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ON THE INVARIANT TRANSLATION APPROXIMATION PROPERTY FOR DISCRETE GROUPS

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Abstract. Recently J. Roe considered the question of whether for a discrete group the reduced group C^*-algebra $C_r^*(\Gamma)$ is the fixed point algebra of $\{\mathrm{Ad}(\rho_t) \mid t \in \Gamma\}$ acting on the uniform Roe algebra $UC_r^*(\Gamma)$. Γ is said to have the invariant translation approximation property in this case. We consider a slight generalization of this property which, for exact Γ, is equivalent to the operator space approximation property of $C_r^*(\Gamma)$. We also give a new characterization of exactness and a short proof of the equivalence of exactness of Γ and exactness of $C_r^*(\Gamma)$ for discrete groups.

1. Introduction

A discrete group Γ has a natural coarse structure which allows us to define the uniform Roe algebra $UC_r^*(\Gamma)$. This algebra may be thought of as the closure of scalar $\Gamma \times \Gamma$-matrices $[\alpha_{s,t}]$ of finite width (i.e. $\{st^{-1} \mid \alpha_{s,t} \neq 0\}$ is finite) with uniformly bounded entries acting on $\ell^2(\Gamma)$. The reduced group C^*-algebra $C_r^*(\Gamma)$ is naturally contained in $UC_r^*(\Gamma)$. Indeed the left translation λ_r on $\ell^2(\Gamma)$ is given by the matrix where $\alpha_{s,t} = 1$ if $st^{-1} = r$ and $\alpha_{s,t} = 0$ otherwise; so the matrices $[\alpha_{s,t}]$ of finite width such that $\alpha_{sr,tr} = \alpha_{s,t}$ for all $s, t, r \in \Gamma$ form precisely the group ring $\mathbb{C}[\Gamma]$. They may also be characterized as those finite width matrices fixed by all automorphisms of the form $\mathrm{Ad}(\rho_t)$, where $t \in \Gamma$ and ρ is the right regular representation of Γ.

Now consider the set of fixed points $UC_r^*(\Gamma)^\Gamma$ of this action in the whole of $UC_r^*(\Gamma)$, not just in the dense subalgebra of finite width matrices. According to [Ro] Γ is said to have the invariant translation approximation property if $C_r^*(\Gamma) = UC_r^*(\Gamma)^\Gamma$. One might wonder whether all discrete groups have this property.

In this note we do not answer this question but consider a slightly stronger invariance property defined as follows. One may define $UC_r^*(\Gamma, S)$ with coefficients in a (concrete) operator space S. $UC_r^*(\Gamma, S)$ is an operator space in general and is a C^*-algebra if S is a C^*-algebra. This very natural idea to allow nontrivial coefficients in the definition of the uniform Roe algebra might be useful elsewhere. $UC_r^*(\Gamma, -)$ may be regarded as a functor (on C^*-algebras or on operator spaces). In Theorem 2.3 we show that Γ is exact iff this functor is exact on C^*-algebras and

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give a quick proof of the result of Kirchberg and Wassermann on the equivalence of exactness of Γ and exactness of $C_r^*(\Gamma)$ for discrete groups ([KW]) together with another characterization of exactness.

Instead of looking at the fixed points in $UC_r^*(\Gamma)$ we consider fixed points in $UC_r^*(\Gamma, S)$. Using results and techniques from [HK] we show that for exact Γ the following conditions are equivalent:

(1) $UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$ for all closed subspaces S of the compact operators $\mathcal{K}$ (on $\ell^2(\mathbb{N})$).
(2) $\big(UC_r^*(\Gamma) \otimes S\big)^\Gamma = C_r^*(\Gamma) \otimes S$ for all closed subspaces S of $\mathcal{K}$.
(3) For any operator space S (not necessarily contained in $\mathcal{K}$) $UC_r^*(\Gamma, S)^\Gamma = \big(UC_r^*(\Gamma) \otimes S\big)^\Gamma = C_r^*(\Gamma) \otimes S$.
(4) $C_r^*(\Gamma)$ has the operator approximation property (OAP) or equivalently Γ has the AP of [HK].

Here and throughout $\otimes$ denotes the minimal tensor product. For group C^*-algebras of discrete groups the OAP implies exactness, and so condition (4) implies the other conditions for all discrete groups Γ. Moreover, the invariant translation approximation property with coefficients implies the one without coefficients, so if Γ has the AP, then $C_r^*(\Gamma) = UC_r^*(\Gamma)^\Gamma$. We also remark that it is not known whether there are exact groups without the AP, but it has been conjectured that $SL_3(\mathbb{Z})$ is such a group (cf. [HK] or [ER]).

