL_\kappa(\mathbb{R})$,*
(iii) *the Thompson's group F, and*
(iv) *the free group of rank c, the continuum.*

Our proofs are completely elementary. We explain the main idea of the proof of Theorem 1.3. Take for example the group $PL_\kappa(\mathbb{R})$. The first step is to realise

this as a subgroup Γ_1 of $PL_\kappa(\mathbb{R})$ having support in $(0,1)$. This is easily achieved by imbedding $\mathbb{R}$ in the interval $(0,1)$. The group Γ_1 can be thought of as a group of piecewise-linear homeomorphisms of the circle. Lifting this back to $\mathbb{R}$ via the covering projection, we now obtain a group $\widetilde{\Gamma}_1$ which no longer has compact support. However each element of this group is quasi-isometric to id. So we conjugate this group by a piecewise-linear homeomorphism whose slope grows exponentially. The result is that the features of each element of $\widetilde{\Gamma}_1$ get magnified, resulting in a quasi-isometry not representing 1. The same trick works for $\widetilde{Diff}(\mathbb{S}^1)$ as well. Parts (iii) and (iv) follow from known embeddings of the relevant groups.

2. Proof of Theorem 1.2

We first establish the following basic observation.

Lemma 2.1. *Let f be a piecewise differentiable homeomorphism of $\mathbb{R}$ with $\Lambda(f) \subset \mathbb{R}^*$ bounded. Then f is a quasi-isometry.*

Proof. Replacing f by $-f$ if necessary, one may assume without loss of generality that f is monotone increasing.

Suppose that $\Lambda(f) \subset (1/M, M)$. If f is differentiable everywhere, then it is an M-quasi-isometry.

Suppose that $B(f) \neq \emptyset$. Let $a \in \mathbb{R}$. For any $b > a$, let $a_1 < \cdots < a_k$ be the points of (a,b) where f is non-differentiable. Then, applying the mean value theorem, $f(b)-f(a) = \sum_{0\le i\le k}(f(a_{i+1})-f(a_i)) = \sum_{0\le i\le k} f'(c_i)(a_{i+1}-a_i)$ for some $c_i \in (a_i, a_{i+1})$. Since $\Lambda(f) \subset (1/M, M)$, it follows that $M^{-1}(b-a) < f(b)-f(a) < M(b-a)$. Since $a, b \in \mathbb{R}$ are arbitrary, we conclude that f is a quasi-isometry. $\square$

One has a well-defined map $\varphi : PL_\delta(\mathbb{R}) \longrightarrow QI(\mathbb{R})$ which is a homomorphism. We now prove that φ is surjective.

Lemma 2.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C-quasi-isometry that preserves the ends of $\mathbb{R}$. Let $x \in \mathbb{R}$. Then* (i) *there exists y such that $y - x \le 4C^2$ is positive integer and $f(y) > f(x)$;* (ii) *there exists v such that $x - v \le 4C^2$ is a positive integer and $f(x) > f(v)$.*

Proof. If $f(x+1) > f(x)$, then $y = x+1$ meets our requirements.

Assume that $f(x) > f(y)$ for all y such that $x+1 \le y < 4C^2 + x$. Let $z \ge x+2$ be the smallest real number such that $z - x$ is a positive integer and $f(z) > f(x) \ge f(z-1)$. Such a z exists since $f(t) \to +\infty$ as $t \to +\infty$. By our assumption $z - x \ge 4C^2 + 1$. Set $u = z - 1$. Then the inequality $(*)$ implies $f(u) < f(x) + C - C^{-1}(u-x) \le f(x) - 3C$ and $f(z) - f(u) < C(z-u) + C = 2C$. Hence $f(z) < f(u) + 2C < f(x) - C$, i.e., $f(z) - f(x) < -C$. This contradicts our hypothesis that $f(z) > f(x)$, completing the proof of part (i). The proof of part (ii) is similar. $\square$

Proof of Theorem 1.2. Since the subgroup $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ that preserves the ends $\{+\infty, -\infty\}$ of $\mathbb{R}$ is of index 2 and since $PL_\delta(\mathbb{R})$ contains elements which are orientation reversing, it suffices to show that $QI^+(\mathbb{R})$ is contained in the image of φ, where $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ is the index 2 subgroup whose elements preserve the ends of $\mathbb{R}$.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C-quasi isometry, with $C > 1$, which preserves the ends of $\mathbb{R}$. We assume, as we may, that C is a positive integer.

