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L^2-SUMMAND VECTORS IN BANACH SPACES

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Abstract. The aim of this paper is to study the set L^2_X of all L^2-summand vectors of a real Banach space X. We provide a characterization of L^2-summand vectors in smooth real Banach spaces and a general decomposition theorem which shows that every real Banach space can be decomposed as an L^2-sum of a Hilbert space and a Banach space without nontrivial L^2-summand vectors. As a consequence, we generalize some results and we obtain intrinsic characterizations of real Hilbert spaces.

1. Background

We say that a Banach space X is smooth if every point x in X satisfies that there exists a unique f in X^* such that $\|f\| = \|x\|$ and $f(x) = \|x\|^2$. If X is a smooth Banach space and x is in X, then we will denote this f by $\mathsf{J}_X(x)$. The mapping $\mathsf{J}_X : X \longrightarrow X^*$ is usually called the duality mapping. Excellent books for consulting the duality mapping are [7] and [8].

A closed subspace M of a real Banach space X is said to be an L^2-summand subspace if there exists another closed subspace N of X verifying $X = (M \oplus N)_2$; in other words, $\|m+n\|^2 = \|m\|^2 + \|n\|^2$ for every m in M and every n in N. The linear projection π_M of X onto M that fixes the elements of M and maps the elements of N to $\{0\}$ is called the L^2-summand projection of X onto M. Notice that N is uniquely determined, and hence so is π_M. Good references for L^2-summand subspaces are [2] and [3].

A vector e of a real Banach space X is an L^2-summand vector if $\mathbb{R}e$ is an L^2-summand subspace. Furthermore, if $e \neq 0$, then there exists a functional e^* in X^*, which is called the L^2-summand functional of e, such that $\|e^*\| = 1/\|e\|$, $e^*(e) = 1$ and $\pi_{\mathbb{R}e}(x) = e^*(x)\, e$ for every x in X.

We want to recall two relevant results about L^2-summand vectors:

(1) In [6], Carlson and Hicks proved that a real Banach space is a Hilbert space if and only if all elements of its unit sphere are L^2-summand vectors.
(2) In [5], Becerra Guerrero and Rodríguez Palacios proved that a real Banach space is a Hilbert space if and only if the subset of its unit sphere whose points are L^2-summand vectors is not rare in the unit sphere.

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2. Geometry of L^2-summand vectors

For a given point in the unit sphere of a real Banach space, we have two ways of studying its geometrical properties: the first one is to find out its rotundity properties (for instance, to be an extreme point); the second way is to find out its smoothness properties (for example, to be a smooth point). The next proposition shows that every L^2-summand vector of norm 1 is a locally uniformly rotund point of the unit ball, that means if $(x_n)_{n\in\mathbb{N}}$ is a sequence in S_X such that $(\|(x_n+e)/2\|)_{n\in\mathbb{N}}$ converges to 1, then $(x_n)_{n\in\mathbb{N}}$ converges to e (where e is the L^2-summand vector). Speaking in rotundity terms, to be a locally uniformly rotund point is the strongest thing that a vector of norm 1 can be. The paper [4] is an excellent reference for these kinds of properties. We prefer to omit the proof, which can be considered as an exercise.

Proposition 2.1. *Let X be a real Banach space. Let e be an L^2-summand vector in S_X. Then, e is a locally uniformly rotund point of B_X.*

The point now would be to study the smoothness properties of L^2-summand vectors. Remember that a point x of the unit sphere of a real Banach space X is said to be a strongly smooth point of the unit ball if the following condition holds: if $(f_n)_{n\in\mathbb{N}}$ is a sequence in S_{X^*} such that $(f_n(e))_{n\in\mathbb{N}}$ converges to 1, then $(f_n)_{n\in\mathbb{N}}$ is convergent. Strongly smooth points can be characterized in terms of differentiability of the norm; that is, these points are exactly the points at which the norm is Fréchet differentiable. As before, speaking in smoothness terms, to be a strongly smooth point is the strongest thing that a vector of norm 1 can be.

