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ISOTOPIC FAMILIES OF CONTACT MANIFOLDS FOR ELLIPTIC PDE

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Abstract. A test for a function to be a solution of an elliptic PDE is given in terms of extensions, as solutions, from the boundaries inside the domains belonging to an isotopic family. It generalizes a result of Ehrenpreis for spheres moved along a straight line.

1. Introduction

The following problem has circulated among a certain group of mathematicians for a long time: given a family of closed Jordan curves in the complex plane and a continuous function admitting a holomorphic extension inside each curve, when is the function holomorphic in the union of the curves? We refer the reader to the articles [11], [12] by L. Zalcman, where this and closely related problems are discussed.

The case of rotation-invariant families was carefully studied in [1], [5]–[8]. However, even for simple families, for instance, circles of constant radius centered on a segment of a straight line, the question remained open until recently, when significant progress was made. Namely, in [2], a complete description of arbitrary continuous one-parameter families of circles, detecting holomorphicity in the above sense, was given for rational functions, and for real-analytic functions. Independently, the real-analytic case, though for special families of circles (constant radius, centers on a segment), was treated by Ehrenpreis in [3]. Soon afterwards, Tumanov [10] solved the case of circles of constant radius, centered on a segment (the strip-problem) for continuous functions. In both [2] and [10] the problem is solved by methods of several complex variables.

In the recently published book [4] (see Section 9.5), Ehrenpreis proposed considering the above question from the point of view of PDE, by replacing the Cauchy-Riemann equations with more general PDE. The idea is to characterize solutions of a PDE in terms of their restrictions to a family of closed hypersurfaces (contact manifolds). The condition is that the restriction to each hypersurface must coincide with the boundary value of a solution in the domain bounded by this hypersurface.

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This extension depends, in general, on the domain, and the contacts on the hypersurfaces are assumed of sufficiently high order, i.e. along with higher-order normal derivatives constituting overdetermined Dirichlet-Neumann boundary data.

Counting the number of parameters involved in the problem shows that the families may be taken to be one-parametric and therefore each family can be viewed as a curve in the space of domains. Thus, in a sense, one would like to integrate along a curve of domains the condition of tangency to a solution at each "point" of the curve to recover a global solution. We refer to the book [4] for a more detailed explanation of the concept of contact manifolds.

In the framework of this concept, Ehrenpreis proved a theorem on characterization of solutions of the elliptic PDE

$$P(x, D)f = 0$$

by their restrictions to spheres $S_t = \{(x_1 - t)^2 + x_2^2 + \cdots + x_n^2 = 1\}$ of radius 1 centered on the x_1-axis ([4, Th. 9.5]). Essentially, the result in [4] says that if a smooth function in the strip $\{x_1 \in (-\infty, \infty) : x_2^2 + \cdots + x_n^2 \leq 1\}$ agrees to order m, on each sphere S_t with a solution F_t of $PF_t = 0$ in a neighborhood of the ball B_t bounded by S_t, then f itself is a solution, $Pf = 0$. We omit here some additional technical conditions and refer the reader to [4] for an exact formulation of the theorem.

The goal of this article is to generalize Ehrenpreis' result from the specific family of spheres, S_t to quite general families of closed hypersurfaces. We also simplify the conditions for agreement of solutions, formulated in [4] in terms of approximations and bounds of derivatives, by requiring the solvabiltiy of an overdetermined Dirichlet-Neumann problem on each contact manifold.

Our result can be regarded as a response to Problem 9.8 ([4, p. 579]), which reads "Develop a complete theory of PDE contact manifolds".

2. Main result

We fix a C^1-isotopy of domains $D_t \in \mathbb{R}^n, \quad t \in I = (-1, 1)$. This family can be exhibited as follows: there is an initial domain $D = D_0$; then $D_t = \omega_t(D)$, where $\omega_t : D \mapsto D_t$ is a diffeomorphism, $\omega_0 = id$ and the family ω_t is continuously differentiable in the parameter t. We assume also that each domain D_t has C^1-boundary ∂D_t and the diffeomorphism ω_t admits a C^1-extension to the closed domain $\overline{D}_t$.

Denote by ν_t the outward unit normal vector on ∂D_t. We say that the isotopy D_t is **transversal** if for each $t \in I$, the inner product

$$\rho_t(u) = \langle \partial_t \omega_t(u), \nu_t(\omega_t(u)) \rangle \neq 0$$

for a dense set of $u \in \partial D$.

