PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 7, Pages 2125–2130
S 0002-9939(06)08458-9
Article electronically published on January 17, 2006

ON MAXIMAL OPERATORS ON k-SPHERES IN $\mathbb{Z}^n$

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(Communicated by Michael T. Lacey)

Abstract. A. Magyar's result on L^p-bounds for a family of operators on k-spheres ($k \geq 3$) in $\mathbb{Z}^n$ is improved to match the corresponding theorem for 2-spheres.

1. Introduction

A maximal spherical operator in $\mathbb{R}^n$ ($n \geq 3$) is given by

$$Af(x) = \sup_{r>0} \left| \int_{\Sigma} f(x - ry) d\sigma(y) \right|, \tag{1}$$

where Σ is the unit sphere in $\mathbb{R}^n$ and σ is normalized rotationally invariant measure on Σ. E. M. Stein [8] proved that there exists a constant $C(n,p)$ such that

$$\|Af\|_p \leq C(n,p) \|f\|_p, \text{ for every } f \in L^p(\mathbb{R}^n), \text{ if } p > \frac{n}{n-1}. \tag{2}$$

Now, let $S_{\lambda,k}$ denote the number of lattice points on the k-sphere of radius $\lambda^{1/k}$ centered at the origin in $\mathbb{Z}^n$, i.e., $S_{\lambda,k}$ is the number of $x \in \mathbb{Z}^n$ such that

$$|x|^k = |x_1|^k + ... + |x_n|^k = \lambda.$$

A discrete analogue of the operator (1) is defined by

$$M^{(k)} f(x) = \sup_{\lambda \in \mathbb{N}} \frac{1}{S_{\lambda,k}} \left| \sum_{|y|^k = \lambda} f(x-y) \right|.$$

Obviously, we may write $\sum_{|y|^k=\lambda} f(x-y) = \sigma_{\lambda,k} * f(x)$, where $\sigma_{\lambda,k}$ is the characteristic function of the given k-sphere.

By a classical interpolation theorem, spherical means

$$A_{\lambda,k} f(x) = \frac{1}{S_{\lambda,k}} \sigma_{\lambda,k} * f(x)$$

Received by the editors February 21, 2005.
2000 *Mathematics Subject Classification.* Primary 42B25, 11P55.
Key words and phrases. Maximal functions, Vinogradov's method.

are contractions in $L^p(\mathbb{Z}^n)$, $1 \le p \le \infty$, since

$$\begin{aligned} \|A_{\lambda,k} f\|_1 &= \frac{1}{S_{\lambda,k}} \sum_{x\in\mathbb{Z}^n} |\sigma_{\lambda,k} * f(x)| = \frac{1}{S_{\lambda,k}} \sum_{x\in\mathbb{Z}^n} \left| \sum_{y\in\mathbb{Z}^n} \sigma_{\lambda,k}(y) f(x-y) \right| \\ &\le \frac{1}{S_{\lambda,k}} \sum_{y\in\mathbb{Z}^n} \sigma_{\lambda,k}(y) \sum_{x\in\mathbb{Z}^n} |f(x-y)| = \|f\|_1 \end{aligned}$$

and

$$|A_{\lambda,k} f(x)| = \frac{1}{S_{\lambda,k}} \left| \sum_{y\in\mathbb{Z}^n} \sigma_{\lambda,k}(y) f(x-y) \right| \le \|f\|_\infty \frac{1}{S_{\lambda,k}} \sum_{y\in\mathbb{Z}^n} \sigma_{\lambda,k}(y) = \|f\|_\infty$$

for every $x \in \mathbb{Z}^n$.

The last line yields that $M^{(k)}$ is of strong (∞,∞) type.

In a search for an appropriate analogue of (2), A. Magyar [4] introduced a family of operators

$$M_{\Lambda,k} f(x) = \sup_{\Lambda \le \lambda < 2\Lambda} |A_{\lambda,k} f(x)|,$$

where $\Lambda \in \mathbb{N}$.

He proved the following two theorems about these operators.

