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UNIQUE CONTINUATION FOR THE SYSTEM OF ELASTICITY IN THE PLANE

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Abstract. We prove the strong unique continuation property for the Lamé system of elastostatics in the plane, $\nabla \cdot \left(\mu \left(\nabla u + \nabla u^t\right)\right) + \nabla \left(\lambda \nabla \cdot u\right) = 0$, with variable Lamé coefficients μ, λ, when μ is Lipschitz and λ is measurable.

1. Introduction

The main purpose of this note is to study the strong unique continuation property for the Lamé system of linearized elastostatics in the plane

$$\nabla \cdot \left(\mu(x) \left(\nabla u + \nabla u^t\right)\right) + \nabla \left(\lambda(x) \nabla \cdot u\right) = 0 \ . \tag{1.1}$$

The functions μ and λ are known as the Lamé coefficients, ∇u is the 2×2 matrix $\left(\partial_i u^j\right)_{i,j=1}^2$ and ∇u^t its transpose $\left(\partial_j u^i\right)_{i,j=1}^2$. The system (1.1) is elliptic when μ and λ are in $L^\infty(\mathbb{R}^2)$ and for some $\kappa > 0$,

$$\mu(x) \geq \kappa \text{ and } 2\mu(x) + \lambda(x) \geq \kappa, \text{ when } x \in \mathbb{R}^2 \ . \tag{1.2}$$

The previous results in the literature are the following: Dehman and Robbiano proved the weak unique continuation property for the system of elastostatics in $\mathbb{R}^n$, $n \geq 2$, when $\lambda, \mu \in \mathcal{C}^\infty(\mathbb{R}^n)$ [6], Ang, Ikehata, Trong and Yamamoto for $\lambda \in \mathcal{C}^2(R^n)$ and $\mu \in \mathcal{C}^3(\mathbb{R}^n)$ [1], Weck for $\lambda, \mu \in \mathcal{C}^2(\mathbb{R}^n)$ ([11], [12]), and Nakamura and Wang for the Lamé system with residual stress for $\lambda, \mu \in \mathcal{C}^{1,1}(\mathbb{R}^n)$ [10]. The strong unique continuation property has been proved in $\mathbb{R}^3$ by C. Lin for twice differentiable Lamé coefficients [8], by Alessandrini and Morasi in $\mathbb{R}^n$, $n \geq 2$, for $\lambda, \mu \in \mathcal{C}^{1,1}(\mathbb{R}^n)$, and by C.L. Lin and J.N. Wang, when λ, μ are Lipschitz functions in $\mathbb{R}^2$ [9]. In this note we prove the following result:

Theorem 1. *Assume that $u = (u_1, u_2) \in W^{1,2}(B)$ satisfies in the sense of distributions* (1.1) *in the unit ball B of the plane, μ is a Lipschitz function, λ is measurable, and for all $k \geq 1$ there is some constant C_k such that*

$$\int_{B_r} |u|^2 \, dx \leq C_k r^k \ , \ \textit{when } 0 < r \leq 1 \ .$$

Then, $u \equiv 0$ in B.

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The same result holds when we add to the system of elasticity (1.1) lower order terms with bounded measurable coefficients.

Scalar elliptic operators in the plane with measurable coefficients, both in divergence or non-divergence form, have the strong unique continuation property and perhaps the same holds for the system (1.1) when λ and μ are measurable. Scalar elliptic operators in the plane in non-divergence form have the strong unique continuation property because its solutions u satisfy, $u_x - iu_y = f \circ \aleph$, where f is some analytic function and $\aleph$ is a certain quasi-conformal homeomorphism of the plane, which verifies

$$N^{-1}|z-w|^{\alpha} \leq |\aleph(z) - \aleph(w)| \leq N|z-w|^{\beta} \ , \text{ when } z, w \in \mathbb{R}^2,$$

for some positive constants N, α and β ([4], [5]). In the divergence case, $u = \Re\,(f \circ \aleph)$, where f and $\aleph$ are as above [3].

