ON THE INVARIANT TRANSLATION APPROXIMATION PROPERTY FOR DISCRETE GROUPS

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Abstract. Recently J. Roe considered the question of whether for a discrete group the reduced group $C^*$-algebra $C_r^*(\Gamma)$ is the fixed point algebra of $\{\text{Ad}(\rho_t) \mid t \in \Gamma\}$ acting on the uniform Roe algebra $UC_r^*(\Gamma)$. $\Gamma$ is said to have the invariant translation approximation property in this case. We consider a slight generalization of this property which, for exact $\Gamma$, is equivalent to the operator space approximation property of $C_r^*(\Gamma)$. We also give a new characterization of exactness and a short proof of the equivalence of exactness of $\Gamma$ and exactness of $C_r^*(\Gamma)$ for discrete groups.

1. Introduction

A discrete group $\Gamma$ has a natural coarse structure which allows us to define the uniform Roe algebra $UC_r^*(\Gamma)$. This algebra may be thought of as the closure of scalar $\Gamma \times \Gamma$-matrices $[\alpha_{s,t}]$ of finite width (i.e. $\{st^{-1} \mid \alpha_{s,t} \neq 0\}$ is finite) with uniformly bounded entries acting on $\ell^2(\Gamma)$. The reduced group $C^*$-algebra $C_r^*(\Gamma)$ is naturally contained in $UC_r^*(\Gamma)$. Indeed the left translation $\lambda_r$ on $\ell^2(\Gamma)$ is given by the matrix where $\alpha_{s,t} = 1$ if $st^{-1} = r$ and $\alpha_{s,t} = 0$ otherwise; so the matrices $[\alpha_{s,t}]$ of finite width such that $\alpha_{sr,tr} = \alpha_{s,t}$ for all $s, t, r \in \Gamma$ form precisely the group ring $\mathbb{C}[\Gamma]$. They may also be characterized as those finite width matrices fixed by all automorphisms of the form $\text{Ad}(\rho_t)$, where $t \in \Gamma$ and $\rho$ is the right regular representation of $\Gamma$.

Now consider the set of fixed points $UC_r^*(\Gamma)^\Gamma$ of this action in the whole of $UC_r^*(\Gamma)$, not just in the dense subalgebra of finite width matrices. According to [Ro] $\Gamma$ is said to have the invariant translation approximation property if $C_r^*(\Gamma) = UC_r^*(\Gamma)^\Gamma$. One might wonder whether all discrete groups have this property.

In this note we do not answer this question but consider a slightly stronger invariance property defined as follows. One may define $UC_r^*(\Gamma,S)$ with coefficients in a (concrete) operator space $S$. $UC_r^*(\Gamma,S)$ is an operator space in general and is a $C^*$-algebra if $S$ is a $C^*$-algebra. This very natural idea to allow nontrivial coefficients in the definition of the uniform Roe algebra might be useful elsewhere. $UC_r^*(\Gamma,-)$ may be regarded as a functor (on $C^*$-algebras or on operator spaces). In Theorem 2.3 we show that $\Gamma$ is exact if this functor is exact on $C^*$-algebras and...
give a quick proof of the result of Kirchberg and Wassermann on the equivalence of exactness of $\Gamma$ and exactness of $C^*_r(\Gamma)$ for discrete groups ([KW]) together with another characterization of exactness.

Instead of looking at the fixed points in $UC^*_r(\Gamma)$ we consider fixed points in $UC^*_r(\Gamma, S)$. Using results and techniques from [HK] we show that for exact $\Gamma$ the following conditions are equivalent:

1. $UC^*_r(\Gamma, S)^\Gamma = C^*_r(\Gamma) \otimes S$ for all closed subspaces $S$ of the compact operators $K$ (on $\ell^2(\mathbb{N})$).
2. $(UC^*_r(\Gamma) \otimes S)^\Gamma = C^*_r(\Gamma) \otimes S$ for all closed subspaces $S$ of $K$.
3. For any operator space $S$ (not necessarily contained in $K$) $UC^*_r(\Gamma, S)^\Gamma = (UC^*_r(\Gamma) \otimes S)^\Gamma = C^*_r(\Gamma) \otimes S$.
4. $C^*_r(\Gamma)$ has the operator approximation property (OAP) or equivalently $\Gamma$ has the AP of [HK].

Here and throughout $\otimes$ denotes the minimal tensor product. For group $C^*$-algebras of discrete groups the OAP implies exactness, and so condition (4) implies the other conditions for all discrete groups $\Gamma$. Moreover, the invariant translation approximation property with coefficients implies the one without coefficients, so if $\Gamma$ has the AP, then $C^*_r(\Gamma) = UC^*_r(\Gamma)^\Gamma$. We also remark that it is not known whether there are exact groups without the AP, but it has been conjectured that $SL_3(\mathbb{Z})$ is such a group (cf. [HK] or [ER]).