According to [2, Lemma 1.4, page 8], the L^2-summand functional associated to an L^2-summand vector is always an L^2-summand vector of the dual, and therefore, a locally uniformly rotund point. Keeping in mind the duality relations between smoothness and rotundity, which can be consulted in [8], the following proposition does not need any proof.

Proposition 2.2. *Let X be a real Banach space. If e is an L^2-summand vector of norm 1, then e is a strongly smooth point of B_X.*

Notice that the L^2-summand functional associated to an L^2-summand vector is a norm-attaining functional which attains its norm at a strongly smooth point, so it is also a strongly ω^*-exposed point of the unit ball of the dual.

The following (and last) result in this section is a natural characterization of L^2-summand vectors which is used later in Theorem 4.1.

Proposition 2.3. *Let X be a real Banach space. Let e be in X. The following assertions are equivalent:*

(1) *e is an L^2-summand vector.*
(2) $\operatorname{span}(\{e,x\})$ *is a Hilbert space for every x in X.*

Proof. Taking into account that the L^2-sum of two Hilbert spaces is always a Hilbert space, it is obvious that if e is an L^2-summand vector, then $\operatorname{span}(\{e,x\})$ is a Hilbert space for every x in X.

Let us show the converse. We can suppose that $\|e\|=1$. First, we will prove that e is a smooth point of B_X. Let f and g in S_{X^*} with $f(e)=g(e)=1$. Let x be in X. Since $Y=\operatorname{span}(\{e,x\})$ is a Hilbert space, we deduce that e is a smooth point of B_Y; therefore $f|_Y=g|_Y$. Since x is arbitrary, we deduce that $f=g$.

Now, show that $X = (\mathbb{R}e \oplus M)_2$ where $M = \ker(f)$ with f in S_{X^*} and $f(e) = 1$. Let m and δ be in M and $\mathbb{R}$, respectively. Since $Z = \operatorname{span}(\{e, m\})$ is a Hilbert space, we have that e is an L^2-summand vector of Z; therefore there exists a 1-codimensional subspace N of Z such that $Z = (\mathbb{R}e \oplus N)_2$. On the other hand, we know that $N = \ker(f|_Z)$; therefore $\|m + \delta e\|^2 = \|m\|^2 + \|\delta e\|^2$, because $m \in N$ and $Z = (\mathbb{R}e \oplus N)_2$. □

To see a complete characterization of L^2-summand vectors involving concepts related to isometric reflections, the paper [1] can be consulted.

3. The particular case of smooth spaces

In this section, we center our attention on smooth Banach spaces getting results about the structure of the set of all L^2-summand vectors. In the next section, we generalize these results removing the smoothness hypothesis.

Lemma 3.1. *Let X be a real smooth Banach space. If e is an L^2-summand vector, then*

$$\mathbb{R}e = \bigcap \{\ker(\mathsf{J}_X(m)) : m \in \ker(\mathsf{J}_X(e))\}.$$

Proof. The mapping $\psi : X \longrightarrow X$, defined by $\psi(m + \delta e) = m - \delta e$ for m in $\ker(\mathsf{J}_X(e))$ and δ in $\mathbb{R}$, is a surjective linear isometry. Then, if $m \in \ker(\mathsf{J}_X(e))$,

$$\mathsf{J}_X(m)(e) = -\mathsf{J}_X(m)(\psi(e)) = -(\mathsf{J}_X(m) \circ \psi)(e) = -\mathsf{J}_X(m)(e).$$

Therefore, $\mathsf{J}_X(m)(e) = 0$.