2. Preliminaries

A C^*-algebra A is *exact* if given any exact sequence $0 \to J \to B \to C \to 0$ of C^*-algebras the sequence $0 \to A \otimes J \to A \otimes B \to A \otimes C \to 0$ is again exact, i.e. $A \otimes -$ is an exact functor. Inspection shows that for arbitrary A the kernel of the map $A \otimes B \to A \otimes C$ is the *Fubini product* $F(A, J) = \{x \in A \otimes B \mid (\varphi \otimes \mathrm{id})(x) \in J \ \forall \varphi \in A^*\}$ (cf. [Wa]). Thus exactness of A is equivalent to requirering $F(A, J) = A \otimes J$ for all pairs $J \subseteq B$, where B is a C^*-algebra and J a closed ideal of B.

Kraus [Kr] considered the same condition for subspaces, i.e. $F(A, S) = A \otimes S$ for all pairs $S \subseteq B$, where B is a C^*-algebra and S a closed subspace of B. (Wassermann had considered this earlier for subalgebras.) He showed (cf. [Kr], [ER]) that this condition, the *slice map property for subspaces*, is equivalent to the *strong operator approximation property* (*strong OAP*) which means the following: for every C^*-algebra B there is a net (ϕ_α) of finite rank maps such that $\phi_\alpha \otimes \mathrm{id}(x) \to x$ for all $x \in A \otimes B$. In particular if A has the strong OAP, then it is exact. A weaker variant is the *operator approximation property* (*OAP*) which requires only a net (ϕ_α) of finite rank maps such that $\phi_\alpha \otimes \mathrm{id}(x) \to x$ for all $x \in A \otimes \mathcal{K}$ ($\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$). It is the operator space version of Grothendieck's AP (cf. [ER]). The OAP is equivalent to the slice map property for closed subspaces of $\mathcal{K}$.

Most regularity properties of C^*-algebras define, when required for $C_r^*(\Gamma)$, reasonable regularity properties for the group Γ, e.g. nuclearity of $C_r^*(\Gamma)$ is equivalent to amenabilty of Γ. Similarly $C_r^*(\Gamma)$ is an exact C^*-algebra iff Γ is amenable at infinity. Another formally stronger property is exactness of Γ defined as follows ([KW]). Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a Γ-action on a C^*-algebra A such that $\alpha_s(J) = J$ for all $s \in \Gamma$; then $\dot{\alpha}_s(a + J) = \alpha_s(a) + J$ defines an action $\dot{\alpha}$ on $B = A/J$. Denoting the restriction of α to J by α again, we obtain the exact sequence of Γ-dynamical

systems $0 \to (J,\alpha) \to (A,\alpha) \to (B,\dot{\alpha}) \to 0$. Γ is said to be *exact* if the corresponding sequence of reduced crossed products $0 \to J \rtimes_{\alpha,r} \Gamma \to A \rtimes_{\alpha,r} \Gamma \to B \rtimes_{\dot{\alpha},r} \Gamma \to 0$ is exact. A discrete group Γ is exact iff $C_r^*(\Gamma)$ is an exact C^*-algebra ([KW]), and we will give a short proof of this fact in Theorem 2.3.

As for the strong OAP and the OAP, Haagerup and Kraus [HK] show that they are equivalent for $C_r^*(\Gamma)$ if Γ is a discrete group. They also establish an equivalent condition in terms of the Fourier algebra $A(\Gamma)$ which they call the approximation property (AP). The Fourier algebra $A(\Gamma)$ is the set of all coefficients $s \mapsto \langle \xi, \lambda_s \eta \rangle$ in the regular representation of Γ with the pointwise product and the norm $\|u\| = \inf\{\|\xi\|\|\eta\| \mid u(s) = \langle \xi, \lambda_s \eta \rangle \text{ for all } s \in \Gamma\}$. Amenability of Γ is equivalent to the existence of a bounded approximate unit in $A(\Gamma)$ (even one consisting of positive definite functions). In order to explain the AP recall that a complex function u on Γ is a *multiplier* of $A(\Gamma)$ if $uv \in A(\Gamma)$ for all $v \in A(\Gamma)$. In $A(\Gamma)$ convergence in norm implies pointwise convergence, and therefore, by the closed graph theorem, such a multiplier u defines a bounded linear map on $A(\Gamma)$. Since $A(\Gamma) = VN(\Gamma)_*$, by duality, it also defines a bounded linear $\overline{M}_u \in B(VN(\Gamma))$, and since $\overline{M}_u$ leaves the group ring $\mathbb{C}[\Gamma]$ invariant we finally obtain a bounded $M_u \in B(C_r^*(\Gamma))$. u is said to be a completely bounded multiplier if M_u is a completely bounded map, i.e. $M_u \in CB(C_r^*(\Gamma))$. The space $M_0A(\Gamma)$ of completely bounded multipliers can then be equipped with the completely bounded norm $\|u\|_{M_0}$. As cited in [HK] (cf. [Pi]) $u \in M_0A(\Gamma)$ iff there is a Hilbert space K and bounded maps $p, q : \Gamma \to K$ such that $u(st^{-1}) = \langle p(s), q(t) \rangle$ for all $s, t \in \Gamma$. Moreover, the completely bounded norm is given by

$$\|u\|_{M_0} = \inf\{\|p\|_\infty \|q\|_\infty \mid \exists K \text{ s.t. } p, q : \Gamma \to K, u(st^{-1}) = \langle p(s), q(t) \rangle \ \forall s, t \in \Gamma\}.$$