2. Preliminaries

A $C^*$-algebra $A$ is exact if given any exact sequence $0 \to J \to B \to C \to 0$ of $C^*$-algebras the sequence $0 \to A \otimes J \to A \otimes B \to A \otimes C \to 0$ is again exact, i.e. $A \otimes -$ is an exact functor. Inspection shows that for arbitrary $A$ the kernel of the map $A \otimes B \to A \otimes C$ is the Fubini product $F(A, J) = \{x \in A \otimes B \mid (\varphi \otimes \text{id})(x) \in J \ \forall \varphi \in A^*\}$ (cf. [Wa]). Thus exactness of $A$ is equivalent to requiring $F(A, J) = A \otimes J$ for all pairs $J \subseteq B$, where $B$ is a $C^*$-algebra and $J$ a closed ideal of $B$.

Kraus [Kr] considered the same condition for subspaces, i.e. $F(A, S) = A \otimes S$ for all pairs $S \subseteq B$, where $B$ is a $C^*$-algebra and $S$ a closed subspace of $B$. (Wassermann had considered this earlier for subalgebras.) He showed (cf. [Kr], [ER]) that this condition, the slice map property for subspaces, is equivalent to the strong operator approximation property (strong OAP) which means the following: for every $C^*$-algebra $B$ there is a net $(\phi_\alpha)$ of finite rank maps such that $\phi_\alpha \otimes \text{id}(x) \to x$ for all $x \in A \otimes B$. In particular if $A$ has the strong OAP, then it is exact. A weaker variant is the operator approximation property (OAP) which requires only a net $(\phi_\alpha)$ of finite rank maps such that $\phi_\alpha \otimes \text{id}(x) \to x$ for all $x \in A \otimes K$ ($K = K(\ell^2(\mathbb{N}))$). It is the operator space version of Grothendieck’s AP (cf. [ER]). The OAP is equivalent to the slice map property for closed subspaces of $K$.

Most regularity properties of $C^*$-algebras define, when required for $C^*_r(\Gamma)$, reasonable regularity properties for the group $\Gamma$, e.g. nuclearity of $C^*_r(\Gamma)$ is equivalent to amenability of $\Gamma$. Similarly $C^*_r(\Gamma)$ is an exact $C^*$-algebra iff $\Gamma$ is amenable at infinity. Another formally stronger property is exactness of $\Gamma$ defined as follows ([KW]). Let $\alpha : \Gamma \to \text{Aut}(A)$ be a $\Gamma$-action on a $C^*$-algebra $A$ such that $\alpha_s(J) = J$ for all $s \in \Gamma$; then $\alpha_s(a + J) = \alpha_s(a) + J$ defines an action $\hat{\alpha}$ on $B = A/J$. Denoting the restriction of $\alpha$ to $J$ by $\alpha$ again, we obtain the exact sequence of $\Gamma$-dynamical
systems $0 \to (J, \alpha) \to (A, \alpha) \to (B, \hat{\alpha}) \to 0$. $\Gamma$ is said to be exact if the corresponding sequence of reduced crossed products $0 \to J \rtimes_{\alpha_r} \Gamma \to A \rtimes_{\alpha_r} \Gamma \to B \rtimes_{\alpha_r} \Gamma \to 0$ is exact. A discrete group $\Gamma$ is exact if $C^*_\alpha(\Gamma)$ is an exact $C^*$-algebra ([KW]), and we will give a short proof of this fact in Theorem 2.3.

As for the strong OAP and the OAP, Haagerup and Kraus [HK] show that they are equivalent for $C^*_\alpha(\Gamma)$ if $\Gamma$ is a discrete group. They also establish an equivalent condition in terms of the Fourier algebra $A(\Gamma)$ which they call the approximation property (AP). The Fourier algebra $A(\Gamma)$ is the set of all coefficients $s \mapsto \langle \xi, \lambda, \eta \rangle$ in the regular representation of $\Gamma$ with the pointwise product and the norm $\|u\| = \inf\{\|\xi\|\|\eta\| \mid u(s) = \langle \xi, \lambda, \eta \rangle \text{ for all } s \in \Gamma\}$. Amenability of $\Gamma$ is equivalent to the existence of a bounded approximate unit in $A(\Gamma)$ (even one consisting of positive definite functions). In order to explain the AP recall that a complex function $u$ is a multiplier of $A(\Gamma)$ if $uv \in A(\Gamma)$ for all $v \in A(\Gamma)$. In $A(\Gamma)$ convergence in norm implies pointwise convergence, and therefore, by the closed graph theorem, such a multiplier $u$ defines a bounded linear map on $A(\Gamma)$. Since $A(\Gamma) = VN(\Gamma)_s$, by duality, it also defines a bounded linear $\overline{M}_u \in B(VN(\Gamma))$, and since $\overline{M}_u$ leaves the group ring $C[\Gamma]$ invariant we finally obtain a bounded $\overline{M}_u \in B(C^*_\alpha(\Gamma))$. $u$ is said to be a completely bounded multiplier if $\overline{M}_u$ is a completely bounded map, i.e. $\overline{M}_u \in CB(C^*_\alpha(\Gamma))$. The space $M_0A(\Gamma)$ of completely bounded multipliers can then be equipped with the completely bounded norm $\|u\|_{M_0\Gamma}$. As cited in [HK] (cf. [P]) $u \in M_0A(\Gamma)$ if there is a Hilbert space $K$ and bounded maps $p, q : \Gamma \to K$ such that $u(st^{-1}) = \langle p(s), q(t) \rangle$ for all $s, t \in \Gamma$. Moreover, the completely bounded norm is given by $\|u\|_{M_0\Gamma} = \inf\{\|p\|_{\infty} \|q\|_{\infty} \mid \exists K \text{ s.t. } p, q : \Gamma \to K, u(st^{-1}) = \langle p(s), q(t) \rangle \forall s, t \in \Gamma\}$.