Let $y \in \bigcap \{\ker(\mathsf{J}_X(m)) : m \in \ker(\mathsf{J}_X(e))\}$ and write $y = n + \lambda e$ with $n \in \ker(\mathsf{J}_X(e))$ and $\lambda \in \mathbb{R}$. Then

$$0 = \mathsf{J}_X(n)(y) = \mathsf{J}_X(n)(n + \lambda e) = \|n\|^2.$$

Therefore, $n = 0$ and $y = \lambda e$. □

Theorem 3.2. *Let X be a real smooth Banach space. Let e be in X. The following assertions are equivalent:*

(1) *e is an L^2-summand vector.*
(2) *$\mathsf{J}_X(x + e) = \mathsf{J}_X(x) + \mathsf{J}_X(e)$ for every x in X.*
(3) *$\mathsf{J}_X(e)(x) = \mathsf{J}_X(x)(e)$ for every x in X.*

Proof. Assume that (1) holds. In order to show (2), it suffices to prove that $\mathsf{J}_X(x) = \mathsf{J}_X(m) + \delta \mathsf{J}_X(e)$ for every $x = m + \delta e \in X$ (where $m \in \ker(\mathsf{J}_X(e))$ and $\delta \in \mathbb{R}$). Pick $n \in \ker(\mathsf{J}_X(e))$ and $\lambda \in \mathbb{R}$. By Hölder's inequality,

$$\begin{aligned} |(\mathsf{J}_X(m) + \delta \mathsf{J}_X(e))(n + \lambda e)| &\leq |\mathsf{J}_X(m)(n)| + |\delta\lambda \mathsf{J}_X(e)(e)| \\ &\leq \|m\|\,\|n\| + \|\delta e\|\,\|\lambda e\| \\ &\leq \sqrt{\|m\|^2 + \|\delta e\|^2}\sqrt{\|n\|^2 + \|\lambda e\|^2} \\ &= \|\mathsf{J}_X(x)\|\,\|n + \lambda e\|. \end{aligned}$$

Therefore, $\|\mathsf{J}_X(m) + \delta \mathsf{J}_X(e)\| \leq \|\mathsf{J}_X(x)\|$. On the other hand,

$$(\mathsf{J}_X(m) + \delta \mathsf{J}_X(e))(m + \delta e) = \|m\|^2 + \|\delta e\|^2 = \|x\|^2.$$

Therefore, $\mathsf{J}_X(x) = \mathsf{J}_X(m) + \delta \mathsf{J}_X(e)$.

Assume that (2) holds. Let $x = m + \delta e \in X$. Then

$$\mathsf{J}_X(e)(x) = \mathsf{J}_X(e)(m + \delta e) = \delta \|e\|^2.$$

Assuming $\delta \neq 0$, we have that

$$\mathsf{J}_X(x)(e) = \mathsf{J}_X(m+\delta e)(e) = \mathsf{J}_X(m)(e) + \delta \mathsf{J}_X(e)(e) = \delta \|e\|^2 .$$

Assume that (3) holds. Fix elements $m \in \ker(\mathsf{J}_X(e)) \setminus \{0\}$ and $\delta \in \mathbb{R}$. Let us check that

$$\|m+\delta e\|^2 = \|\delta e\|^2 + \|m\|^2 .$$

Let us consider the mapping

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & \gamma & \longmapsto & f(\gamma) = \|m+\gamma e\|. \end{array}$$

The function f is differentiable, and, for all $\gamma \in \mathbb{R}$, the derivative of f at γ is

$$f'(\gamma) = \mathsf{J}_X\left(\frac{m+\gamma e}{\|m+\gamma e\|}\right)(e) .$$

Assuming, without loss of generality, that $\|e\| = 1$, we get

$$f'(\gamma) = \mathsf{J}_X\left(\frac{m+\gamma e}{\|m+\gamma e\|}\right)(e) = \mathsf{J}_X(e)\left(\frac{m+\gamma e}{\|m+\gamma e\|}\right) = \frac{\gamma}{\|m+\gamma e\|} = \frac{\gamma}{f(\gamma)}.$$

By integration, $f(\gamma)^2 = \gamma^2 + f(0)^2$ for all $\gamma \in \mathbb{R}$, and taking $\gamma = \delta$ we obtain $\|m+\delta e\|^2 = \|\delta e\|^2 + \|m\|^2$. □

Notice that this last theorem indicates that, in the case of smoothness, the set of all L^2-summand vectors is a closed subspace. This raises the question if it is possible to get that in general Banach spaces. The answer to this is affirmative.