Since all $u \in M_0A(\Gamma)$ are bounded functions, every summable sequence of scalars $(\alpha_s) \in \ell^1(\Gamma)$ defines a functional on $M_0A(\Gamma)$ by $\omega_{(\alpha_s)}(u) = \sum_s \alpha_s u(s)$. Let $Q(\Gamma)$ be the closure of $\{\omega_{(\alpha_s)} \mid (\alpha_s) \in \ell^1(\Gamma)\}$ in $M_0A(\Gamma)^*$. Then Γ is said to have the *approximation property* (*AP*) if there is a net $(u_\alpha) \subseteq A(\Gamma)$ converging to 1 in the $\sigma(M_0A(\Gamma), Q(\Gamma))$-topology. By ([HK], 2.1) $C_r^*(\Gamma)$ has the OAP iff Γ has the AP. It is pointed out in [HK] that one may assume $u_\alpha \in A_c(\Gamma)$, i.e. all the u_α to have finite support.

A matrix $[\alpha_{s,t}]_{s,t \in \Gamma}$ is said to have *width* F, where $F \subseteq \Gamma$ is a subset if $\alpha_{s,t} \neq 0$ implies $st^{-1} \in F$. If $[\alpha_{s,t}]_{s,t \in \Gamma}$ has finite width (i.e. width F for some finite $F \subseteq \Gamma$) and there is $M \geq 0$ such that $|\alpha_{s,t}| \leq M$ for all $s, t \in \Gamma$, then this matrix defines a bounded operator acting on $\ell^2(\Gamma)$. Such matrices form a $*$-algebra and the uniform Roe algebra $UC_r^*(\Gamma)$ is the closure of it in $B(\ell^2(\Gamma))$. We have $UC_r^*(\Gamma) = \ell^\infty(\Gamma) \rtimes_{\tau,r} \Gamma$, where τ is the translation action on $\ell^\infty(\Gamma)$. $UC_r^*(\Gamma)$ is closely related to exactness of Γ. More precisely Guentner, Kaminker ([GK]) and Ozawa ([Oz]) showed the following important result (cf. [Pi] for an excellent exposition).

Theorem 2.1. *For a discrete group Γ the following conditions are equivalent:*

(1) *$C_r^*(\Gamma)$ is exact.*
(2) *$UC_r^*(\Gamma)$ is nuclear.*
(3) *There exists a net of finite width positive definite kernels $k_\alpha : \Gamma \times \Gamma \to \mathbb{C}$ such that for all $\varepsilon > 0$ and every finite subset $F \subseteq \Gamma$ there is α_0 such that $|k_\alpha(s,t) - 1| < \varepsilon$ whenever $st^{-1} \in F$ and $\alpha \geq \alpha_0$.*

In (3) k_α positive definite means $\sum_{s,t} k_\alpha(s,t) \bar{\alpha}_s \alpha_t \geq 0$ for all $(\alpha_s)_{s \in \Gamma}$ of finite support. Condition (3) is quite similar to classical amenability which is equivalent

to the existence of a net of positive definite functions on Γ with analogous properties. Another equivalent condition says that Γ is amenable at infinity, i.e. acts amenably on a compact space.

By a GNS-type construction attributed to Kolmogoroff any positive definite kernel $k : \Gamma \times \Gamma \to \mathbb{C}$ is of the form $k(s,t) = \langle v(s), v(t)\rangle$, where $v : \Gamma \to K$ is a map into a suitable Hilbert space K. Thus $V(e_s) = e_s \otimes v(s)$ defines a linear map $V : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes K$ such that $\|V^*V\| = \sup\{\|v(s)\|^2 \mid s \in \Gamma\}$, and if $X \in B(\ell^2(\Gamma))$ has the matrix representation $[\alpha_{s,t}]$, then $V^*(X \otimes 1)V$ has the matrix representation $[k(s,t)\alpha_{s,t}]$. Thus $\Theta_k([\alpha_{s,t}]) = [k(s,t)\alpha_{s,t}]$ defines a completely positive map (Schur multiplier) $\Theta_k \in B(B(\ell^2(\Gamma)))$ which is used in [Oz]. If k has finite width, then $\Theta_k(B(\ell^2(\Gamma)))$ consists of finite width matrices. Moreover, if (k_α) is as in Theorem 2.1(3), then $\Theta_{k_\alpha}(x) \to x$ for $x \in UC_r^*(\Gamma)$ since this is true for finite width matrices and $\|\Theta_{k_\alpha}\| \to 1$ as $\alpha \to \infty$.

