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ALL n-COTILTING MODULES ARE PURE-INJECTIVE

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Abstract. We prove that all n-cotilting R-modules are pure-injective for any ring R and any $n \geq 0$. To achieve this, we prove that ${}^{\perp_1}U$ is a covering class whenever U is an R-module such that ${}^{\perp_1}U$ is closed under products and pure submodules.

1. Introduction

Tilting theory has been developed as an important tool in the representation theory of algebras. In that context, tilting modules are usually assumed to be finite dimensional. However, some of the results have recently been extended to general modules over arbitrary associative unital rings, with interesting applications to finitistic dimension conjectures (see [2] and [15]). In contrast to the finite dimensional case, cotilting modules form a larger class in general than duals of tilting modules, [6].

So a natural question arises whether each cotilting module is at least pure-injective, that is, a direct summand in a dual module (where duals are considered in the sense of modules of characters for general rings, or vector space duals for algebras over a field). An affirmative answer has important consequences: for example, each cotilting class is then a covering class, [9].

The pure-injectivity of all 1-cotilting modules was first proved in the particular setting of abelian groups, and modules over Dedekind domains, as a consequence of their classification by Eklof, Göbel and Trlifaj in [10] and [9].

The crucial step towards a general solution was the proof of pure-injectivity of all 1-cotilting modules over any ring by Bazzoni, [4]. In [5], she was able to prove pure-injectivity of all n-cotilting modules, $n \geq 0$, modulo one of the following conjectures where (B) is weaker than (A):

(A) If U is an R-module such that ${}^{\perp_1}U$ is closed under products and pure submodules, then ${}^{\perp_1}U$ is closed under direct limits.

(B) If U is an R-module such that ${}^{\perp_1}U$ is closed under products and pure submodules, then ${}^{\perp_1}U$ is a special precovering class.

Recently, Conjecture (A) has been proved for countable rings and for divisible modules U over Prüfer domains by Bazzoni, Göbel and Strüngmann in [7]. A stronger version of Conjecture (B) was proved for any ring, but under the additional

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set theoretic assumption of Gödel's axiom of constructibility, by Šaroch and Trlifaj in [14].

In the present paper, we prove Conjecture (A) in ZFC, thus proving that all n-cotilting modules over any ring are pure-injective.

2. Preliminaries

Let R be a unital associative ring. All the modules will be left R-modules. For a class of modules $\mathcal{C}$ and $i \geq 1$, denote by ${}^{\perp_i}\mathcal{C}$ the class of all modules X such that $\mathrm{Ext}^i_R(X, C) = 0$ for all $C \in \mathcal{C}$. Dually, $\mathcal{C}^{\perp_i} = \{X \mid \mathrm{Ext}^i_R(C, X) = 0 \text{ for all } C \in \mathcal{C}\}$. We will write ${}^{\perp_i}U$ instead of ${}^{\perp_i}\{U\}$ for a single module U. Note that then ${}^{\perp_{i+1}}U = {}^{\perp_i}U'$ where U' is a cosyzygy of U.

A (non-strictly) increasing chain of sets $(S_\alpha \mid \alpha < \lambda)$ indexed by ordinals less than λ is called *smooth* if $S_\mu = \bigcup_{\alpha<\mu} S_\alpha$ for all limit ordinals $\mu < \lambda$. A smooth chain $(M_\alpha \mid \alpha < \lambda)$ of submodules of a module M is called a *filtration* of M if $M_0 = 0$ and $M = \bigcup_{\alpha<\lambda} M_\alpha$.

The following lemma is well known (see e.g. [8, Proposition XII.1.14]):

Lemma 1. *Let M, U be modules such that M has a filtration $(M_\alpha \mid \alpha < \lambda)$ with $M_{\alpha+1}/M_\alpha \in {}^{\perp_1}U$ for all $\alpha < \lambda$. Then $M \in {}^{\perp_1}U$.*

Let $\mathcal{C}$ be a class of modules. Then a homomorphism $f : C \to M$ is called a *special $\mathcal{C}$-precover* of M if f is epic and $\mathrm{Ker}\, f \in \mathcal{C}^{\perp_1}$. The class $\mathcal{C}$ is called *special precovering* if every module has a special $\mathcal{C}$-precover. The term comes from the fact that whenever $f : C \to M$ is a special precover and $g : C' \to M$ is any homomorphism such that $C' \in \mathcal{C}$, then g factorizes through f. Therefore, special precovers are indeed special instances of precovers as defined for example in [16]. A special $\mathcal{C}$-precover f is called a *$\mathcal{C}$-cover* if in addition $g : C \to C$ is an automorphism whenever $fg = f$. A *covering class* is defined in an obvious way.