Set $x_0 = 0$. We define $x_k \in \mathbb{Z}$ for any integer k as follows: Let $k \geq 1$. Having defined x_{k-1} inductively, choose $x_k > x_{k-1}$ to be the smallest integer such that $f(x_k) > f(x_{k-1})$. For any negative integer k, we define x_k analogously (by downward induction) as the greatest integer such that $x_k < x_{k+1}$ and $f(x_k) < f(x_{k+1})$.

Set $y_k := x_{C^3 k}$, and let $B := \{y_k |\ k \in \mathbb{Z}\} \subset \mathbb{Z}$. By Lemma 2.2, we see that B is a discrete subset of $\mathbb{R}$ which is $4C^5$-dense in $\mathbb{R}$. Note that for any $k \in \mathbb{Z}$, $y_k - y_{k-1} \geq C^3$.

Since $f(y_k) > f(y_{k-1})$ for all $k \in \mathbb{Z}$, there exists a unique piecewise-linear homeomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g(y_k) = f(y_k)$ and is *linear* on the interval $[y_{k-1}, y_k]$ for every $k \in \mathbb{Z}$. We claim that g has bounded slopes. Since g is linear on each of the intervals $[y_{k-1}, y_k]$, we need only bound $\frac{g(y_k)-g(y_{k-1})}{y_k - y_{k-1}}$. Indeed,

$$\frac{g(y_k) - g(y_{k-1})}{y_k - y_{k-1}} = \frac{f(y_k) - f(y_{k-1})}{y_k - y_{k-1}} < C + \frac{C}{y_k - y_{k-1}} \leq C + C^{-2}$$

as $y_k - y_{k-1} \geq C^3$. Similarly,

$$\frac{g(y_k) - g(y_{k-1})}{y_k - y_{k-1}} > C^{-1} - C^{-2}.$$

It follows that $\Lambda(g) \subset [C^{-1} - C^{-2}, C + C^{-2}]$ and $g \in PL_\delta(\mathbb{R})$.

Since f and g agree on the quasi-dense set B, we see that $[f] = [g]$. This completes the proof. □

Remark 2.3. (i) By setting $g(y_k)$ equal to a rational number sufficiently close to $f(y_k)$ in the above proof, we see that since $y_k \in \mathbb{Z}$, the element $g \in PL_\delta(\mathbb{R})$ has rational slopes. Consequently it follows that φ restricted to the subgroup $PL_{\mathbb{Q}}^{\mathbb{Q}^*}(\mathbb{R})$ of $PL_\delta(\mathbb{R})$ consisting of those $g \in PL_\delta(\mathbb{R})$ having slopes in $\mathbb{Q}^*$ and $B(g)$ contained in $\mathbb{Q}$ is surjective.

(ii) The kernel of φ contains the group of all piecewise-linear homeomorphisms which have slope 1 outside a compact interval. This latter group equals the derived group $PLF'(\mathbb{R})$, where $PLF(\mathbb{R})$ denotes the subgroup of $PL_\delta(\mathbb{R})$ consisting of homeomorphisms f for which $B(f)$ is finite. Also $PL_\kappa(\mathbb{R}) = PLF''(\mathbb{R})$. See [2].

3. Proof of Theorem 1.3

Let $h_1 : \mathbb{R} \longrightarrow (0,1)$ be the homeomorphism defined by $h_1(-x) = 1 - h_1(x)$ for every $x \in \mathbb{R}$, $h_1(n) = 1 - 1/(n+2)$ for each integer $n \geq 0$, and is linear on each interval $[n, n+1]$ for $n \in \mathbb{Z}$. If f is any compactly supported (piecewise-linear) homeomorphism of $\mathbb{R}$, then $h_1 \circ f \circ h_1^{-1}$ is a compactly supported (piecewise-linear) homeomorphism of $(0,1)$. Since $\mathbb{S}^1 = I/\{0,1\}$, we also get an embedding $\bar{\eta} : PL_\kappa(\mathbb{R}) \longrightarrow PL(\mathbb{S}^1)$, where $\bar{\eta}(f)$ is defined to be the extension of $h_1 \circ f \circ h_1^{-1}$ to $\mathbb{S}^1$. We define $\eta : PL_\kappa(\mathbb{R}) \longrightarrow PL_\delta(\mathbb{R})$ as the imbedding $f \mapsto \eta(f)$, where $\eta(f)(n) = n$ for $n \in \mathbb{Z}$ and $\eta(f)(x) = n + h_1 f h_1^{-1}(x - n)$ for $n < x < n+1$.