Since all $u \in M_0A(\Gamma)$ are bounded functions, every summable sequence of scalars $(\alpha_s) \in \ell^1(\Gamma)$ defines a functional on $M_0A(\Gamma)$ by $\omega(\alpha_s)(u) = \sum_s \alpha_s u(s)$. Let $Q(\Gamma)$ be the closure of $\{\omega(\alpha_s) \mid (\alpha_s) \in \ell^1(\Gamma)\}$ in $M_0A(\Gamma)^\ast$. Then $\Gamma$ is said to have the approximation property (AP) if there is a net $(u_\alpha) \subseteq A(\Gamma)$ converging to 1 in the $\sigma(M_0A(\Gamma), Q(\Gamma))$-topology. By ([HK], 2.1) $C^*_\alpha(\Gamma)$ has the OAP iff $\Gamma$ has the AP. It is pointed out in [HK] that one may assume $u_\alpha \in A_c(\Gamma)$, i.e. all the $u_\alpha$ to have finite support.

A matrix $[\alpha_{s,t}]_{s,t \in \Gamma}$ is said to have width $F$, where $F \subseteq \Gamma$ is a subset if $\alpha_{s,t} \neq 0$ implies $st^{-1} \in F$. If $[\alpha_{s,t}]_{s,t \in \Gamma}$ has finite width (i.e. width $F$ for some finite $F \subseteq \Gamma$) and there is $M \geq 0$ such that $|\alpha_{s,t}| \leq M$ for all $s, t \in \Gamma$, then this matrix defines a bounded operator acting on $\ell^2(\Gamma)$. Such matrices form an $*$-algebra and the uniform Roe algebra $UC^*_\alpha(\Gamma)$ is the closure of it in $B(\ell^2(\Gamma))$. We have $UC^*_\alpha(\Gamma) = \ell^\infty(\Gamma) \rtimes_{\alpha_r} \Gamma$, where $\sigma$ is the translation action on $\ell^\infty(\Gamma)$. $UC^*_\alpha(\Gamma)$ is closely related to exactness of $\Gamma$. More precisely Guentner, Kaminker ([K]) and Ozawa ([O]) showed the following important result (cf. [P] for an excellent exposition).

**Theorem 2.1.** For a discrete group $\Gamma$ the following conditions are equivalent:

1. $C^*_\alpha(\Gamma)$ is exact.
2. $UC^*_\alpha(\Gamma)$ is nuclear.
3. There exists a net of finite width positive definite kernels $k_\alpha : \Gamma \times \Gamma \to \mathbb{C}$ such that for all $\varepsilon > 0$ and every finite subset $F \subseteq \Gamma$ there is $\alpha_0$ such that $|k_\alpha(s, t) - 1| < \varepsilon$ whenever $st^{-1} \in F$ and $\alpha \geq \alpha_0$.

In (3) $k_\alpha$ positive definite means $\sum_{s,t} k_\alpha(s, t)\overline{\alpha_s}\alpha_t \geq 0$ for all $(\alpha_s)_{s \in \Gamma}$ of finite support. Condition (3) is quite similar to classical amenability which is equivalent
to the existence of a net of positive definite functions on $\Gamma$ with analogous properties. Another equivalent condition says that $\Gamma$ is amenable at infinity, i.e. acts amenably on a compact space.

By a GNS-type construction attributed to Kolmogoroff any positive definite kernel $k : \Gamma \times \Gamma \to \mathbb{C}$ is of the form $k(s,t) = \langle v(s), v(t) \rangle$, where $v : \Gamma \to K$ is a map into a suitable Hilbert space $K$. Thus $V(e_s) = e_s \otimes v(s)$ defines a linear map $V : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes K$ such that $\|V v\| = \text{sup}\{\|v(s)\|^2 \mid s \in \Gamma\}$, and if $X \in B(\ell^2(\Gamma))$ has the matrix representation $[a_{s,t}]$, then $V^* (X \otimes 1)V$ has the matrix representation $[k(s,t) a_{s,t}]$. Thus $\Theta_k([a_{s,t}]) = [k(s,t) a_{s,t}]$ defines a completely positive map (Schur multiplier) $\Theta_k \in B(B(\ell^2(\Gamma)))$ which is used in [OZ]. If $k$ has finite width, then $\Theta_k(B(\ell^2(\Gamma)))$ consists of finite width matrices. Moreover, if $(k_\alpha)$ is as in Theorem 2.1(3), then $\Theta_{k_\alpha}(x) \to x$ for $x \in \mathcal{UC}^*\ell^2(\Gamma)$ since this is true for finite width matrices and $\|\Theta_{k_\alpha}\| \to 1$ as $\alpha \to \infty$.