Theorem A. *If $n > 4$ and $p > \frac{n}{n-2}$, then*

$$\|M_{\Lambda,2} f\|_{L^p(\mathbb{Z}^n)} \le c_{n,p,2} \|f\|_{L^p(\mathbb{Z}^n)},$$

where the constant $c_{n,p,2}$ is independent of Λ.

Theorem B. *Let $k \ge 3$, $K = 2^{k-1}$. Then for $n > 4kK$, $p > \frac{n}{n-4kK}$ we have*

$$\|M_{\Lambda,k} f\|_p \le c_{n,p,k} \|f\|_p,$$

where the constant $c_{n,p,k}$ is independent of Λ.

Theorem A has served as a building block in the recent proof of the definite $M^{(2)}$ result by A. Magyar, E. Stein and S. Wainger.

Theorem C ([5])**.** *The maximal operator $M^{(2)}$ is a bounded operator on $L^p(\mathbf{Z}^n)$, for $p > \frac{n}{n-2}$, when $n > 4$.*

The purpose of this note is to reduce the bounds on n and p in Theorem B, so that the obtained result coincides with Theorem A in case $k = 2$. This is achieved through refinement of Magyar's reasoning on minor arcs. For large values of k (i.e. $k \ge 14$) Vinogradov's estimates of trigonometric sums are applied.

2. Main result

Theorem 1. *Let $k \ge 3$, $\varrho = \left[8k^2(\ln k + 0.5 \ln\ln k + 1.3)\right]^{-1}$, $n > A = \min\{\frac{k}{\rho}, kK\}$ and $p > \frac{2}{2-\theta}$, where $K = 2^{k-1}$ and $\theta = \frac{A}{n}$. Then, the sequence of operators $\{M_{\Lambda,k} : \Lambda \in \mathbb{N}\}$ is uniformly bounded on $L^p(\mathbb{Z}^n)$.*

Proof. The number $S_{\lambda,k}$ is closely related to the number of representations of λ as a sum $x_1^k + ... + x_n^k$ of nonnegative integers. From the results of Hardy-Littlewood [1, p. 290] and Vinogradov [9, p. 83] on the asymptotic formula in Waring's problem, we know that there exist constants $c_{n,k}$ and $C_{n,k}$ such that

$$c_{n,k} \lambda^{\frac{n}{k}-1} \le S_{\lambda,k} \le C_{n,k} \lambda^{\frac{n}{k}-1}, \text{ for } \lambda > N(n,k).$$

The L^1-bound for an operator $M_{\Lambda,k}$ is straightforward: $\|M_{\Lambda,k}\|_1 \le \Lambda \|f\|_1$.

For $p > 1$, it can be shown [4, p. 312] that

$$\|M_{\Lambda,k} f\|_p \le c_{n,k} \Lambda^{-\frac{n}{k}+1} \int_0^1 \|s_t * f\|_p \, dt, \tag{3}$$

where $s_t(x) = e^{2\pi i |x|^k (t+i\varepsilon)}$, with $\varepsilon = \frac{1}{2\pi\Lambda}$.

In case $p = 2$, Parseval's formula gives us

$$\|s_t * f\|_2 \le \sup_{\xi \in \Pi^n} |\widehat{s}_t(\xi)| \, \|f\|_2 , \tag{4}$$

where Π^n is the unit cube in $\mathbb{R}^n$ and $\widehat{s}_t$ is the Fourier transform of s_t. Finding an appropriate estimate of $\widehat{s}_t = \widehat{s}_{t,k,\varepsilon}$ and, consequently, of the integral $\int_0^1 \|s_t * f\|_2 \, dt$ constitutes the essence of the proof of the theorem.

We have come to the point where a suitable partition of the interval $[0,1]$ comes into play.

For a natural number N, the Farey sequence of order N is a finite sequence of irreducible rational fractions $\frac{a}{q}$ $(0 \le a \le q \le N)$ arranged in ascending order.