Let $h_0 : \mathbb{R} \longrightarrow \mathbb{R}$ be the piecewise-linear homeomorphism defined as follows: $h_0(-x) = -h_0(x)\ \forall x \in \mathbb{R}, h_0(x) = x$ for $0 \leq x \leq 1$ and maps the interval $[n, n+1]$ onto $[2^{n-1}, 2^n]$ linearly for each positive integer n.

Suppose $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is an orientation-preserving piecewise-linear homeomorphism or a diffeomorphism. Let $\widetilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ be any lift of f so that $p \circ \widetilde{f} = f \circ p$, where $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ is the covering projection $t \mapsto \exp(2\pi\sqrt{-1}t)$. Then $[\widetilde{f}] = 1$ in

$QI(\mathbb{R})$. (Indeed one has $\widetilde{f}(x+n) = n + \widetilde{f}(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ and so $|\widetilde{f} - id| \leq |\widetilde{f}(0)| + 1$.)

Let Γ be one of the groups $PL(\mathbb{S}^1)$ or $Diff(\mathbb{S}^1)$ and let $\widetilde{\Gamma}$ be the group of homeomorphisms of $\mathbb{R}$ which are lifts of elements of Γ with respect to the covering projection p. For $\widetilde{f} \in \widetilde{\Gamma}$ set $f_0 := h_0 \widetilde{f} h_0^{-1}$. Clearly, $\widetilde{f} \mapsto f_0$ is a monomorphism of groups $\widetilde{\Gamma} \longrightarrow Homeo(\mathbb{R})$. We claim that for any $\widetilde{f} \in \widetilde{\Gamma}$, f_0 is a quasi-isometry. To see this, we assume without loss of generality that $\widetilde{f}$ is orientation preserving. It is clear that f_0 is differentiable outside a discrete subset of $\mathbb{R}$. We claim that f_0 has bounded slopes. Since f_0 has continuous derivatives on each interval on which f_0 has derivatives, it suffices to show that the set $\{f_0'(t)\}$ as t varies in $\mathbb{R} \setminus B$ is bounded, where B is any discrete set which contains $B(f_0)$. We set $B := B(h_0) \cup h_0 B(\widetilde{f}) \cup h_0 \widetilde{f}^{-1} B(h_0)$.

Let $0 < m < M$ be such that $m < \widetilde{f}'(x) < M$ for $x \in \mathbb{R}$. Let $t \in \mathbb{R} \setminus B$ and set $s = h_0^{-1}(t), u = \widetilde{f}(s)$ so that $h_0^{-1}, \widetilde{f}, h_0$ are differentiable at t, s, u, respectively. Consequently f_0 is differentiable at t.

Since $|u - s| = |\widetilde{f}(s) - s| < q$, where $q := [|\widetilde{f}(0)|] + 2$, we see that $2^{-q} < h_0'(u)/h_0'(s) < 2^q$. Using the chain rule, it follows that $f_0'(t) = h_0'(u)\widetilde{f}'(s)(h_0^{-1})'(t) = \widetilde{f}'(s)h_0'(u)/h_0'(s)$ lies in the interval $(2^{-q}m, 2^q M)$. It follows from Lemma 2.1 that f_0 is a quasi-isometry.

It is clear that the map $\psi : \widetilde{\Gamma} \longrightarrow QI(\mathbb{R})$ defined as $\widetilde{f} \mapsto [f_0]$ is a homomorphism.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We use the above notations throughout the proof.