If $S \subseteq B(H)$ is a closed subspace (i.e. a concrete operator space) we may define the operator space $\mathcal{UC}^*_\ell(\Gamma, S)$ as the closure of finite width matrices $[a_{s,t}]_{s,t \in \Gamma}$, where $a_{s,t} \in S$ and $\|a_{s,t}\|$ is uniformly bounded for all $s, t \in \Gamma$ acting on $\ell^2(\Gamma) \otimes H$. Clearly $\mathcal{UC}^*_\ell(\Gamma) \otimes S \subseteq \mathcal{UC}^*_\ell(\Gamma, S)$. Similar to tensor product functors like $\mathcal{UC}^*_\ell(\Gamma)$, we may regard $\mathcal{UC}^*_\ell(\Gamma, -)$ as a functor on the category of $C^*$-algebras and $*$-homomorphisms, and will say that $\mathcal{UC}^*_\ell(\Gamma, -)$ is an exact functor if for every exact sequence $0 \to J \to A \to B \to 0$ of $C^*$-algebras the induced sequence $0 \to \mathcal{UC}^*_\ell(\Gamma, J) \to \mathcal{UC}^*_\ell(\Gamma, A) \to \mathcal{UC}^*_\ell(\Gamma, B) \to 0$ is exact. In Theorem 2.3 we will see that exactness of $\Gamma$ and of $\mathcal{UC}^*_\ell(\Gamma, -)$ are equivalent. The proof uses Theorem 2.1 and ideas from [KW] and [K]. We start with some preliminaries.

Let $0 \to (J, \alpha) \to (A, \alpha) \xrightarrow{p_\alpha} (B, \dot{\alpha}) \to 0$ be an exact sequence of $\Gamma$-dynamical systems. Denote by $p_\alpha$ the surjective $*$-homomorphism $p_\alpha : A \rtimes_{\alpha, r} \Gamma \to B \rtimes_{\dot{\alpha}, r} \Gamma$ induced by $p$. $A \rtimes_{\alpha, r} \Gamma$ may be regarded naturally as a subalgebra of $\mathcal{UC}^*_\ell(\Gamma, A)$ since for $a \in A$ and $s \in \Gamma$ the element $a\lambda_s \in A \rtimes_{\alpha, r} \Gamma$ corresponds to the matrix $\sum_{t \in \Gamma} e_{s-1} : a \otimes a_{-1}(a) \in \mathcal{UC}^*_\ell(\Gamma, A)$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit in $J$ consisting of positive contractions. Then $\tilde{e}_\lambda = \sum_{t \in \Gamma} e_{s,t} \otimes a_{-1}(a) e_\lambda$ corresponds to the element $e_\lambda$ in the regular representation and is an approximate unit for $J \rtimes_{\alpha, r} \Gamma$. Since $e_\lambda(a\lambda_s) \tilde{e}_\lambda \subseteq J \rtimes_{\alpha, r} \Gamma$ and $e_\lambda(a\lambda_s) \tilde{e}_\lambda - a\lambda_s$ for all $a \in J$ and $s \in \Gamma$, it follows that $x \in J \rtimes_{\alpha, r} \Gamma$ lies in $J \rtimes_{\alpha, r} \Gamma$ iff $x\lambda x \tilde{e}_\lambda \to x$ as $\lambda \to \infty$.

Let $\ell^\infty(\Lambda, A)$ be the bounded functions on $\Lambda$ with values in $A$ and $c_0(\Lambda, A) = \{(a_\lambda) \in \ell^\infty(\Lambda, A) \mid a_\lambda \to 0 \text{ as } \lambda \to \infty\}$. Denoting the quotient $C^*$-algebra $\ell^\infty(\Lambda, A)/c_0(\Lambda, A)$ by $q^\infty(\Lambda, A)$ we obtain the exact sequence

$$0 \to c_0(\Lambda, A) \to \ell^\infty(\Lambda, A) \xrightarrow{q} q^\infty(\Lambda, A) \to 0$$

and the induced surjective $*$-homomorphism

$$q : \mathcal{UC}^*_\ell(\Gamma, \ell^\infty(\Lambda, A)) \to \mathcal{UC}^*_\ell(\Gamma, q^\infty(\Lambda, A)).$$