A module U is called *n-cotilting*, where $n \geq 0$ is a natural number, if:

(1) $\mathrm{inj.dim}\, U \leq n$,
(2) $\mathrm{Ext}^i_R(U^\kappa, U) = 0$ for all $i \geq 1$ and all cardinals κ,
(3) there is an injective cogenerator W and an exact sequence $0 \to U_m \to \cdots \to U_1 \to U_0 \to W \to 0$ such that all U_j's are direct summands of some products of copies of U for all $0 \leq j \leq m$.

A class $\mathcal{A}$ is *n-cotilting* if $\mathcal{A} = \bigcap_{i\geq 1} {}^{\perp_i}U$ for some n-cotilting module U. In addition, we have adopted the following notation: Let M be a module. Then $PE(M)$ denotes the pure-injective hull of M.

Let $(M_\alpha \mid \alpha < \lambda)$ be a family of modules indexed by ordinal numbers less than λ. Then $\prod^b_{\alpha<\lambda} M_\alpha$ denotes the (pure) submodule of the direct product formed by the elements with a bounded support in λ. When $M_\alpha \cong M$ for all $\alpha < \lambda$, the corresponding "bounded power" is denoted by $M^{<\lambda}$.

Let M be a module, I a set, and let κ be a cardinal number. Then the submodule of M^I consisting of the elements with supports of cardinality $< \kappa$ is denoted $M^{[I;\kappa]}$.

3. Special embeddings into pure-injective modules

The aim of this section is to embed a module into a pure-injective module in such a way that we know more about the structure of the cokernel.

Lemma 2. *Let R be a ring and M a module. Then there is an increasing (non-smooth) chain of modules M_λ indexed by ordinal numbers, and homomorphisms $S_\lambda : \prod_{\alpha<\lambda} M_\alpha \to M_\lambda$, such that*

(a) *$M_0 = M$, $M_{\lambda+1} = M_\lambda$ for each ordinal λ,*
(b) *$M_\lambda / \bigcup_{\alpha<\lambda} M_\alpha \cong \prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha$ for each limit ordinal λ,*
(c) *the embeddings $M_\mu \subseteq M_\lambda$ are pure for each $\mu < \lambda$,*
(d) *the restrictions $S_\lambda \restriction M_\alpha : M_\alpha \to M_\lambda$ to any direct summand of the product $\prod_{\alpha<\lambda} M_\alpha$ are just the inclusions $M_\alpha \subseteq M_\lambda$ (that is, each S_λ extends the summation map $\bigoplus_{\alpha<\lambda} M_\alpha \to M_\lambda$),*
(e) *$S_\lambda \restriction \prod_{\alpha<\mu} M_\alpha = S_\mu$ for each $\mu < \lambda$.*

Proof. We will construct the modules M_λ by induction. By (a), $M_0 = M$ and $S_0 = 0$. If $\lambda = \mu + 1$, then $M_\lambda = M_\mu$ and $S_\lambda : (\prod_{\alpha<\mu} M_\alpha) \oplus M_\mu \to M_\lambda$ is just the coproduct homomorphism of S_μ and $id : M_\mu \to M_\lambda$.

Now, let λ be a limit ordinal. By induction hypothesis, $\tilde{S} = \bigcup_{\alpha<\lambda} S_\alpha$ is a well-defined homomorphism $\prod^b_{\alpha<\lambda} M_\alpha \to \bigcup_{\mu<\lambda} M_\mu$. Let us define M_λ and S_λ by the following push-out:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod^b_{\alpha<\lambda} M_\alpha & \xrightarrow{\subseteq} & \prod_{\alpha<\lambda} M_\alpha & \longrightarrow & \prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \tilde{S}} & & \Big\downarrow{\scriptstyle S_\lambda} & & \Big\| & & \\
0 & \longrightarrow & \bigcup_{\mu<\lambda} M_\mu & \xrightarrow{\subseteq} & M_\lambda & \longrightarrow & \prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha & \longrightarrow & 0
\end{array}$$

Then (b), (d), (e) are obvious. Moreover, $\bigcup_{\mu<\lambda} M_\mu$ is a pure submodule of M_λ, since the upper left horizontal map is a pure inclusion, thus (c) follows. $\square$