This means that the set of points where the direction of the transformation ω_t is tangent to the boundary ∂D_t is nowhere dense. A simple example of non-transversal isotopy is rotation of a ball because in this case the vector of the transformation is tangent to the boundary sphere at each point. On the other hand any family of translations in a constant direction e of a strictly convex domain D, $D_t = te + D$ (in particular, translation of a ball as in Ehrenpreis' theorem), is a transversal isotopy. Indeed, in this case, the direction vector $\partial_t \omega_t(u) = e$ can be tangent only to the surface on a subset of lower dimension.

We make the following additional assumption of regularity.

$(*)$ Each solution $u \in C^m(\overline{D})$ to the equation $L^*u = 0$ in the closed domain $\overline{D}$ can be approximated with respect to the C^m-norm by solutions in a neighborhood of $\overline{D}$. Here L^* is the formal adjoint operator. Sufficient conditions for $(*)$ to be fulfilled are given in the book of Tarkhanov [9, Ch. 6]. For instance, $(*)$ holds if the domain D is smooth and the operator L has real-analytic coefficients.

Our main result is the following.

Theorem 2.1. *Let $D_t = \omega_t(D)$ be a smooth transversal isotopy of domains in $\mathbb{R}^n$, $D = D_0$. Set*

$$\Omega = \bigcup_{t \in I} \partial D_t,$$

and let

$$L = P(x, D) = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha$$

be an elliptic partial differential operator of order $2m$ with smooth coefficients defined in the domain Ω and satisfying the condition $(*)$. Suppose that the function $f \in C^{2m}(\Omega)$ has the property that on each boundary ∂D_t, it is tangent to order m to a solution in the domain D_t, meaning that for each $t \in I$, the Dirichlet-Neumann boundary problem:

$$LF_t(x) = 0, \quad x \in D_t,$$

with the boundary conditions

$$\partial^j_{\nu_t} F_t(x) = \partial^j_{\nu_t} f(x), \quad x \in \partial D_t, \quad j = 0, 1, \cdots, (m-1), m,$$

has a solution $F_t \in C^m(\overline{D}_t)$.

Then f is a global solution, $Lf(x) = 0$, for $x \in \Omega$.

Note that the well-determined boundary data for the elliptic operator L of order $2m$ include m derivatives of orders from 0 to $m-1$. However the boundary data in Theorem 2.1 are overdetermined, as they contain the extra m-th derivative; therefore requiring solvability of the Cauchy problem in the above theorem imposes a nontrivial condition on f.

Here is a particular case of Theorem 2.1, for the Laplace operator:

Corollary 2.2. *Let D_t and Ω be as in Theorem 2.1 and let f be a C^2-function in Ω. If for each t, the function f coincides to order 1 on ∂D_t with its Poisson integral in D_t, then f is harmonic in Ω.*

3. Proof of the main result

We fix a transversal smooth isotopy of domains in $\mathbb{R}^n$, $D_t = \omega_t(D), t \in I, \omega_0 = id$. Sometimes we will find it to be more convenient writing $\omega(t, x)$ instead of $\omega_t(x)$. We also fix a function $f \in C^{2m}(\Omega)$ which has for each t an extension F_t in D_t such that $\partial^j_{\nu_t}(f - F_t) = 0$ on ∂D_t for $j = 0, \cdots, m$.

We need the following lemma.

Lemma 3.1. *Let G be a smooth function in $\Omega = \bigcup_{t \in T} D_t$. Then*

$$\frac{d}{dt} \int_{D_t} G(x) dV(x)|_{t=0} = \int_{\partial D_0} G(x) \rho_0(x) dS(x), \tag{3.1}$$

where dS is the surface measure on ∂D_0 and the function ρ_t is defined in Section 2.

Proof. The change of variables $x = \omega_t(u)$ in $\int_{D_t} G(x)\, dx$ gives

$$\int_D G(\omega_t(u))|J_t(u)|dV(u), \tag{3.2}$$

where $J_t(u) = J(t,u)$ is the Jacobian $det(\partial\omega_t/\partial u)$. Since the Jacobians do not vanish and therefore preserve the sign, we can omit the absolute value in (3.2).