Following ideas in [K] we define the completely bounded map

$$\delta : A \rtimes_{\alpha, r} \Gamma \to \mathcal{UC}^*_\ell(\Gamma, \ell^\infty(\Lambda, A))$$

by $\delta(x)_\lambda = x - x\lambda x \tilde{e}_\lambda$, where we think of $\mathcal{UC}^*_\ell(\Gamma, \ell^\infty(\Lambda, A)) \subseteq \ell^\infty(\Lambda, \mathcal{UC}^*_\ell(\Gamma, A))$ as the closure of families $(x_\lambda)$, where all $x_\lambda \in \mathcal{UC}^*_\ell(\Gamma, A)$ have the same finite width. Then it is clear that $\delta(a\lambda_s) \in \mathcal{UC}^*_\ell(\Gamma, \ell^\infty(\Lambda, A))$ for all $a \in A$ and $s \in \Gamma$, and so $\delta(A \rtimes_{\alpha, r} \Gamma) \subseteq \mathcal{UC}^*_\ell(\Gamma, \ell^\infty(\Lambda, A))$. 

Lemma 2.2. For $x \in A \rtimes_{\alpha,r}\Gamma$ consider the following conditions:

1. $x \in J \rtimes_{\alpha,r}\Gamma$.
2. $\delta(x) \in UC^*_r(\Gamma,c_0(\Lambda,A))$.
3. $x \in \ker(p_\alpha : A \rtimes_{\alpha,r}\Gamma \to B \rtimes_{\alpha,r}\Gamma)$.
4. $x_{s,t} \in J$ for all $s,t \in \Gamma$, where $x_{s,t}$ denote the matrix elements of $x$.
5. $\delta(x) \in \ker(\hat{q} : UC^*_r(\Gamma,\ell^\infty(\Lambda,A)) \to UC^*_r(\Gamma,\ell^\infty(\Lambda,A)))$.

Then (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Proof. (1) $\Leftrightarrow$ (2): Follows from the fact that $\hat{\epsilon}_\lambda \hat{e}_\lambda \xrightarrow{\lambda \to \infty} x$ iff $x \in J \rtimes_{\alpha,r}\Gamma$.

(3) $\Leftrightarrow$ (4): This is evident if we note that $p_\alpha : A \rtimes_{\alpha,r}\Gamma \to B \rtimes_{\alpha,r}\Gamma$ is simply the restriction of $\hat{\rho} : UC^*_r(\Gamma,A) \to UC^*_r(\Gamma,B)$ and the kernel of $\hat{\rho}$ consists precisely of those matrices with entries in $J$.

(4) $\Leftrightarrow$ (5): We have

$$\ker \hat{q} = \{ X \in UC^*_r(\Gamma,\ell^\infty(\Lambda,A)) \mid x_{s,t} \in c_0(\Lambda,A) \text{ for all } s,t \in \Gamma \}.$$ 

Regarding $\delta(x)$ as in its definition as a family $(\delta(x)_\lambda)_{\lambda \in \Lambda}$, we have

$$\delta(x)_{\lambda,s,t} = x_{s,t} - \alpha_{s-r}(e_\lambda) x_{s,t} \alpha_{t-r}(e_\lambda).$$

$(\alpha_s(e_\lambda))_{\lambda \in \Lambda}$ and $(\alpha_t(e_\lambda))_{\lambda \in \Lambda}$ are also approximate units in $J$, whenever $s,t \in \Gamma$ are arbitrary and fixed so that

$$x_{s,t} \in J \iff \delta(x)_{\lambda,s,t} \to 0 \text{ as } \lambda \to \infty$$
or $x_{s,t} \in J$ iff $(\lambda \to \delta(x)_{\lambda,s,t}) \in c_0(\Lambda,A)$. This proves (4) $\Leftrightarrow$ (5). \qed

We are now ready to prove our characterizations of exactness. Condition (3) of the following theorem will be used later.

Theorem 2.3. For a discrete group the following conditions are equivalent:

1. $\Gamma$ is exact.
2. $C^*_r(\Gamma)$ is exact.
3. For all Hilbert spaces $H$ and closed subspaces $S \subseteq B(H)$,

$$UC^*_r(\Gamma,S) = \{ x \in UC^*_r(\Gamma,B(H)) \mid x_{s,t} \in S \text{ for all } s,t \in \Gamma \}.$$ 

4. $UC^*_r(\Gamma,-)$ is an exact functor.

Proof. (1) $\Rightarrow$ (2): This is clear since $0 \to J \rtimes_{\alpha,r}\Gamma \to A \rtimes_{\alpha,r}\Gamma \to B \rtimes_{\alpha,r}\Gamma \to 0$ is just $0 \to J \otimes C^*_r(\Gamma) \to A \otimes C^*_r(\Gamma) \to B \otimes C^*_r(\Gamma) \to 0$ when $\alpha$ is the trivial action $\alpha_s = \text{id}$ for all $s$.