Lemma 3. *Let R be a ring and M a module. Let λ be an ordinal, let*

$$\sum_{j\in J} a_{ij} x_j = y_i, \quad y_i \in M, \quad i < \lambda, \tag{1}$$

be a system of equations in any (finite or infinite) number of unknowns x_j, $j \in J$, and let M_λ be the module corresponding to M and λ from the previous lemma. If (1) *is finitely satisfied in M, then it is satisfied in M_λ.*

Proof. Suppose that (1) is finitely satisfied. We will construct by induction partial solutions $x_j^\mu \in M_\mu$, $j \in J$, of the first μ equations such that

$$x_j^\mu = S_\mu\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu}\big). \tag{2}$$

We will set $x_j^0 = 0$ for each $j \in J$ by definition. If μ is non-zero finite, there is a solution of the first μ equations by the assumption and (2) is trivially satisfied. Since for any μ infinite: $\operatorname{card}(\mu) = \operatorname{card}(\mu + 1)$, we can find a solution of the first $\mu + 1$ equations just by renumbering the equations and using the induction hypothesis. Then

$$\begin{aligned} x_j^{\mu+1} &= (x_j^{\mu+1} - x_j^\mu) + x_j^\mu = (x_j^{\mu+1} - x_j^\mu) + S_\mu\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu}\big) \\ &= S_{\mu+1}\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu+1}\big). \end{aligned}$$

Now let μ be a limit ordinal. We will consider (2) as a definition of x_j^μ. Then for arbitrary $i < \mu$:

$$\begin{aligned}\sum_{j\in J} a_{ij} x_j^\mu &= S_\mu\big(\sum_{j\in J} a_{ij}\,(x_j^{\alpha+1} - x_j^\alpha)_{\alpha<\mu}\big) = S_{i+1}\big(\sum_{j\in J} a_{ij}\,(x_j^{\alpha+1} - x_j^\alpha)_{\alpha<i+1}\big)\\ &= \sum_{j\in J} a_{ij} S_{i+1}\big((x_j^{\alpha+1} - x_j^\alpha)_{\alpha<i+1}\big) = \sum_{j\in J} a_{ij} x_j^{i+1} = y_i.\end{aligned}$$

Thus, x_j^μ is a solution of the first μ equations, and subsequently x_j^λ, $j \in J$, is a solution of the whole system. □

Corollary 4. *Let R be a ring, M a module and $\kappa = \max\{\aleph_0, \operatorname{card}(R)\}$. Let $(N_\alpha \mid \alpha \le \kappa^+)$ be a smooth chain of modules defined via: $N_0 = 0$, $N_1 = M$, $N_{\alpha+1}$ is the κ-th member of the chain from Lemma 2 when starting with the module N_α. Then N_{κ^+} is pure-injective.*

Proof. It is sufficient to prove that every system of linear equations

$$\sum_{j\in J} a_{ij} x_j = y_i, \quad y_i \in N_{\kappa^+}, \quad i < \kappa,$$

in unknowns x_j, $j \in J$, which is finitely satisfied in N_{κ^+} is satisfied in N_{κ^+} [8, V.1.2]. But all the y_i's are actually included in N_μ for some $\mu < \kappa^+$, thus the system is satisfied in $N_{\mu+1}$ by the preceding lemma. □

4. Cotilting modules

First, we need the following two set-theoretic lemmas that hold in ZFC. The first one was proven in [11] for the special case $\kappa = \aleph_0$. The second one is a straightforward generalization of [4, 2.3].

Lemma 5. *Let κ be an infinite regular cardinal. Then for every cardinal μ there is a cardinal $\lambda \ge \mu$ such that $\lambda^\kappa = 2^\lambda$ and $\lambda^\alpha = \lambda$ for each $\alpha < \kappa$.*

Proof. Let κ, μ be as above, and let λ be the union of the smooth chain $(\mu_i \mid i < \kappa)$ defined by $\mu_0 = \mu$ and $\mu_{i+1} = 2^{\mu_i}$. Then clearly λ is of cofinality κ and $\nu < \lambda$ implies $2^\nu < \lambda$. The power set $\mathcal{P}(\lambda)$ embeds in an obvious way in $\prod_{i<\kappa} \mathcal{P}(\mu_i)$, hence $2^\lambda \le \lambda^\kappa$. If $\alpha < \kappa$, then the range of any map $\alpha \to \lambda$ is actually contained in some μ_i, thus $\lambda^\alpha = \operatorname{card}(\bigcup_{i<\kappa} \mu_i^\alpha) \le \lambda$. □