Differentiating (3.2) in t at $t = 0$ and taking into account that $J(0,x) = 1$, we obtain

$$\frac{d}{dt}\int_{D_t} G(x)dV(x)|_{t=0} = \int_D \sum_{k=1}^n \partial_k G(u)\partial_t\omega_k(0,u)dV(u) + \int_D G(u)\partial_t J(0,u)dV(u). \tag{3.3}$$

Rewrite the integrand in the first integral in the right-hand side of (3.3) as

$$\sum_{k=1}^n \partial_k[G(u)\partial_t\omega_k(0,u)] - G(u)div[\partial_t\omega(0,u)]. \tag{3.4}$$

Now, since the Jacobian matrix of ω_0 is the identity matrix, we have for the derivative of the Jacobian in the second integral in (3.3)

$$\begin{aligned}\partial_t J(0,u) &= \partial_t det[\nabla\omega_1(t,u),\cdots,\nabla\omega_n(t,u)]\\ &= \sum_{k=1}^n det[\nabla\omega_1(0,u),\cdots,\partial_t\nabla_k(0,u),\cdots,\nabla\omega_n(0,u)]\\ &= \sum_{k=1}^n \partial_k\partial_t\omega_k(0,u) = div[\partial_t\omega(0,u)]. \qquad (3.5)\end{aligned}$$

Plugging (3.4) and (3.5) into (3.3) and canceling (3.5) with the *div* term in (3.4) yields that the left-hand side in (3.3) equals

$$\int_D \{\sum_{k=1}^n \partial_k[G(u)\partial_t\omega_k(0,u)]\}dV(u),$$

which by Green's formula is equal to

$$\begin{aligned}&\int_{\partial D}\{\sum_{k=1}^n \nu_k(u)G(u)\partial_t\omega_k(0,u)\}dS(u)\\ &= \int_{\partial D} G(u) < \nu(u), \partial_t\omega(0,u) > dS(u)\end{aligned}$$

where $\nu = \nu_0$ is the unit normal vector on the boundary ∂D but this is just the right-hand side in (3.1). The lemma is proved.

Let

$$(L^*\psi)(x) = \sum_{|\alpha|\le 2m} (-1)^{|\alpha|}D^\alpha(a_\alpha(x)\psi(x))$$

be the formally adjoint operator, which is also elliptic. It is well known that the well-posed Cauchy boundary problem involves m Cauchy data; that is, the equation $L^*\psi = 0$ has in D a unique solution u satisfying any m prescribed boundary conditions $\partial_\nu^j\psi = \psi_j, \quad j = 0,\cdots,m-1$, on ∂D.

Denote by C_j the Dirichlet system of boundary operators (see, e.g. [9, p. 298]), for which Green's formula holds:

$$\int_D (L\phi \ \ \psi - \phi L^*\psi)dV = \int_{\partial D} (\sum_{j=0}^{2m-1} C_j\phi \ \ \partial_\nu^j \psi)dS.$$

When the function ϕ is smooth in a neighborhood of the domain D, C_j acts on ϕ as a differential operator of order not greater than $2m-1-j$.

Lemma 3.2. *On the boundary ∂D_t, the identities $C_j(f-F_t)=0$ hold for $j=0,\cdots,m-2$.*

Proof. First we assume a condition, slightly stronger than the condition $(*)$ of approximation, namely we assume that every solution in $\overline{D}$ extends to a solution in a neighborhood of D. This occurs, for instance, when both the boundary of the domain D and the coefficients of the operator L are real-analytic.

By changing the parameter t, it is clear that it suffices to prove the identity for $t=0$, i.e., for the domain D. Take a function ψ which solves the equation $L^*\psi=0$ in a neighborhood of the domain D. Apply Green's formula for the domain D_t and the functions $\phi = f - F_t$ and ψ. Since

$$LF_t = L^*\psi = 0$$

in D_t, we obtain

$$\int_{D_t} (Lf)\psi dV = \int_{\partial D_t} [\sum_{j=0}^{m-2} C_j(f-F_t) \ \ \partial_{\nu_t}^j \psi]dS. \tag{3.6}$$

We only need to explain why the index j in the sum on the right-hand side of (3.6) does not exceed $m-2$. Indeed, by the main condition of Theorem 2.1, $f-F_t$ vanishes on ∂D_t with all derivatives up to the order m. Since *ord* $C_j \le 2m-1-j$, we have $C_j(f-F_t)=0$ as long as $2m-1-j \le m$, i.e., when $j \ge m-1$.