(2) $\Rightarrow$ (3): Clearly $\subseteq$ in (3) always holds. For the reverse inclusion note that if $C^*_r(\Gamma)$ is exact, then there exists a net $(k_\alpha)$ as in Theorem 2.1(3), and the Schur multiplier $\Theta_{k_\alpha}$ defines a completely positive map on $B(L^2(\Gamma) \otimes H)$ such that $\Theta_{k_\alpha}(x \to x$ is norm for all $x \in UC^*_r(\Gamma,B(H))$. Moreover, all the images

$$\Theta_{k_\alpha} \left( \{ x \in UC^*_r(\Gamma,B(H)) \mid x_{s,t} \in S \text{ for all } s,t \in \Gamma \} \right)$$

consist of finite width matrices only, which must have entries in $S$ whose norms are uniformly bounded. Therefore they form subsets of $UC^*_r(\Gamma,S)$. Since $UC^*_r(\Gamma,S)$ is norm closed and $\Theta_{k_\alpha}(x \to x$ for $x \in UC^*_r(\Gamma,B(H))$, the claim follows.

(3) $\Rightarrow$ (4): Let $0 \to J \to A \xrightarrow{p} B \to 0$ be any exact sequence of C*-algebras (without a $\Gamma$-action). $p$ induces a surjective $\ast$-homomorphism $\bar{\rho} : UC^*_r(\Gamma,A) \to UC^*_r(\Gamma,B)$, where $\ker \bar{\rho} = \{ x \in UC^*_r(\Gamma,A) \mid x_{s,t} \in J \text{ for all } s,t \in \Gamma \}$. Thinking of
A as being faithfully represented on a Hilbert space $H$, i.e. $A \subseteq B(H)$, we have $UC^*_r(\Gamma, J) \subseteq \ker \hat{\rho} \subseteq \{ x \in UC^*_r(\Gamma, B(H)) \mid x_{s,t} \in J \text{ for all } s, t \in \Gamma \} = UC^*_r(\Gamma, J)$ by assumption, hence $\ker \hat{\rho} = UC^*_r(\Gamma, J)$.

(4) $\Rightarrow$ (1): Let $0 \to (J, \alpha) \to (A, \alpha) \to (B, \hat{\alpha}) \to 0$ be an exact sequence of $\Gamma$-dynamical systems and $p_\alpha : A \rtimes_{\alpha,r} \Gamma \to B \rtimes_{\hat{\alpha},r} \Gamma$ the induced surjective $*$-homomorphism. Let $0 \to \sigma_0(\Lambda, A) \to \ell^\infty(\Lambda, A) \xrightarrow{\tilde{q}} q^\infty(\Lambda, A) \to 0$ be the exact sequence before Lemma 2.2. Then by assumption the sequence

$$0 \to UC^*_r\left(\Gamma, \sigma_0(\Lambda, A)\right) \to UC^*_r\left(\Gamma, \ell^\infty(\Lambda, A)\right) \xrightarrow{\tilde{q}} UC^*_r\left(\Gamma, q^\infty(\Lambda, A)\right) \to 0$$

is exact. Suppose that $x \in \ker p_\alpha$. Then by Lemma 2.2(3)$\Rightarrow$(5) we know that $\delta(x) \in \ker \tilde{q} = UC^*_r(\Gamma, \sigma_0(\Lambda, A))$ and by Lemma 2.2(2)$\Rightarrow$(1) it follows that $x \in J \rtimes_{\alpha,r} \Gamma$ and hence $\Gamma$ is an exact group. 

Now we get back to the OAP, respectively AP for $\Gamma$. Let us first note that a completely bounded multiplier $u \in M_0 A(\Gamma)$ defines a completely bounded map $\hat{M}_u \in CB(UC^*_r(\Gamma, S))$ for any operator space $S \subseteq B(H)$ by $M_u([x_{s,t}]) = [u(st^{-1})x_{s,t}]$. (Here $CB$ stands for completely bounded linear maps.) Indeed, as in the definition of Schur multipliers, let $K$ be a Hilbert space and $p, q : \Gamma \to K$ be bounded with $u(st^{-1}) = \langle p(s), q(t) \rangle$ for all $s, t \in \Gamma$. Define bounded linear maps $V, W : \ell^2(\Gamma) \otimes H \to \ell^2(\Gamma) \otimes H \otimes K$ by

$$V(e_s \otimes \xi) = e_s \otimes \xi \otimes p(s) \text{ and } W(e_s \otimes \xi) = e_s \otimes \xi \otimes q(s),$$

where $(e_s) \subseteq \ell^2(\Gamma)$ is the canonical orthonormal basis in $\ell^2(\Gamma)$ and $\xi \in H$. Then $V^* (x \otimes 1_K) W = \hat{M}_u(x)$ for all $x \in UC^*_r(\Gamma, S)$. This map is clearly completely bounded with cb-norm $\leq \|V\|\|W\|$. It follows that $\|\hat{M}_u\| = \|u\|_{M_0}$ and sup$_S \|\hat{M}_u\| = \|u\|_{M_0}$, where the supremum is taken over all operator spaces.