Lemma 6. *Let λ, κ be cardinals such that $\lambda^\kappa = 2^\lambda$ and $\lambda^\alpha = \lambda$ for each $\alpha < \kappa$. Then there is a family $\mathcal{S}$ of subsets of λ of cardinality κ such that*

(a) $\operatorname{card}(\mathcal{S}) = 2^\lambda$,
(b) $\operatorname{card}(X \cap Y) < \kappa$ *for each pair of distinct elements $X, Y \in \mathcal{S}$.*

Proof. Let D denote the disjoint union of the sets λ^α for all $\alpha < \kappa$. Then $\operatorname{card}(D) = \lambda$. Define a map $F : \lambda^\kappa \to \mathcal{P}(D)$ by $F(f) = \{(f \restriction \alpha) \mid \alpha < \kappa\}$. Then clearly $\operatorname{card}(F(f)) = \kappa$ and $\operatorname{card}(F(f) \cap F(g)) < \kappa$ for each distinct $f, g \in \lambda^\kappa$. The family $\mathcal{S}$ arises just by applying bijections between λ and D, and between λ^κ and 2^λ. □

The following lemma is a generalization of [4, 2.5] (which deals with the case of $\kappa = \aleph_0$):

Lemma 7. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then for any regular cardinal κ, $M \in {}^{\perp_1}U$ implies $M^\kappa/M^{<\kappa} \in {}^{\perp_1}U$.*

Proof. Let λ be a cardinal such that $\lambda^\kappa = 2^\lambda$ and $\lambda^\alpha = \lambda$ for each $\alpha < \kappa$. Consider a family $\mathcal{S}$ of subsets of λ as in Lemma 6. For each $X \in \mathcal{S}$, let $\eta_X : M^X \to M^\lambda/M^{[\lambda;\kappa]}$ be the composition of the canonical embedding $M^X \to M^\lambda$ with the canonical projection. Denote the module $M^\kappa/M^{<\kappa}$ by N. Then clearly $\operatorname{Im}\eta_X \cong N$ and $\operatorname{Ker}\eta_X = M^{[X;\kappa]}$. Moreover, it is easy to see that the sum $\sum_{X\in\mathcal{S}} \operatorname{Im}\eta_X$ is actually a direct sum.

Next, denote by V the preimage of $\sum_{X\in\mathcal{S}} \operatorname{Im}\eta_X$ in M^λ. We claim that V is a pure submodule of M^λ. In fact, $x \in V$ if and only if the support of x is a subset of some union of the form $G \cup X_1 \cup \cdots \cup X_n$, where $X_1, \ldots, X_n$ are finitely many elements of $\mathcal{S}$ and $\operatorname{card}(G) < \kappa$. Thus, any system of finitely many linear equations $\sum_{j\le m} a_{ij}x_j = y_i$ with all the y_i's in V that can be solved in M^λ has a solution with supports of x_i's inside the union of the supports of y_i's, therefore it has a solution in V.

Now suppose that $M \in {}^{\perp_1}U$. Then $V \in {}^{\perp_1}U$ as well, and we have a short exact sequence of the form

$$0 \to M^{[\lambda;\kappa]} \to V \to N^{(\mathcal{S})} \to 0$$

and the corresponding induced exact sequence

$$\operatorname{Hom}_R(M^{[\lambda;\kappa]}, U) \to \operatorname{Ext}^1_R(N^{(\mathcal{S})}, U) \to 0.$$

We can always choose λ so that in addition $\lambda \ge \operatorname{card}(\operatorname{Hom}_R(M^\mu, U))$ for each $\mu < \kappa$ using Lemma 5. Let $\mathcal{L}$ denote the set of all the subsets of λ of cardinality $< \kappa$. Then any homomorphism $f : M^{[\lambda;\kappa]} \to U$ is uniquely determined by its restrictions to M^Z, Z running through all elements of $\mathcal{L}$. Therefore,

$$\operatorname{card}(\operatorname{Hom}_R(M^{[\lambda;\kappa]}, U)) \le \prod_{Z\in\mathcal{L}} \operatorname{card}(\operatorname{Hom}_R(M^Z, U)) \le \lambda^{\operatorname{card}(\mathcal{L})}.$$

Moreover, $\operatorname{card}(\mathcal{L}) \le \operatorname{card}(\bigcup_{\mu<\kappa} \lambda^\mu) = \lambda$. Hence $\operatorname{card}(\operatorname{Hom}_R(M^{[\lambda;\kappa]}, U)) \le 2^\lambda$. On the other hand, if $\operatorname{Ext}^1_R(N, U) \ne 0$, then $\operatorname{card}(\operatorname{Ext}^1_R(N^{(\mathcal{S})}, U)) \ge 2^{\operatorname{card}(\mathcal{S})} = 2^{2^\lambda}$, a contradiction with the existence of an epimorphism. Thus $N \in {}^{\perp_1}U$. □

The following lemma generalizes [5, 3.7, part 2]. The proof is essentially the same as in [5].