Our improvement of the result by C.L. Lin and J.N. Wang in [9] closely follows their arguments. In the next section the reader will realize that the difference between the two arguments is a simple algebraic calculation generalizing the following fact:

Assume that μ is a constant, λ is measurable, and $u = (u_1, u_2)$ is a solution of (1.1) *in B. Then, $f(z) = \mu\left(\partial_1 u_2 - \partial_2 u^1\right) - i\,(2\mu + \lambda)\,(\partial_1 u_1 + \partial_2 u_2)$ is an analytic function in B.*

Here and in the sequel we use the following notation: (x_1, x_2) denotes a point in the plane, $\partial_1 = \frac{\partial}{\partial x_1}$, and $\partial_2 = \frac{\partial}{\partial x_2}$.

2. An algebraic calculation

Proof of Theorem 1. The Lamé equations in the plane are

$$\begin{aligned} &2\partial_1(\mu\partial_1 u_1) + \partial_2(\mu(\partial_1 u_2 + \partial_2 u_1)) + \partial_1(\lambda(\partial_1 u_1 + \partial_2 u_2)) = 0 \ , \\ &\partial_1(\mu(\partial_1 u_2 + \partial_2 u_1)) + 2\partial_2(\mu\partial_2 u_2) + \partial_2(\lambda(\partial_1 u_1 + \partial_2 u_2)) = 0 \ , \end{aligned} \tag{2.1}$$

and can be rewritten as

$$\begin{aligned} &\partial_2(\mu(\partial_1 u_2 + \partial_2 u_1)) - 2\partial_1(\mu\partial_2 u_2) + \partial_1((2\mu + \lambda)(\partial_1 u_1 + \partial_2 u_2)) = 0 \ , \\ &\partial_1(\mu(\partial_1 u_2 + \partial_2 u_1)) - 2\partial_2(\mu\partial_1 u_1) + \partial_2((2\mu + \lambda)(\partial_1 u_1 + \partial_2 u_2)) = 0 \ . \end{aligned} \tag{2.2}$$

On the other hand,

$$\begin{aligned} -2\partial_1(\mu\partial_2 u_2) &= -2\partial_2(\mu\partial_1 u_2) + 2\partial_2\mu\partial_1 u_2 - 2\partial_1\mu\partial_2 u_2 \ , \\ -2\partial_2(\mu\partial_1 u_1) &= -2\partial_1(\mu\partial_2 u_1) + 2\partial_1\mu\partial_2 u_1 - 2\partial_2\mu\partial_1 u_1 \ . \end{aligned}$$

Plugging these formulas in (2.2) we get the following system of equations:

$$\begin{aligned} &-\partial_2(\mu(\partial_1 u_2 - \partial_2 u_1)) + \partial_1((2\mu + \lambda)(\partial_1 u_1 + \partial_2 u_2)) + 2\partial_2\mu\partial_1 u_2 - 2\partial_1\mu\partial_2 u_2 = 0 \ , \\ &\partial_1(\mu(\partial_1 u_2 - \partial_2 u_1)) + \partial_2((2\mu + \lambda)(\partial_1 u_1 + \partial_2 u_2)) + 2\partial_1\mu\partial_2 u_1 - 2\partial_2\mu\partial_1 u_1 = 0 \ . \end{aligned} \tag{2.3}$$

Setting $v_1 = u_1$, $v_2 = u_2$, $v_3 = \mu(\partial_1 u_2 - \partial_2 u_1)$ and $v_4 = (2\mu + \lambda)(\partial_1 u_1 + \partial_2 u_2)$, it follows from (2.3) and the formulas

$$\partial_1 u_2 = \frac{v_3}{\mu} + \partial_2 v_1 \quad , \quad \partial_1 u_1 = \frac{v_4}{2\mu + \lambda} - \partial_2 v_2 \ , \tag{2.4}$$

that $v_1,\ v_2,\ v_3$ and v_4 satisfy the system of equations

$$\begin{cases} \partial_1 v_1 + \partial_2 v_2 - \frac{v_4}{2\mu+\lambda} = 0 \ , \\ \partial_1 v_2 - \partial_2 v_1 - \frac{v_3}{\mu} = 0 \ , \\ \partial_1 v_3 + \partial_2 v_4 + 2\partial_1\mu\partial_2 v_1 + 2\partial_2\mu\partial_2 v_2 - \frac{2\partial_2\mu}{2\mu+\lambda} v_4 = 0 \ , \\ \partial_1 v_4 - \partial_2 v_3 + 2\partial_2\mu\partial_2 v_1 - 2\partial_1\mu\partial_2 v_2 + \frac{2\partial_2\mu}{\mu} v_3 = 0 \ , \end{cases} \tag{2.5}$$