For convergence properties of $\hat{M}_u_n$ (as $\alpha \to \infty$), where $(u_\alpha)$ is a net in $A_\alpha(\Gamma)$, we can now use the same kind of arguments as in [HK] for tensor products which we sketch for convenience. By a Hahn-Banach type argument the point-norm closure and the point-weak closure of the subspace $\{ M_u \mid u \in A_\alpha(\Gamma) \} \subseteq B(UC^*_r(\Gamma, S))$ are identical. Thus the identity map is in the point-norm closure of this subspace iff there is a net $(u_\alpha)$ in $A_\alpha(\Gamma)$ such that $\varphi(\hat{M}_u_n(x)) \to \varphi(x)$ for all $\varphi \in UC^*_r(\Gamma, S)^*$ and $x \in UC^*_r(\Gamma, S)$. Now let us view $u \mapsto \varphi(\hat{M}_u(x))$ as a bounded linear functional $\omega_{\varphi,x} \in M_0 A(\Gamma)^*$. Then $u_\alpha \to 1$ in the $\sigma(M_0 A(\Gamma), Q(\Gamma))$-topology implies $\varphi(\hat{M}_u_n(x)) \to \varphi(x)$ for all $\varphi \in UC^*_r(\Gamma, S)^*$ and $x \in UC^*_r(\Gamma, S)$, provided we can show the following.

**Lemma 2.4.** $\omega_{\varphi,x}$ is in $Q(\Gamma)$ for all $\varphi \in UC^*_r(\Gamma, S)^*$ and $x \in UC^*_r(\Gamma, S)$.

**Proof.** Note first that $\|\hat{M}_u\| \leq \|u\|_{M_0}$ implies $\|\omega_{\varphi,x}\| \leq \|\varphi\||\|x\|$. Since $Q(\Gamma)$ is complete we may thus assume that $x$ has finite width. But then $\omega_{\varphi,x}(u)$ actually only depends on the value of the function $u : \Gamma \to \mathbb{C}$ on a finite subset of $\Gamma$ which clearly means that $u \mapsto \omega_{\varphi,x}(u)$ is in $Q(\Gamma)$ in this case. 

Thus the existence of $(u_\alpha)$ in $A_\alpha(\Gamma)$ with $u_\alpha \to 1$ in the $\sigma(M_0 A(\Gamma), Q(\Gamma))$-topology guarantees that we may choose $(u_\alpha)$ so that $\hat{M}_u_n(x) \to x$ for all $x \in UC^*_r(\Gamma, S)$. 


3. The invariant translation approximation property and the AP

Note that if \( S \subseteq B(H) \) is a concrete operator space, then \( \text{Ad}(\lambda_t \otimes \text{id}) \) and \( \text{Ad}(\rho_t \otimes \text{id}) \) both leave \( UC^*_r(\Gamma, S) \subseteq B(\ell^2(\Gamma) \otimes H) \) invariant. We say that a discrete group \( \Gamma \) has the operator invariant translation approximation property if for any closed subspace \( S \) of the compact operators \( \mathcal{K} \) (on \( \ell^2(\mathbb{N}) \)) the equality
\[
UC^*_r(\Gamma, S)^\Gamma = C^*_r(\Gamma) \otimes S
\]
is obtained, where \( UC^*_r(\Gamma, S)^\Gamma \) denotes the set of elements in \( UC^*_r(\Gamma, S) \) which are fixed under \( \text{Ad}(\rho_t \otimes \text{id}) \) for all \( t \in \Gamma \). If \( \Gamma \) is exact we may replace this condition by
\[
(UC^*_r(\Gamma) \otimes S)^\Gamma = C^*_r(\Gamma) \otimes S.
\]

**Lemma 3.1.** Suppose that \( \Gamma \) is exact and \( S \subseteq B(H) \) is an arbitrary closed subspace. Then \( UC^*_r(\Gamma, S)^\Gamma = (UC^*_r(\Gamma) \otimes S)^\Gamma \).

**Proof.** Let \((k_u)\) be as in Theorem 2.1(3). We show first that \( \Theta_{k_u}(UC^*_r(\Gamma, S)^\Gamma) \subseteq UC^*_r(\Gamma) \otimes S \), where \( \otimes \) denotes the algebraic tensor product. Given \( x \in UC^*_r(\Gamma, S)^\Gamma \) we have \( x_{s,t} = x_{st,1r} \in S \) for all \( s, t, r \in \Gamma \). So \( x_{s,t} \) only depends on the value of \( st^{-1} \). In particular \( x_{s,t} = x_{st^{-1},e} \) for all \( s, t \in \Gamma \). Let \( F \subseteq \Gamma \) be finite and \( x^F \) the element obtained from \( x \) by replacing \( x_{s,t} \) by 0 whenever \( st^{-1} \notin F \). Then \( x^F = \sum_{r \in F} \lambda_r \otimes x_{r,e} \) is in \( \mathbb{C}[\Gamma] \otimes S \), where \( \mathbb{C}[\Gamma] \) is the group ring. Moreover, \( \Theta_{k_u}(x) = \Theta_{k_u}(x^F) \) provided \( k_u \) has width 1. The completely bounded operator \( \Theta_{k_u}(x)^F = \sum_{\alpha \in F} \Theta_{k_u}(\lambda_{\alpha}) \otimes x_{\alpha,e} \in UC^*_r(\Gamma) \otimes S \), thus \( \Theta_{k_u}(UC^*_r(\Gamma, S)^\Gamma) \subseteq UC^*_r(\Gamma) \otimes S \) for all \( \alpha \). Since \( \Theta_{k_u}(x) \rightarrow x \) for \( x \in UC^*_r(\Gamma, S) \), it follows that \( UC^*_r(\Gamma, S)^\Gamma = UC^*_r(\Gamma) \otimes S \cap UC^*_r(\Gamma, S)^\Gamma = (UC^*_r(\Gamma) \otimes S)^\Gamma \). \( \Box \)