Let

$$V_{\frac{a}{q}} = \left[\frac{a}{q} - \frac{1}{q(q+q'')}, \frac{a}{q} + \frac{1}{q(q+q')} \right],$$

where $\frac{a''}{q''} < \frac{a}{q} < \frac{a'}{q'}$ are three consecutive terms in the Farey sequence. This implies that any $t \in V_{\frac{a}{q}}$ can be written in the form $t = \frac{a}{q} + \tau$, with $|\tau| \le \frac{1}{Nq}$. Intervals $V_{\frac{a}{q}}$ are non-overlapping and their union is precisely $[0,1]$ (cf. [2]).

Our choice of N will depend on A.

Case I. Suppose $3 \le k \le 13$. Then, $A = kK$. Let us take $N = \left\lfloor \Lambda^{1-\frac{1}{k}} \right\rfloor + 1$. We will call $V_{\frac{a}{q}}$ a major arc if $q \le \Lambda^{1/k}$. Otherwise, $V_{\frac{a}{q}}$ is a minor arc.

Obviously,

$$\int_0^1 \|s_t * f\|_p \, dt = \sum_{q \le \Lambda^{1/k}} \int_{V_{\frac{a}{q}}} \|s_t * f\|_p \, dt + \sum_{\Lambda^{1/k} < q \le N} \int_{V_{\frac{a}{q}}} \|s_t * f\|_p \, dt. \tag{5}$$

We will first deal with minor arcs.

Let $\langle x, \xi \rangle$ denote the scalar product of x and ξ in $\mathbb{R}^n$. The fact that we have

$$\widehat{s}_t(\xi) = \sum_{x \in \mathbb{Z}^n} e^{2\pi i (|x|^k (t+i\varepsilon) + \langle x, \xi \rangle)} = \prod_{j=1}^n \left(\sum_{x_j \in \mathbb{Z}} e^{2\pi i (|x_j|^k (t+i\varepsilon) + x_j \xi_j)} \right)$$

reduces the problem to the one-dimensional case.

Substituting $-\xi$ for ξ in $\sum_{x \in \mathbb{Z}} e^{2\pi i (|x|^k (t+i\varepsilon) + x\xi)}$, we see that it suffices to estimate

$$u_{t,k,\varepsilon}(\xi) = \sum_{x=0}^{\infty} e^{2\pi i (x^k (t+i\varepsilon) + x\xi)}.$$

Let us split the last sum into two parts: the L-th partial sum and the tail.

For a fixed $p > \frac{2n}{2n-kK}$ and α given by $\frac{1}{p} = 1 - \frac{\alpha}{2}$ we have $\alpha > \frac{kK}{n}$. Take a δ such that $4n\delta < \frac{n\alpha - kK}{kK}$. Let $L = \left\lfloor \Lambda^{\frac{1}{k}+\delta} \right\rfloor + 1$. Then, for the tail, we have

$$\left| \sum_{x=L+1}^{\infty} e^{2\pi i (x^k (t+i\varepsilon) + x\xi)} \right| \leq \int_L^{\infty} e^{-2\pi\varepsilon x^k} dx.$$

Recall that $\varepsilon = \frac{1}{2\pi\Lambda}$. Thus the integral on the right-hand side is bounded by

$$\Lambda^{\frac{1}{k}} \int_{L\Lambda^{-\frac{1}{k}}}^{\infty} e^{-t} dt \leq e^{\frac{1}{k}\ln\Lambda - \Lambda^{\delta}},$$

since $L\Lambda^{-\frac{1}{k}} > \Lambda^{\delta}$.