If $S \subseteq B(H)$ is a closed subspace (i.e. a concrete operator space) we may define the operator space $UC_r^*(\Gamma, S)$ as the closure of finite width matrices $[a_{s,t}]_{s,t\in\Gamma}$, where $a_{s,t} \in S$ and $\|a_{s,t}\|$ is uniformly bounded for all $s,t \in \Gamma$ acting on $\ell^2(\Gamma) \otimes H$. Clearly $UC_r^*(\Gamma) \otimes S \subseteq UC_r^*(\Gamma, S)$. Similar to tensor product functors like $C_r^*(\Gamma) \otimes -$, we may regard $UC_r^*(\Gamma, -)$ as a functor on the category of C^*-algebras and $*$-homomorphisms, and will say that $UC_r^*(\Gamma, -)$ is an exact functor if for every exact sequence $0 \to J \to A \to B \to 0$ of C^*-algebras the induced sequence $0 \to UC_r^*(\Gamma, J) \to UC_r^*(\Gamma, A) \to UC_r^*(\Gamma, B) \to 0$ is exact. In Theorem 2.3 we will see that exactness of Γ and of $UC_r^*(\Gamma, -)$ are equivalent. The proof uses Theorem 2.1 and ideas from [KW] and [Ki]. We start with some preliminaries.

Let $0 \to (J, \alpha) \to (A, \alpha) \xrightarrow{p} (B, \dot\alpha) \to 0$ be an exact sequence of Γ-dynamical systems. Denote by p_α the surjective $*$-homomorphism $p_\alpha : A \rtimes_{\alpha,r} \Gamma \to B \rtimes_{\dot\alpha,r} \Gamma$ induced by p. $A \rtimes_{\alpha,r} \Gamma$ may be regarded naturally as a subalgebra of $UC_r^*(\Gamma, A)$ since for $a \in A$ and $s \in \Gamma$ the element $a\lambda_s \in A \rtimes_{\alpha,r} \Gamma$ corresponds to the matrix $\sum_{t\in\Gamma} e_{t,s^{-1}t} \otimes \alpha_{t^{-1}}(a) \in UC_r^*(\Gamma, A)$. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit in J consisting of positive contractions. Then $\hat e_\lambda = \sum_{t\in\Gamma} e_{t,t} \otimes \alpha_{t^{-1}}(e_\lambda)$ corresponds to the element e_λ in the regular representation and is an approximate unit for $J \rtimes_{\alpha,r} \Gamma$. Since $\hat e_\lambda (A \rtimes_{\alpha,r} \Gamma) \hat e_\lambda \subseteq J \rtimes_{\alpha,r} \Gamma$ and $\hat e_\lambda (a\lambda_s) \hat e_\lambda \to a\lambda_s$ for all $a \in J$ and $s \in \Gamma$, it follows that $x \in A \rtimes_{\alpha,r} \Gamma$ lies in $J \rtimes_{\alpha,r} \Gamma$ iff $\hat e_\lambda x \hat e_\lambda \to x$ as $\lambda \to \infty$.

Let $\ell^\infty(\Lambda, A)$ be the bounded functions on Λ with values in A and $c_0(\Lambda, A) = \{(a_\lambda) \in \ell^\infty(\Lambda, A) \mid a_\lambda \to 0 \text{ as } \lambda \to \infty\}$. Denoting the quotient C^*-algebra $\ell^\infty(\Lambda, A)/c_0(\Lambda, A)$ by $q^\infty(\Lambda, A)$ we obtain the exact sequence

$$0 \longrightarrow c_0(\Lambda, A) \longrightarrow \ell^\infty(\Lambda, A) \xrightarrow{q} q^\infty(\Lambda, A) \longrightarrow 0$$

and the induced surjective $*$-homomorphism

$$\hat q : UC_r^*\big(\Gamma, \ell^\infty(\Lambda, A)\big) \longrightarrow UC_r^*\big(\Gamma, q^\infty(\Lambda, A)\big).$$

Following ideas in [Ki] we define the completely bounded map

$$\delta : A \rtimes_{\alpha,r} \Gamma \longrightarrow UC_r^*\big(\Gamma, \ell^\infty(\Lambda, A)\big)$$

by $\delta(x)_\lambda = x - \hat e_\lambda x \hat e_\lambda$, where we think of $UC_r^*\big(\Gamma, \ell^\infty(\Lambda, A)\big) \subseteq \ell^\infty\big(\Lambda, UC_r^*(\Gamma, A)\big)$ as the closure of families (x_λ), where all $x_\lambda \in UC_r^*(\Gamma, A)$ have the same finite width. Then it is clear that $\delta(a\lambda_s) \in UC_r^*\big(\Gamma, \ell^\infty(\Lambda, A)\big)$ for all $a \in A$ and $s \in \Gamma$, and so $\delta(A \rtimes_{\alpha,r} \Gamma) \subseteq UC_r^*\big(\Gamma, \ell^\infty(\Lambda, A)\big)$.