Choose ψ to be the solution to the Dirichlet-Neumann problem:

$$L^*\psi = 0$$

with the boundary data on ∂D:

$$\psi = \partial_\nu \psi = \cdots = \partial_\nu^{m-2}\psi = 0, \ \ \partial_\nu^{m-1}\psi = h, \tag{3.7}$$

where h is arbitrary.

Now differentiate both sides of (3.6) with respect to t at the point $t=0$. By Lemma 1, the left-hand side becomes after differentiation

$$\int_{\partial D} (Lf)\psi\rho_0 dS = 0,$$

since $\psi = 0$ on ∂D. As far as the right-hand side is concerned, we first change the variable $u = \omega_t(x)$ in the surface integral and then differentiate the integrand in t. The operator of differentiation in t at $t=0$ is defined by the vector field

$$X = \langle \partial_t \omega(0,u), \nabla \rangle,$$

where ∇ is the gradient in the variable u. This vector field can be decomposed as

$$X = T + \lambda \partial_\nu,$$

where T is a tangential vector field on ∂D, ν is the unit normal vector to ∂D and λ is the inner product of X and ν, that is, $\lambda = \rho_0(x)$.

Differentiation with respect to the variable t throws up many terms, but only one of them remains. Indeed, since all the derivatives of ψ up to order $m-2$ vanish on ∂D, the only nonzero term after differentiation in the surface integral in (3.6) may be $C_{m-2}(f-F_0)\ X\partial_\nu^{m-2}\psi$. Moreover, since the tangential derivative $T\ \partial_\nu^{m-2}\ \psi = 0$ we have

$$X\ \partial_\nu^{m-2}\psi = \rho\partial_\nu^{m-1}\psi = \rho h.$$

Thus, we have

$$\int_{\partial D} C_{m-2}(f-F_0)\ \rho_0 h dS = 0, \tag{3.8}$$

and since h is arbitrary and ρ vanishes on a nowhere dense set, we conclude that $C_{m-2}(f-F_0) = 0$ on ∂D.

This argument applies to any domain D_t; therefore the summation in the formula (3.6) extends to $m-3$.

Now we can repeat the argument, choosing ψ, a solution to the adjoint equation $L^*\psi = 0$, with the boundary conditions on ∂D: $\partial_\nu^j\psi = 0,\ j = 0, \cdots, m-3, \partial_\nu^{m-2}\psi = h$, where h is arbitrary. Such a solution exists, and is non-unique as the $(m-1)$-th normal derivative is not specified. Differentiating in t and repeating the above argument we obtain that $C_{m-3}(f-F_t) = 0$ on ∂D_t.

Proceeding this way, we complete the proof of Lemma 3.2 under the assumption that any solution of the adjoint equation in $\overline{D}$ extends as a solution in a neighborhood of $\overline{D}$. If only an approximation (condition $(*)$) is assumed, then in the proof above we choose a sequence ψ_k of solutions to $L^*\psi_k = 0$ in a neighborhood of $\overline{D}$, approximating with derivatives a solution ψ of the boundary problem (3.7) in D. Then after differentiation in the parameter t we let k tend to ∞ and arrive at the identity (3.8) which, in turn, leads to the desired conclusions. Lemma 3.2 is proved.

Proof of Theorem 2.1. By Lemma 3.2 and the definition of the adjoint operator we have

$$\int_D (Lf)\psi dV = \int_D fL^*\psi dV.$$

Choose ψ in the kernel of the adjoint operator, $L^*\psi = 0$. Then the right-hand side vanishes, so we have

$$\int_D (Lf)\psi\ dV = 0.$$

Apply Lemma 3.1 to $G = (Lf)\psi$. Then we obtain

$$\int_{\partial D} Lf(x)\psi(x)\rho_0(x)dS(x) = 0.$$

However, the boundary value ψ on ∂D can be chosen arbitrarily; therefore $(Lf)\rho_0 = 0$ on ∂D, and the condition of transversality implies that $Lf = 0$ on ∂D. Clearly, any domain D_t in the family can be chosen as the initial one; therefore the same is true for any domain D_t, so $Lf = 0$ on the union $\Omega = \bigcup_{t\in I} D_t$. The proof is complete.

Remark. Examination of the proof shows that if we give up the condition of transversality, what we can claim in Theorem 2.1 is that f is a solution, $Lf = 0$, in the closure of the domain $\{x = \omega_t(u) : u \in D,\ t \in I,\ \rho_t(u) \neq 0,\}$, that is, in the domain where the isotopy transformation ω_t acts on the boundary in a non-tangential direction.

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