However, the function $f(u) = \frac{1}{k}\ln u - u^{\delta},\ u > 0$, takes an absolute maximum at the point $u = (k\delta)^{-\frac{1}{\delta}}$. Hence

$$\left| \sum_{x=L+1}^{\infty} e^{2\pi i (x^k (t+i\varepsilon) + x\xi)} \right| \leq e^{\frac{1}{k\delta}\left(\ln\frac{1}{k\delta} - 1\right)} = C(k,\delta) = C\left(n,k,p\right). \tag{6}$$

Now, let $t \in V_{\frac{a}{q}}$, with $q > \Lambda^{\frac{1}{k}}$. Putting $t = \frac{a}{q} + \tau$ into the L-th partial sum of $u_{t,k,\varepsilon}$ and applying Abel partial summation, we get

$$\begin{aligned} \left| \sum_{x=0}^{L} e^{2\pi i (x^k \frac{a}{q} + x\xi)} e^{2\pi i x^k (\tau + i\varepsilon)} \right| &\leq C(k,\delta) \max_{l \leq L} \left| T_l\left(\xi\right) \right| \ln\Lambda \\ &\leq C(k,\delta) \max_{l \leq L} \left| T_l\left(\xi\right) \right| \Lambda^{\delta}, \end{aligned} \tag{7}$$

where $T_l\left(\xi\right) = \sum_{x=0}^{l} e^{2\pi i (x^k \frac{a}{q} + x\xi)}$.

On these arcs, we will make use of Weyl's inequality [6, Th. 4.3]. For $l \leq L$ this will give

$$\begin{aligned} \left| T_l\left(\xi\right) \right| &\leq C(k,\delta) l^{1+\delta} \left(l^{-1} + q^{-1} + l^{-k} q \right)^{\frac{1}{K}} \leq C(k,\delta) \Lambda^{\left(\frac{1}{k}+\delta\right)(1+\delta) - \frac{1}{kK}} \\ &\leq C(k,\delta) \Lambda^{\frac{1}{k} + 3\delta - \frac{1}{kK}}. \end{aligned} \tag{8}$$

Therefore, according to relations (6), (7) and (8), in n dimensions we will have (on minor arcs)

$$\left| \widehat{s}_t(\xi) \right| \leq C\left(n,k,p\right) \Lambda^{\frac{n}{k} + 4n\delta - \frac{n}{kK}}, \text{ uniformly in } \xi \in \Pi^n.$$

Interpolating this L^2-bound (see (4)) with the trivial L^1-bound

$$\left\| s_t * f \right\|_1 \leq C\left(n,k\right) \Lambda^{\frac{n}{k}} \left\| f \right\|_1,$$

we obtain

$$\left\| s_t * f \right\|_p \leq C\left(n,k,p\right) \Lambda^{\frac{n}{k} + 4n\delta\alpha - \frac{n\alpha}{kK}} \left\| f \right\|_p,$$

for t belonging to a minor arc. Hence,

$$\Lambda^{-\frac{n}{k}+1} \int_{V_{\frac{a}{q}}} \left\| s_t * f \right\|_p dt \leq C\left(n,k,p\right) \Lambda^{1 + 4n\delta\alpha - \frac{n\alpha}{kK}} \frac{1}{Nq} \left\| f \right\|_p \tag{9}$$

if $V_{\frac{a}{q}}$ is a minor arc.

On the other hand, scrutinizing the proof of Magyar's estimate [4, Lemma 2] of the integrals over major arcs, we obtain

$$\Lambda^{-\frac{n}{k}+1}\int_{V_{\frac{a}{q}}}\|s_t * f\|_p\,dt \le C\,(n,k,p)\,q^{-\frac{n\alpha}{k}}\,\|f\|_p \tag{10}$$

if $V_{\frac{a}{q}}$ is a major arc. ($K = 2^{k-1}$ in [4, Lemma 2] may be replaced by k, since $\left|\sum_{x=0}^{q-1} e^{2\pi i(x^k\frac{a}{q}-x\frac{l}{q})}\right| \le C\,(k)\,q^{1-\frac{1}{k}}$, uniformly in l, according to Hua's theorem [7].)

From (3), (5) , (9) and (10), we deduce

$$\|M_{\Lambda,k}f\|_p \le C(n,k,p)\left(\sum_{q\le\Lambda^{\frac{1}{k}}}\varphi(q)q^{-\frac{n\alpha}{k}} + \Lambda^{1+4n\delta\alpha-\frac{n\alpha}{kK}}\sum_{\Lambda^{\frac{1}{k}}<q\le N}\frac{\varphi(q)}{Nq}\right)\|f\|_p\,.$$

Notice that the first sum in the parentheses on the right-hand side is a partial sum of a convergent series since $\varphi(q)q^{-\frac{n\alpha}{k}} \le q^{1-\frac{n\alpha}{k}}$ and $\frac{n\alpha}{k} > 2$.