Lemma 8. *Let $\mathcal{C}$ be a class of modules closed under pure submodules and products. Assume in addition that there is a limit ordinal λ such that $M \in \mathcal{C}$ implies $M^\lambda/M^{<\lambda} \in \mathcal{C}$. Then $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \in \mathcal{C}$ for any family $(M_\alpha \mid \alpha < \lambda)$ of modules of $\mathcal{C}$.*

Proof. Let us denote $W = \prod^b_{\alpha<\lambda} M_\alpha$ and let $\varepsilon_\alpha : M_\alpha \to W$ be the canonical embeddings. Since W is a pure submodule of $\prod_{\alpha<\lambda} M_\alpha$, we get $W \in \mathcal{C}$ and $W^\lambda/W^{<\lambda} \in \mathcal{C}$. Denote by $f : \prod_{\alpha<\lambda} M_\alpha \to W^\lambda/W^{<\lambda}$ the composition of the product of the maps ε_α with the canonical projection. Then the kernel of f is exactly $\prod^b_{\alpha<\lambda} M_\alpha$ and the induced embedding $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \to W^\lambda/W^{<\lambda}$ is pure. Thus $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \in \mathcal{C}$. □

Now, we are able to extend Lemma 7 to all limit ordinals:

Lemma 9. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then for any limit ordinal λ, if $(M_\alpha \mid \alpha < \lambda)$ is a family of modules of ${}^{\perp_1}U$, then $\prod_{\alpha<\lambda} M_\alpha / \prod^b_{\alpha<\lambda} M_\alpha \in {}^{\perp_1}U$.*

Proof. In the view of the preceding lemma, it is sufficient to prove, by induction on λ, that $M \in {}^{\perp_1}U$ implies $M^\lambda/M^{<\lambda} \in {}^{\perp_1}U$. If λ is a regular cardinal, and this is in particular the case when $\lambda = \aleph_0$, then we use Lemma 7. If λ is not a regular cardinal, then there is a limit ordinal $\mu < \lambda$ and an increasing continuous map $f : \mu \to \lambda$ with an unbounded range and such that $f(0) = 0$. Let us denote $M_\alpha = M^{f(\alpha+1)\setminus f(\alpha)}$ for each $\alpha < \mu$. Then obviously $M^\lambda/M^{<\lambda} \cong \prod_{\alpha<\mu} M_\alpha / \prod^b_{\alpha<\mu} M_\alpha$, and the latter module is contained in ${}^{\perp_1}U$ by the induction hypothesis. □

Proposition 10. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then $M \in {}^{\perp_1}U$ implies $PE(M)/M \in {}^{\perp_1}U$.*

Proof. By Lemmas 1 and 9, $M_\lambda/M \in {}^{\perp_1}U$ whenever $M \in {}^{\perp_1}U$ for all M_λ in Lemma 2. Thus, using this and Corollary 4, M purely embeds into the pure injective module N_{κ^+} and $N_{\kappa^+}/M \in {}^{\perp_1}U$. Therefore, $PE(M)/M$ is isomorphic to a direct summand of N_{κ^+}/M, [12, Theorem 4.20]. Hence $PE(M)/M \in {}^{\perp_1}U$. □

Finally, we are ready to prove both the conjectures (A) and (B). The proof of Theorem 11 given here is inspired by the proof of Conjecture (A) in [7].