4. L^2-summand vectors and Hilbert spaces

Let us begin this section doing some considerations about L^2-summand subspaces. Let X be a real Banach space. Let us consider a closed subspace Y of X and a closed subspace M of Y. It can be easily checked that:

(1) If M is an L^2-summand subspace of X, then M is an L^2-summand subspace of Y.
(2) If M is an L^2-summand subspace of Y and Y is an L^2-summand subspace of X, then M is an L^2-summand subspace of X.

On the other hand, it is also easy to check that if M and N are L^2-summand subspaces of X, then M is contained in N if and only if $\ker(\pi_N)$ is contained in $\ker(\pi_M)$.

Theorem 4.1. *Let X be a real Banach space. Let u be an L^2-summand vector. Then:*

(1) *If F is an L^2-summand subspace, then* $\operatorname{span}(F \cup \{u\})$ *is an L^2-summand subspace.*
(2) *If v is another L^2-summand vector, then $u+v$ is an L^2-summand vector.*

Proof. First of all, (2) is a consequence of (1) together with the fact that $\operatorname{span}(\{u,v\})$ is a Hilbert space. So, let us do the proof of (1). Firstly, let us show that

$$\operatorname{span}(F \cup \{u\}) \cap (\ker(\pi_F) \cap \ker(\pi_{\mathbb{R}u})) = \{0\} .$$

Let z be in $\operatorname{span}(F \cup \{u\}) \cap (\ker(\pi_F) \cap \ker(\pi_{\mathbb{R}u}))$. We can write $z = y + \alpha u$ with y in F and α in $\mathbb{R}$. Then, $\alpha u = z - y$ with z in $\ker(\pi_F)$ and y in F; therefore

$\|\alpha u\|^2 = \|z\|^2 + \|y\|^2$. On the other hand, $y = z - \alpha u$ with z in $\ker(\pi_{\mathbb{R}u})$; therefore $\|y\|^2 = \|z\|^2 + \|\alpha u\|^2$. Then, $\|\alpha u\|^2 = 2\|z\|^2 + \|\alpha u\|^2$; therefore, $z = 0$.

Next, let us show that

$$\operatorname{span}(F \cup \{u\}) + (\ker(\pi_F) \cap \ker(\pi_{\mathbb{R}u})) = X.$$

We can assume that u is not in F, which means that $\ker(\pi_F)$ is not contained in $\ker(\pi_{\mathbb{R}u})$. Since

$$\mathbb{R}(u - \pi_F(u)) + F = \operatorname{span}(F \cup \{u\})$$

it suffices to check that

$$(\ker(\pi_F) \cap \ker(\pi_{\mathbb{R}u})) + \mathbb{R}(u - \pi_F(u)) = \ker(\pi_F).$$

To get this, the only thing we have to prove is that $u - \pi_F(u)$ is not in $\ker(\pi_{\mathbb{R}u})$, because $\ker(\pi_F) \cap \ker(\pi_{\mathbb{R}u})$ is a 1-codimensional subspace of $\ker(\pi_F)$. Suppose that $u - \pi_F(u)$ is in $\ker(\pi_{\mathbb{R}u})$. Then, $\|u\|^2 = \|u - \pi_F(u)\|^2 + \|\pi_F(u)\|^2$ and $\|\pi_F(u)\|^2 = \|\pi_F(u) - u\|^2 + \|u\|^2$; therefore $u = \pi_F(u) \in F$, which is a contradiction.