Lemma 2.2. *For $x \in A \rtimes_{\alpha,r} \Gamma$ consider the following conditions:*

(1) $x \in J \rtimes_{\alpha,r} \Gamma$.
(2) $\delta(x) \in UC_r^*\big(\Gamma, c_0(\Lambda, A)\big)$.
(3) $x \in \ker\big(p_\alpha : A \rtimes_{\alpha,r} \Gamma \to B \rtimes_{\dot\alpha,r} \Gamma\big)$.
(4) $x_{s,t} \in J$ *for all* $s, t \in \Gamma$, *where* $x_{s,t}$ *denote the matrix elements of* x.
(5) $\delta(x) \in \ker\Big(\hat q : UC_r^*\big(\Gamma, \ell^\infty(\Lambda, A)\big) \longrightarrow UC_r^*\big(\Gamma, q^\infty(\Lambda, A)\big)\Big)$.

Then (1) $\Leftrightarrow$ (2) *and* (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Proof. (1) $\Leftrightarrow$ (2): Follows from the fact that $\hat e_\lambda x \hat e_\lambda \overset{\lambda\to\infty}{\longrightarrow} x$ iff $x \in J \rtimes_{\alpha,r} \Gamma$.

(3) $\Leftrightarrow$ (4): This is evident if we note that $p_\alpha : A \rtimes_{\alpha,r} \Gamma \to B \rtimes_{\dot\alpha,r} \Gamma$ is simply the restriction of $\hat p : UC_r^*(\Gamma, A) \longrightarrow UC_r^*(\Gamma, B)$ and the kernel of $\hat p$ consists precisely of those matrices with entries in J.

(4) $\Leftrightarrow$ (5): We have

$$\ker \hat q = \big\{X \in UC_r^*\big(\Gamma, \ell^\infty(\Lambda, A)\big) \mid X_{s,t} \in c_0(\Lambda, A) \text{ for all } s, t \in \Gamma\big\}.$$

Regarding $\delta(x)$ as in its definition as a family $\big(\delta(x)_\lambda\big)_{\lambda\in\Lambda}$, we have

$$\delta(x)_{\lambda;s,t} = x_{s,t} - \alpha_{s^{-1}}(e_\lambda) x_{s,t} \alpha_{t^{-1}}(e_\lambda).$$

$\big(\alpha_{s^{-1}}(e_\lambda)\big)_{\lambda\in\Lambda}$ and $\big(\alpha_{t^{-1}}(e_\lambda)\big)_{\lambda\in\Lambda}$ are also approximate units in J, whenever $s, t \in \Gamma$ are arbitrary and fixed so that

$$x_{s,t} \in J \Longleftrightarrow \delta(x)_{\lambda;s,t} \to 0 \text{ as } \lambda \to \infty$$

or $x_{s,t} \in J$ iff $\big(\lambda \mapsto \delta(x)_{\lambda;\,s,t}\big) \in c_0(\Lambda, A)$. This proves (4) $\Leftrightarrow$ (5). □

We are now ready to prove our characterizations of exactness. Condition (3) of the following theorem will be used later.

Theorem 2.3. *For a discrete group the following conditions are equivalent:*

(1) Γ *is exact.*
(2) $C_r^*(\Gamma)$ *is exact.*
(3) *For all Hilbert spaces* H *and closed subspaces* $S \subseteq B(H)$,

$$UC_r^*(\Gamma, S) = \big\{x \in UC_r^*(\Gamma, B(H)) \mid x_{s,t} \in S \text{ for all } s, t \in \Gamma\big\}.$$

(4) $UC_r^*(\Gamma, -)$ *is an exact functor.*

Proof. (1) $\Rightarrow$ (2): This is clear since $0 \to J \rtimes_{\alpha,r} \Gamma \to A \rtimes_{\alpha,r} \Gamma \to B \rtimes_{\dot\alpha,r} \Gamma \to 0$ is just $0 \to J \otimes C_r^*(\Gamma) \to A \otimes C_r^*(\Gamma) \to B \otimes C_r^*(\Gamma) \to 0$ when α is the trivial action $\alpha_s = \mathrm{id}$ for all s.