Theorem 11. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then ${}^{\perp_1}U$ is closed under pure epimorphic images.*

Proof. It suffices to prove that, whenever $i : Y \to X$ is a pure monomorphism such that $X \in {}^{\perp_1}U$, and $f : Y \to U$ is any homomorphism, then there is a homomorphism $g : X \to U$ such that $f = gi$. But in this case $Y \in {}^{\perp_1}U$ and $PE(Y)/Y \in {}^{\perp_1}U$ too (Proposition 10). Thus, there are homomorphisms $h : X \to PE(Y)$ and $k : PE(Y) \to U$ such that $j = hi$ and $f = kj$, where j is the embedding of Y into $PE(Y)$. The composition kh yields the desired map g. □

Corollary 12. *Let R be a ring and U a module such that ${}^{\perp_1}U$ is closed under pure submodules and products. Then ${}^{\perp_1}U$ is a covering class.*

Proof. This follows by [5, Proposition 5.4] and [9, Theorem 5]. □

The following is the main result of our paper:

Theorem 13. *Let R be an arbitrary ring, $n \geq 0$, and U an n-cotilting module. Then U is pure-injective.*

Proof. This is immediate from Corollary 12 and [5, Theorem 5.5]. □

From [1, Theorem 4.1] and [3, Proposition 3.5], we get

Corollary 14. *Let U be an n-cotilting module over an arbitrary ring such that $n \geq 1$, and let U' be a cosyzygy of U. Then $\bigcap_{i\geq 1} {}^{\perp_i}U'$ is an $(n-1)$-cotilting class.*

Remark. It is possible to state Lemma 7 more generally with just a small change in the proof: If $\mathcal{U}$ is a class of modules such that ${}^{\perp_1}\mathcal{U}$ is closed under products and pure submodules, then $M \in {}^{\perp_1}\mathcal{U}$ implies $M^\kappa/M^{<\kappa} \in {}^{\perp_1}\mathcal{U}$ for any regular cardinal κ. The subsequent statements in this paper generalize in a similar way so we can consider a class of modules $\mathcal{U}$ instead of a single module U everywhere to Corollary 12. This fact was recently used by Šaroch and Trlifaj in [14] to improve the characterization of cotilting cotorsion pairs from [1], dropping out the assumption of the completeness of a cotorsion pair.

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References

[1] L. Angeleri Hügel and F. Coelho, *Infinitely generated tilting modules of finite projective dimension*, Forum Math. **13** (2001), 239–250. MR1813669 (2002b:16009)
[2] L. Angeleri Hügel, D. Herbera and J.Trlifaj, *Tilting modules and Gorenstein rings*, to appear in Forum. Math.
[3] S. Bazzoni, *A characterisation of n-cotilting and n-tilting modules*, J. Algebra **273** (2004), 359–372. MR2032465
[4] S. Bazzoni, *Cotilting modules are pure-injective*, Proc. Amer. Math. Soc. **131** (2003), 3665–3672. MR1998172 (2004f:16049)
[5] S. Bazzoni, *n-cotilting modules and pure-injectivity*, Bull. London Math. Soc. **36** (2004), 599–612. MR2070436 (2005d:16012)
[6] S. Bazzoni, *Tilting and cotilting modules over Prüfer domains*, preprint.
[7] S. Bazzoni, R. Göbel and L. Strüngmann, *Pure injectivity of n-cotilting modules: the Prüfer and the countable case*, to appear in Archiv der Mathematik.
[8] P. C. Eklof and A. H. Mekler, *Almost Free Modules*, North-Holland Math. Library, Elsevier, Amsterdam, 1990. MR1055083 (92e:20001)
[9] P. C. Eklof and J. Trlifaj, *Covers induced by Ext*, J. Algebra **231** (2000), 640–651. MR1778163 (2001f:16021)
[10] R. Göbel and J. Trlifaj, *Cotilting and a hierarchy of almost cotorsion groups*, J. Algebra **224** (2000), 110–122. MR1736696 (2001a:20100)
[11] P. Griffith, *On a subfunctor of Ext*, Arch. Math. XXI (1970), 17–22. MR0262356 (41:6964)
[12] M. Prest, *Model theory and modules*, London Math. Soc. Lect. Note Ser. **130**, Cambridge University Press, 1988. MR0933092 (89h:03061)
[13] L. Salce, *Cotorsion theories for abelian groups*, Symposia Math. **23** (1979), Academic Press, 11–32. MR0565595 (81j:20078)
[14] J. Šaroch and J. Trlifaj, *Completeness of cotorsion pairs*, preprint.
[15] J. Trlifaj, *Infinite dimensional tilting modules and cotorsion pairs*, to appear in the Handbook of Tilting Theory (London Math. Soc. Lect. Notes).
[16] J. Xu, *Flat Covers of Modules*, Lecture Notes in Math. 1634, Springer-Verlag, New York, 1996. MR1438789 (98b:16003)

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