Lastly, let us show that

$$(\operatorname{span}(F \cup \{u\}) \oplus (\ker(\pi_F) \cap \ker(\pi_{\mathbb{R}u})))_2 = X.$$

Let p and q be in $\operatorname{span}(F \cup \{u\})$ and $\ker(\pi_F) \cap \ker(\pi_{\mathbb{R}u})$, respectively. We can write $p = m + \lambda u$ with m in F and λ in $\mathbb{R}$, and $m = h + \delta u$ with h in $\ker(\pi_{\mathbb{R}u})$ and δ in $\mathbb{R}$. Then

$$\begin{aligned}
\|h\|^2 + \|\delta u\|^2 + \|q\|^2 &= \|m\|^2 + \|q\|^2 \\
&= \|m + q\|^2 \\
&= \|(h + q) + \delta u\|^2 \\
&= \|h + q\|^2 + \|\delta u\|^2 .
\end{aligned}$$

Therefore, $\|h + q\|^2 = \|h\|^2 + \|q\|^2$. Finally

$$\begin{aligned}
\|p + q\|^2 &= \|(h + q) + (\lambda + \delta) u\|^2 \\
&= \|h + q\|^2 + \|(\lambda + \delta) u\|^2 \\
&= \|h\|^2 + \|q\|^2 + \|(\lambda + \delta) u\|^2 \\
&= \|h\|^2 + \|(\lambda + \delta) u\|^2 + \|q\|^2 \\
&= \|h + (\lambda + \delta) u\|^2 + \|q\|^2 \\
&= \|m + \lambda u\|^2 + \|q\|^2 \\
&= \|p\|^2 + \|q\|^2
\end{aligned}$$

and the proof is done. □

Theorem 4.2. *Let X be a real Banach space. Then, L^2_X is a Hilbert subspace of X.*

Proof. The only thing we have to prove is that L^2_X is closed. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in L^2_X which is convergent to some element u in X. We can suppose, for every n in $\mathbb{N}$, that $\|u_n\| = \|u\| = 1$. There exists a subnet $(u^*_\alpha)_{\alpha\in I}$ of $(u^*_n)_{n\in\mathbb{N}}$ which is ω^*-convergent to some element f in B_{X^*}. Since $\|x\|^2 = \|u^*_\alpha(x) u_\alpha\|^2 + \|x - u^*_\alpha(x) u_\alpha\|^2$ for every x in X and for every α in I, we deduce that $\|x\|^2 = \|f(x) u\|^2 + \|x - f(x) u\|^2$ for every x in X. Therefore, $\mathbb{R}u$ is an L^2-summand subspace of X. Notice that $f = u^*$. □

At this point, we know that, in the category of all real Banach spaces, the set of all L^2-summand vectors is a Hilbert subspace. But, we can ask another question: is it an L^2-summand subspace? The answer is yes as we will show in the next theorem.

Theorem 4.3. *Let X be a real Banach space. There exist two closed subspaces H and E of X such that:*

(1) *H is a Hilbert space.*
(2) *If e is not in H, then e is not an L^2-summand vector of X.*
(3) *If e is in $E\setminus\{0\}$, then e is not an L^2-summand vector of E.*
(4) *$(H\oplus E)_2 = X$.*

Proof. We will take $H=\mathsf{L}^2_X$. Let us consider the set

$$\mathcal{L}=\left\{M\in\mathcal{P}(X) : M\subseteq H \text{ and is an } \mathsf{L}^2\text{-summand subspace of } X\right\}$$

with the natural order given by the inclusion. Suppose that $\mathcal{L}$ has a maximal element M. Every h in H satisfies that $\operatorname{span}(M\cup\{h\})$ is an element of $\mathcal{L}$ and contains M, so h is in M. Therefore, we deduce that $M=H$, and the only thing we have to prove is that $\mathcal{L}$ has a maximal element.