(2) $\Rightarrow$ (3): Clearly $\subseteq$ in (3) always holds. For the reverse inclusion note that if $C_r^*(\Gamma)$ is exact, then there exists a net (k_α) as in Theorem 2.1(3), and the Schur multiplier Θ_{k_α} defines a completely positive map on $B(\ell^2(\Gamma) \otimes H)$ such that $\Theta_{k_\alpha}(x) \to x$ is norm for all $x \in UC_r^*(\Gamma, B(H))$. Moreover, all the images $\Theta_{k_\alpha}\Big(\big\{x \in UC_r^*(\Gamma, B(H)) \mid x_{s,t} \in S \text{ for all } s, t \in \Gamma\big\}\Big)$ consist of finite width matrices only, which must have entries in S whose norms are uniformly bounded. Therefore they form subsets of $UC_r^*(\Gamma, S)$. Since $UC_r^*(\Gamma, S)$ is norm closed and $\Theta_{k_\alpha}(x) \to x$ for $x \in UC_r^*(\Gamma, B(H))$, the claim follows.

(3) $\Rightarrow$ (4): Let $0 \to J \to A \xrightarrow{p} B \to 0$ be any exact sequence of C^*-algebras (without a Γ-action). p induces a surjective $*$-homomorphism $\hat p : UC_r^*(\Gamma, A) \longrightarrow UC_r^*(\Gamma, B)$, where $\ker \hat p = \{x \in UC_r^*(\Gamma, A) \mid x_{s,t} \in J \text{ for all } s, t \in \Gamma\}$. Thinking of

A as being faithfully represented on a Hilbert space H, i.e. $A \subseteq B(H)$, we have $UC_r^*(\Gamma, J) \subseteq \ker \hat{p} \subseteq \{x \in UC_r^*(\Gamma, B(H)) \mid x_{s,t} \in J \text{ for all } s,t \in \Gamma\} = UC_r^*(\Gamma, J)$ by assumption, hence $\ker \hat{p} = UC_r^*(\Gamma, J)$.

(4) $\Rightarrow$ (1): Let $0 \to (J, \alpha) \to (A, \alpha) \to (B, \dot{\alpha}) \to 0$ be an exact sequence of Γ-dynamical systems and $p_\alpha : A \rtimes_{\alpha,r} \Gamma \to B \rtimes_{\dot{\alpha},r} \Gamma$ the induced surjective $*$-homomorphism. Let $0 \to c_0(\Lambda, A) \to \ell^\infty(\Lambda, A) \xrightarrow{q} q^\infty(\Lambda, A) \to 0$ be the exact sequence before Lemma 2.2. Then by assumption the sequence

$$0 \longrightarrow UC_r^*\Big(\Gamma, c_0(\Lambda, A)\Big) \longrightarrow UC_r^*\Big(\Gamma, \ell^\infty(\Lambda, A)\Big) \xrightarrow{\hat{q}} UC_r^*\Big(\Gamma, q^\infty(\Lambda, A)\Big) \longrightarrow 0$$

is exact. Suppose that $x \in \ker p_\alpha$. Then by Lemma 2.2(3)$\Rightarrow$(5) we know that $\delta(x) \in \ker \hat{q} = UC_r^*\big(\Gamma, c_0(\Lambda, A)\big)$ and by Lemma 2.2(2)$\Rightarrow$(1) it follows that $x \in J \rtimes_{\alpha,r} \Gamma$ and hence Γ is an exact group. $\square$

Now we get back to the OAP, respectively AP for Γ. Let us first note that a completely bounded multiplier $u \in M_0A(\Gamma)$ defines a completely bounded map $\hat{M}_u \in CB(UC_r^*(\Gamma, S))$ for any operator space $S \subseteq B(H)$ by $\hat{M}_u([x_{s,t}]) = [u(st^{-1})x_{s,t}]$. (Here CB stands for completely bounded linear map.) Indeed, as in the definition of Schur multipliers, let K be a Hilbert space and $p, q : \Gamma \to K$ be bounded with $u(st^{-1}) = \langle p(s), q(t)\rangle$ for all $s, t \in \Gamma$. Define bounded linear maps $V, W : \ell^2(\Gamma) \otimes H \to \ell^2(\Gamma) \otimes H \otimes K$ by

$$V(e_s \otimes \xi) = e_s \otimes \xi \otimes p(s) \text{ and } W(e_s \otimes \xi) = e_s \otimes \xi \otimes q(s),$$

where $(e_s) \subseteq \ell^2(\Gamma)$ is the canonical orthonormal basis in $\ell^2(\Gamma)$ and $\xi \in H$. Then $V^*(x \otimes 1_K)W = \hat{M}_u(x)$ for all $x \in UC_r^*(\Gamma, S)$. This map is clearly completely bounded with cb-norm $\leq \|V\|\|W\|$. It follows that $\|\hat{M}_u\| \leq \|u\|_{M_0}$ and $\sup_S \|\hat{M}_u\| = \|u\|_{M_0}$, where the supremum is taken over all operator spaces.