which can be written in matrix form as

$$\partial_1 V + J\partial_2 V + MV = 0 \ , \tag{2.6}$$

where $V = (v_1, v_2, v_3, v_4)^t$ and J, M are the 4×4 matrices

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2\partial_1\mu & 2\partial_2\mu & 0 & 1 \\ 2\partial_2\mu & -2\partial_1\mu & -1 & 0 \end{pmatrix} \quad , \quad M = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2\mu+\lambda} \\ 0 & 0 & -\frac{1}{\mu} & 0 \\ 0 & 0 & 0 & -\frac{2\partial_2\mu}{2\mu+\lambda} \\ 0 & 0 & \frac{2\partial_2\mu}{\mu} & 0 \end{pmatrix} \quad .$$

The Cauchy-Riemann equations and (2.3) show that the claim at the end of the Introduction does hold.

If $u = (u_1, u_2) \in W^{1,2}(B)$ is a solution in the sense of distributions of (1.1), the last two identities in (2.5) give

$$\partial_{\bar z}(v_3 - iv_4) = f \text{ in } B \ , \tag{2.7}$$

where $\partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2)$ is the Cauchy-Riemann operator and $|f| \le N|\nabla u|$ for some $N > 0$ depending on the constant κ in (1.2) and $\|\nabla\mu\|_{L^\infty(\mathbb{R}^2)}$. The gradient of the inverse of the Cauchy-Riemann operator is a Calderón-Zygmund operator, and so (2.7) and standard elliptic regularity implies

$$\int_{B_r} |\nabla v_3|^2 + |\nabla v_4|^2 \, dx \le \frac{N}{r^2} \int_{B_{2r}} |\nabla u|^2 \, dx \ , \text{ when } 0 < r \le 1/2 \ .$$

This argument, the Cacciopoli-type inequality

$$\int_{B_r} |\nabla u + \nabla u^t|^2 \, dx \le \frac{N}{r^2} \int_{B_{2r}} |u|^2 \, dx \ , \text{ when } 0 < r \le 1/2,$$

and Korn's inequality [7]

$$2\int_B |\nabla\varphi|^2 \, dx \le \int_B |\nabla\varphi + \nabla\varphi^t|^2 \, dx \ , \text{ when } \varphi = (\varphi_1, \varphi_2) \in W_0^{1,2}(B),$$

imply that when $u = (u_1, u_2) \in W^{1,2}(B)$ satisfies the conditions in Theorem 1, the following holds: $V \in W^{1,2}_{loc}(B)$, V verifies pointwise almost everywhere the system (2.6) and has a zero of infinite order at the origin i.e., for each $k \ge 1$ there is some constant C_k such that

$$\int_{B_r} |V|^2 \, dx \le C_k r^k \ , \text{ when } 0 < r \le 1 \ .$$

At this point we recall the following result which proves Theorem 1. The interested reader will find its proof between the lines in [9].

Theorem 2. *Assume that $V \in W^{1,2}_{loc}(B)$, $V = (v_1, v_2, v_3, v_4)^t$ is a solution in the unit ball $B \subset \mathbb{R}^2$ of a first order system of the form* (2.6)*, where the 4×4 matrices J and M verify $J, M \in L^\infty(\mathbb{R}^2)$, and*

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 1 \\ \gamma & \delta & -1 & 0 \end{pmatrix}$$

for some measurable functions α, β, γ and δ. Then, if V has zero of infinite order at the origin, $V \equiv 0$ in B. □

The claim after Theorem 1 follows in the same way from Theorem 2, the corresponding analog of (2.2), and the formulas (2.4).

As stated in the Introduction, This proof only differs from the arguments in [9] on an algebraic calculation. The arguments in [9] do not imply Theorem 1 because they choose as the components of V the functions, $v_1 = u_1$, $v_2 = u_2$, $v_3 = \partial_1 u_2 - \partial_2 u_1$ and $v_4 = \partial_1 u_1 + \partial_2 u_2$. With these choices, V is a solution of an elliptic system of the type (2.6) but with matrices J and M, which depend on $\nabla\mu$ and $\nabla\lambda$.

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