(i) We prove that $\psi : \widetilde{\Gamma} \longrightarrow QI(\mathbb{R})$ is a monomorphism where $\Gamma = PL(\mathbb{S}^1)$ or $Diff(\mathbb{S}^1)$. Suppose that $\widetilde{f} \in \widetilde{\Gamma}$, $\widetilde{f} \neq id$. We shall show that that $|f_0 - id|$ is unbounded. Choose x in the interval $[0, 1)$ such that $\widetilde{f}(x) \neq x$. Set $k = [\widetilde{f}(x)]$ so that $\widetilde{f}(x) = k + y$, $0 \leq y < 1$. Replacing $\widetilde{f}$ by its inverse if necessary, we assume without loss of generality that $x < \widetilde{f}(x)$. This implies that $k \geq 0$ with equality only if $y > x$. For any positive integer n, we have $f_0(2^n + 2^n x) = h_0 \widetilde{f} h_0^{-1}(2^n + 2^n x) = h_0 \widetilde{f}(n + 1 + x) = h_0(n + 1 + \widetilde{f}(x)) = h_0(n + 1 + k + y) = 2^{n+k} + 2^{n+k} y$.

If $k = 0$, then $y > x$ and so $f_0(2^n + 2^n x) - (2^n + 2^n x) = 2^n(y - x)$. Thus $|f_0 - id|$ is unbounded.

If $k > 0$, then $f_0(2^n + 2^n x) - (2^n + 2^n x) = 2^{n+k} + 2^{n+k} y - 2^n - 2^n x \geq 2^{n+1} - 2^n - 2^n x = 2^n(1 - x)$. As $0 \leq x < 1$, again it follows that $|f_0 - id|$ is unbounded.

(ii) As observed earlier, $\eta : PL_\kappa(\mathbb{R}) \longrightarrow PL_\delta(\mathbb{R})$ is a monomorphism. It is evident that the image of η is contained in $\widetilde{PL}(\mathbb{S}^1)$. Since $\psi : \widetilde{PL}(\mathbb{S}^1) \longrightarrow QI(\mathbb{R})$ is a monomorphism by (i), assertion (ii) follows.

(iii) Now statement (iii) follows from (ii) above and the fact that Thompson's group F is isomorphic to the subgroup of $PL_\kappa(\mathbb{R})$ of all piecewise-linear homeomorphisms which have support in $[0, 1]$ having break points contained in the set of dyadic rationals in $[0, 1]$ and slopes contained in the multiplicative subgroup of $\mathbb{R}^*$ generated by 2.

(iv) To prove (iv), recall that Grabowski [8] has shown that the free group of rank c the continuum embeds in the group of compactly supported C^k diffeomorphisms $(1 \leq k \leq \infty)$ of any positive-dimensional manifold. In particular, this is true of

$Diff(\mathbb{S}^1)$. It follows easily that $\widetilde{Diff}(\mathbb{S}^1)$ also contains a free group of rank the continuum. By part (i), this completes the proof. □

Lemma 3.1. *The group $QI^+(\mathbb{R})$ is torsion-free.*

Proof. Let $f \in PL_\delta(\mathbb{R})$ be such that $[f] \neq 1 \in QI^+(\mathbb{R})$. Thus $f - id$ is unbounded. Choose a sequence (a_n) of real numbers such that $a_n \to +\infty$ as $n \to +\infty$ and $|f(a_n) - a_n| \to +\infty$. Let $k > 1$ be any integer. Suppose that $f(a_n) > a_n$. Since f is order preserving, for each n we have $a_n < f(a_n) < \cdots < f^k(a_n)$. In particular $f^k(a_n) - a_n > f(a_n) - a_n$. Similarly, $a_n - f^k(a_n) > a_n - f(a_n)$ in case $a_n > f(a_n)$. Therefore $|f^k(a_n) - a_n| > |f(a_n) - a_n|$ $\forall n$ and hence $f^k - id$ is unbounded. Hence $[f^k] \neq 1$ in $QI^+(\mathbb{R})$ for $k > 1$. □

Remark 3.2. Thompson's group G does not imbed in $QI(\mathbb{R})$ since it has an element of order 3, whereas it follows from Lemma 3.1 that all torsion elements in $QI(\mathbb{R})$ are of order 2.

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Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, India
E-mail address: sankaran@imsc.res.in