For convergence properties of $\hat{M}_{u_\alpha}$ (as $\alpha \to \infty$), where (u_α) is a net in $A_c(\Gamma)$, we can now use the same kind of arguments as in [HK] for tensor products which we sketch for convenience. By a Hahn-Banach type argument the point-norm closure and the point-weak closure of the subspace $\{\hat{M}_u \mid u \in A_c(\Gamma)\} \subseteq B(UC_r^*(\Gamma, S))$ are identical. Thus the identity map is in the point-norm closure of this subspace iff there is a net $(u_\alpha) \subseteq A_c(\Gamma)$ such that $\varphi(\hat{M}_{u_\alpha}(x)) \to \varphi(x)$ for all $\varphi \in UC_r^*(\Gamma, S)^*$ and $x \in UC_r^*(\Gamma, S)$. Now let us view $u \mapsto \varphi(\hat{M}_u(x))$ as a bounded linear functional $\omega_{\varphi,x} \in M_0A(\Gamma)^*$. Then $u_\alpha \to 1$ in the $\sigma(M_0A(\Gamma), Q(\Gamma))$-topology implies $\varphi(\hat{M}_{u_\alpha}(x)) \to \varphi(x)$ for all $\varphi \in UC_r^*(\Gamma, S)^*$ and $x \in UC_r^*(\Gamma, S)$, provided we can show the following.

Lemma 2.4. *$\omega_{\varphi,x}$ is in $Q(\Gamma)$ for all $\varphi \in UC_r^*(\Gamma, S)^*$ and $x \in UC_r^*(\Gamma, S)$.*

Proof. Note first that $\|\hat{M}_u\| \leq \|u\|_{M_0}$ implies $\|\omega_{\varphi,x}\| \leq \|\varphi\|\|x\|$. Since $Q(\Gamma)$ is complete we may thus assume that x has finite width. But then $\omega_{\varphi,x}(u)$ actually only depends on the value of the function $u : \Gamma \to \mathbb{C}$ on a finite subset of Γ which clearly means that $u \mapsto \omega_{\varphi,x}(u)$ is in $Q(\Gamma)$ in this case. $\square$

Thus the existence of (u_α) in $A_c(\Gamma)$ with $u_\alpha \to 1$ in the $\sigma(M_0A(\Gamma), Q(\Gamma))$-topology guarantees that we may choose (u_α) so that $\hat{M}_{u_\alpha}(x) \to x$ for all $x \in UC_r^*(\Gamma, S)$.

3. The invariant translation approximation property and the AP

Note that if $S \subseteq B(H)$ is a concrete operator space, then $\mathrm{Ad}(\lambda_t \otimes \mathrm{id})$ and $\mathrm{Ad}(\rho_t \otimes \mathrm{id})$ both leave $UC_r^*(\Gamma, S) \subseteq B(\ell^2(\Gamma) \otimes H)$ invariant. We say that a discrete group Γ has the *operator invariant translation approximation property* if for any closed subspace S of the compact operators $\mathcal{K}$ (on $\ell^2(\mathbb{N})$) the equality

$$UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$$

is obtained, where $UC_r^*(\Gamma, S)^\Gamma$ denotes the set of elements in $UC_r^*(\Gamma, S)$ which are fixed under $\mathrm{Ad}(\rho_t \otimes \mathrm{id})$ for all $t \in \Gamma$. If Γ is exact we may replace this condition by $\big(UC_r^*(\Gamma) \otimes S\big)^\Gamma = C_r^*(\Gamma) \otimes S$:

Lemma 3.1. *Suppose that Γ is exact and $S \subseteq B(H)$ is an arbitrary closed subspace. Then $UC_r^*(\Gamma, S)^\Gamma = \big(UC_r^*(\Gamma) \otimes S\big)^\Gamma$.*

Proof. Let (k_α) be as in Theorem 2.1(3). We show first that $\Theta_{k_\alpha}\big(UC_r^*(\Gamma, S)^\Gamma\big) \subseteq UC_r^*(\Gamma) \odot S$, where $\odot$ denotes the algebraic tensor product. Given $x \in UC_r^*(\Gamma, S)^\Gamma$ we have $x_{s,t} = x_{sr,tr} \in S$ for all $s, t, r \in \Gamma$. So $x_{s,t}$ only depends on the value of st^{-1}. In particular $x_{s,t} = x_{st^{-1},e}$ for all $s, t \in \Gamma$. Let $F \subseteq \Gamma$ be finite and x^F the element obtained from x by replacing $x_{s,t}$ by 0 whenever $st^{-1} \notin F$. Then $x^F = \sum_{r \in F} \lambda_r \otimes x_{r,e}$ is in $\mathbb{C}[\Gamma] \odot S$, where $\mathbb{C}[\Gamma]$ is the group ring. Moreover, $\Theta_{k_\alpha}(x) = \Theta_{k_\alpha}(x^F)$ provided k_α has width F. But $\Theta_{k_\alpha}(x^F) = \sum_{r \in F} \Theta_{k_\alpha}(\lambda_r) \otimes x_{r,e} \in UC_r^*(\Gamma) \odot S$, thus $\Theta_{k_\alpha}\big(UC_r^*(\Gamma, S)^\Gamma\big) \subseteq UC_r^*(\Gamma) \odot S$ for all α. Since $\Theta_{k_\alpha}(x) \to x$ for $x \in UC_r^*(\Gamma, S)$, it follows that $UC_r^*(\Gamma, S)^\Gamma = UC_r^*(\Gamma) \otimes S \cap UC_r^*(\Gamma, S)^\Gamma = \big(UC_r^*(\Gamma) \otimes S\big)^\Gamma$. $\square$