The second term in the parentheses is also bounded, uniformly in Λ, since the exponent $1+4n\delta\alpha-\frac{n\alpha}{kK}$ is negative by the choice we made for δ.

Case II. Now, suppose $k \ge 14$. Then, $A = \frac{k}{\rho}$. Let again $p > \frac{2}{2-\theta}$ and $\alpha \in (0,1)$ be such that $\frac{1}{p} = 1-\frac{\alpha}{2}$. Obviously, $\alpha > \theta = \frac{k}{n\rho}$. Choose $\delta > 0$ such that $\beta = 1-\frac{n\rho\alpha}{k}+\delta n\alpha\,(2-\rho) < 0$. This is possible since $\frac{n\rho\alpha}{k} > 1$. As before, $L = \left\lfloor \Lambda^{\frac{1}{k}+\delta}\right\rfloor + 1$.

We shall take $N = \left\lfloor L^{k-\frac{1}{k}}\right\rfloor + 1$ and replace (5) above with

$$\int_0^1 \|s_t * f\|_p\,dt = \sum_{q\le L^{1/k}}\int_{V_{\frac{a}{q}}}\|s_t * f\|_p\,dt + \sum_{L^{1/k}<q\le N}\int_{V_{\frac{a}{q}}}\|s_t * f\|_p\,dt. \tag{5'}$$

As far as the first sum on the right-hand side is concerned, it suffices to notice that (10) remains unchanged. The point of bifurcation in our argument is again the treatment of the second sum. We turn our attention to (7) above. Applying Vinogradov's theorem [9, Th. 3 on p. 66] at the point $\left(\frac{a}{q},0,...,0,\xi\right) \in \mathbb{R}^k$, $q \ge L^{\frac{1}{k}} \ge l^{\frac{1}{k}}$, we obtained

$$|T_l\,(\xi)| \le C\,(k)\,l^{1-\rho} \quad \text{for } l \ge k^k \quad \text{(see [3])}.$$

The last inequality together with (6) and (7) give us

$$|u_{t,k,\epsilon}| \le C\,(k,\delta)\,L^{1-\rho}\Lambda^{\delta}$$

in the one-dimensional case and, respectively,

$$|\widehat{s}_t(\xi)| \le C\,(n,k,\delta)\,L^{n(1-\rho)}\Lambda^{n\delta} \le C\,(n,k,\delta)\,\Lambda^{\left(\frac{1}{k}+\delta\right)n(1-\rho)+n\delta},$$

where $\xi \in \Pi^n$ and t belongs to a minor arc. Hence, by interpolation,

$$\int_{V_{\frac{a}{q}}}\|s_t * f\|_p\,dt \le C\,(n,k,p)\,\Lambda^{(1-\alpha)\frac{n}{k}+\alpha\left(\frac{1}{k}+\delta\right)n(1-\rho)+\alpha n\delta}\frac{1}{Nq}\,\|f\|_p$$

on every minor arc $V_{\frac{a}{q}}$.

Thus,

$$\Lambda^{-\frac{n}{k}+1}\int_{V_{\frac{a}{q}}}\|s_t * f\|_p\,dt \le C\,(n,k,p)\,\Lambda^{1-\frac{n\rho\alpha}{k}+\delta n\alpha(2-\rho)}\frac{1}{Nq}\,\|f\|_p \tag{11}$$

$$= C\,(n,k,p)\,\Lambda^{\beta}\frac{1}{Nq}\,\|f\|_p$$

on minor arcs.

From (3), (5′), (10) and (11), we get

$$\|M_{\Lambda,k}f\|_p \le C(n,k,p)\,\|f\|_p\,.$$

The proof is complete.

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