Let $(M_\alpha)_{\alpha\in I}$ be a chain of $\mathcal{L}$. Let us consider

$$M=\bigcup_{\alpha\in I} M_\alpha \quad\text{and}\quad L=\bigcap_{\alpha\in I}\ker(\pi_{M_\alpha}).$$

If we show that $X=(\operatorname{cl}(M)\oplus L)_2$, then we will have proved that $\operatorname{cl}(M)\in\mathcal{L}$, and thus $\operatorname{cl}(M)$ is an upper bound for the chain $(M_\alpha)_{\alpha\in I}$. In that case, Zorn's lemma allows us to deduce the existence of maximal elements in $\mathcal{L}$.

Taking limits, it is pretty easy to show that $\operatorname{cl}(M)\cap L=\{0\}$. Let us show that $\operatorname{cl}(M)+L=X$. Let $x\in X$ with $\|x\|=1$. For every $\alpha\in I$, there exist $x_\alpha\in M_\alpha$ and $z_\alpha\in\ker(\pi_{M_\alpha})$ such that $x=x_\alpha+z_\alpha$. Since $\operatorname{cl}(M)$ is a Hilbert space and $(x_\alpha)_{\alpha\in I}$ is a net in the unit ball of $\operatorname{cl}(M)$, we deduce that there exists a subnet $(x_\beta)_{\beta\in J}$ of $(x_\alpha)_{\alpha\in I}$ which is ω-convergent to some $x_0\in\operatorname{cl}(M)$. If we fix $\gamma\in I$, then for every $\beta\geq\gamma$, $x-x_\beta=z_\beta\in\ker\left(\pi_{M_\beta}\right)\subseteq\ker\left(\pi_{M_\gamma}\right)$. Therefore, $x-x_0\in\ker\left(\pi_{M_\gamma}\right)$, and, since γ is arbitrary, we deduce that $x-x_0\in\bigcap\{\ker(\pi_{M_\alpha}) : \alpha\in I\}=L$. As a consequence, $x\in\operatorname{cl}(M)+L$. Finally, taking limits again, it can be proved without difficulty that $X=(\operatorname{cl}(M)\oplus L)_2$. □

As a corollary, we obtain a characterization of real Hilbert spaces, which does not need any proof if we take into account the previous theorem.

Corollary 4.4. *Let X be a real Banach space. Then:*

(1) *If E is a closed subspace of L^2_X, then E is an L^2-summand subspace of X and $\ker(\pi_E)=\bigcap\{\ker(\pi_{\mathbb{R}e}) : e\in E\}$.*
(2) *A necessary and sufficient condition for X to be a Hilbert space is that $\bigcap\{\ker(\pi_{\mathbb{R}u}) : u\in\mathsf{L}^2_X\}=\{0\}$.*
(3) *L^2_X can never be a 1-codimensional subspace of X.*

Finally, we provide two examples recalling the decomposition of a real Banach space given by the set of its L^2-summand vectors:

(1) For every real Hilbert space H, we can construct a real Banach space X such that $\mathsf{L}^2_X=H$ and $\ker\left(\pi_{\mathsf{L}^2_X}\right)$ is an infinite-dimensional transitive Banach space. Indeed, it is known that there exists a transitive real Banach space Y that it is not a Hilbert space (a good reference for Banach spaces with

transitive norm is [5]). Then, $\mathsf{L}^2_Y = \{0\}$. As a consequence, $X = (H \oplus Y)_2$ is the required space.

(2) The space c_0 does not contain any non-trivial L^2-summand vector according to Proposition 2.1, because its unit sphere is free of extreme points.

(3) The space ℓ_p^n, with $p \in [1, \infty] \setminus \{2\}$ and $n \geq 2$, does not contain any non-trivial L^2-summand vector according to Theorem 3.2 when $1 < p < \infty$ and Proposition 2.1 when $p \in \{1, \infty\}$. We refer to [2, Theorem 1.3, page 8] or [3] for a wider perspective of these examples.

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