Finally we can prove our result.

Theorem 3.2. *For a discrete exact group Γ the following conditions are equivalent:*

1. *Γ has the operator invariant translation approximation property.*
2. *$UC_r^*(\Gamma, S)^\Gamma = \big(UC_r^*(\Gamma) \otimes S\big)^\Gamma = C_r^*(\Gamma) \otimes S$ for any closed subspace $S \subseteq \mathcal{K}$.*
3. *$UC_r^*(\Gamma, S)^\Gamma = \big(UC_r^*(\Gamma) \otimes S\big)^\Gamma = C_r^*(\Gamma) \otimes S$ for any Hilbert space H and any closed subspace $S \subseteq B(H)$.*
4. *Γ has the AP.*

Proof. (1) $\Leftrightarrow$ (2): follows from Lemma 3.1.

(1) $\Rightarrow$ (4): given Γ exact with the operator invariant translation approximation property, it suffices to show that $C_r^*(\Gamma)$ has the slice map property for closed subspaces $S \subseteq \mathcal{K}$. Thus given such an S we have to show that $F(C_r^*(\Gamma), S) = \{x \in C_r^*(\Gamma) \otimes \mathcal{K} \mid (\varphi \otimes \mathrm{id})(x) \in S \text{ for all } \varphi \in C_r^*(\Gamma)^*\}$ equals $C_r^*(\Gamma) \otimes S$. Now $C_r^*(\Gamma) \otimes \mathcal{K} \subseteq UC_r^*(\Gamma, \mathcal{K})$, so we may view $x \in F(C_r^*(\Gamma), S)$ as a $\Gamma \times \Gamma$-matrix with entries in $\mathcal{K}$. Since $(\langle e_s, \cdot\, e_t\rangle \otimes \mathrm{id})(x) = x_{s,t}$, all its entries must be in S and if all entries $x_{s,t}$ are in S, then one checks that $x \in F(C_r^*(\Gamma), S)$. Since Γ is exact Theorem 2.3(3) now implies that $F(C_r^*(\Gamma), S) = C_r^*(\Gamma) \otimes \mathcal{K} \cap UC_r^*(\Gamma, S)$, and this equals $UC_r^*(\Gamma, \mathcal{K})^\Gamma \cap UC_r^*(\Gamma, S) = UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$ if Γ has the operator invariant translation approximation property.

(4) $\Rightarrow$ (3): let $(u_\alpha) \subseteq A_c(\Gamma)$ (all of finite support) such that $\hat{M}_{u_\alpha}(x) \to x$ for all $x \in UC_r^*(\Gamma, B(H))$. $\hat{M}_{u_\alpha}$ commutes with $\mathrm{Ad}\rho_t$ for all $t \in \Gamma$, so for all closed subspaces $S \subseteq B(H)$ the completely bounded operator $\hat{M}_{u_\alpha} \in CB\big(UC_r^*(\Gamma, S)\big)$ maps invariant elements into invariant elements. Moreover, $\hat{M}_{u_\alpha}\big(UC_r^*(\Gamma, S)^\Gamma\big) \subseteq \mathbb{C}[\Gamma] \odot S$ by a similar argument as in the proof of Lemma 3.1. Since $\hat{M}_{u_\alpha}(x) \to x$

whenever $x \in UC_r^*(\Gamma, S)$, it follows that $UC_r^*(\Gamma, S)^\Gamma \subseteq C_r^*(\Gamma) \otimes S$, i.e. $UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$.

(3) $\Rightarrow$ (1): obvious. $\square$

Remark. Is it true that if Γ is a discrete group, then $\big(UC_r^*(\Gamma) \otimes S\big)^\Gamma = UC_r^*(\Gamma)^\Gamma \otimes S$ for all closed subspaces $S \subseteq \mathcal{K}$? If so, then Roe's invariant translation approximation property is equivalent to the OAP at least if Γ is exact. We leave this question open.

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