Finally we can prove our result.

**Theorem 3.2.** For a discrete exact group \( \Gamma \) the following conditions are equivalent:

1. \( \Gamma \) has the operator invariant translation approximation property.
2. \( UC^*_r(\Gamma, S)^\Gamma = (UC^*_r(\Gamma) \otimes S)^\Gamma = C^*_r(\Gamma) \otimes S \) for any closed subspace \( S \subseteq \mathcal{K} \).
3. \( UC^*_r(\Gamma, S)^\Gamma = (UC^*_r(\Gamma) \otimes S)^\Gamma = C^*_r(\Gamma) \otimes S \) for any Hilbert space \( H \) and any closed subspace \( S \subseteq B(H) \).
4. \( \Gamma \) has the AP.

**Proof.** (1) \( \Leftrightarrow \) (2): follows from Lemma 3.1.

(1) \( \Rightarrow \) (4): given \( \Gamma \) exact with the operator invariant translation approximation property, it suffices to show that \( C^*_r(\Gamma) \) has the slice map property for closed subspaces \( S \subseteq \mathcal{K} \). Thus given such an \( S \) we have to show that \( F(C^*_r(\Gamma, S)) = \{ x \in C^*_r(\Gamma) \otimes \mathcal{K} \mid (\varphi \otimes \text{id})(x) \in S \text{ for all } \varphi \in C^*_r(\Gamma)^* \} \) equals \( C^*_r(\Gamma) \otimes S \). Now \( C^*_r(\Gamma) \otimes \mathcal{K} \subseteq UC^*_r(\Gamma, \mathcal{K}) \), so we may view \( x \in F(C^*_r(\Gamma, S)) \) as a \( \Gamma \times \Gamma \)-matrix with entries in \( \mathcal{K} \). Since \( (\epsilon_{s,t} \otimes \epsilon_{t,1}) \otimes \text{id})(x) = x_{s,t} \), all its entries must be in \( S \) and if all entries \( x_{s,1} \) are in \( S \), then one checks that \( x \in F(C^*_r(\Gamma, S)) \). Since \( \Gamma \) is exact Theorem 2.3(3) now implies that \( F(C^*_r(\Gamma, S)) = C^*_r(\Gamma) \otimes \mathcal{K} \cap UC^*_r(\Gamma, S) \), and this equals \( UC^*_r(\Gamma, \mathcal{K})^\Gamma \cap UC^*_r(\Gamma, S) = UC^*_r(\Gamma, S)^\Gamma = C^*_r(\Gamma) \otimes S \) if \( \Gamma \) has the operator invariant translation approximation property.

(4) \( \Rightarrow \) (3): let \((u_n) \subseteq A_c(\Gamma) \) (all of finite support) such that \( \hat{M}_{u_n}(x) \rightarrow x \) for all \( x \in UC^*_r(\Gamma, B(H)) \). \( \hat{M}_{u_n} \) commutes with \( \text{Ad}\rho_t \) for all \( t \in \Gamma \), so for all closed subspaces \( S \subseteq B(H) \) the completely bounded operator \( \hat{M}_{u_n} \in CB(UC^*_r(\Gamma, S)) \) maps invariant elements into invariant elements. Moreover, \( \hat{M}_{u_n}(UC^*_r(\Gamma, S)^\Gamma) \subseteq \mathbb{C}[\Gamma] \otimes S \) by a similar argument as in the proof of Lemma 3.1. Since \( \hat{M}_{u_n}(x) \rightarrow x \)
whenever $x \in UC^*_r(\Gamma, S)$, it follows that $UC^*_r(\Gamma, S)^\Gamma \subseteq C^*_r(\Gamma) \otimes S$, i.e. $UC^*_r(\Gamma, S)^\Gamma = C^*_r(\Gamma) \otimes S$.

(3) $\Rightarrow$ (1): obvious. $\square$

**Remark.** Is it true that if $\Gamma$ is a discrete group, then $(UC^*_r(\Gamma) \otimes S)^\Gamma = UC^*_r(\Gamma^\Gamma) \otimes S$ for all closed subspaces $S \subseteq K$? If so, then Roe’s invariant translation approximation property is equivalent to the OAP at least if $\Gamma$ is exact. We leave this question open.

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