PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 8, Pages 2419–2425
S 0002-9939(06)08207-4
Article electronically published on March 14, 2006

BOREL CARDINALITIES BELOW c_0

MICHAEL RAY OLIVER

(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. The Borel cardinality of the quotient of the power set of the natural numbers by the ideal $\mathcal{Z}_0$ of asymptotically zero-density sets is shown to be the same as that of the equivalence relation induced by the classical Banach space c_0. We also show that a large collection of ideals introduced by Louveau and Veličkovič, with pairwise incomparable Borel cardinality, are all Borel reducible to c_0. This refutes a conjecture of Hjorth and has facilitated further work by Farah.

1. THE IDEAL OF DENSITY IS EQUIREDUCIBLE WITH c_0

1.1. **Origin of the question.** When investigating whether Borel reductions (see Definition 1.3 below) exist between given equivalence relations, it is sometimes convenient to replace one equivalence relation with a combinatorially simpler one which is known to be reducible in both directions with the given equivalence relation. For example, consider the equivalence relation on $\mathbb{R}^\omega$ induced by the action of ℓ^1 by coordinatewise addition (let us refer to this equivalence relation simply as ℓ^1). Hjorth [Hjo00] has shown that if $E \leq_B \ell^1$, then either $\ell^1 \leq_B E$, or E is reducible to an equivalence relation all of whose equivalence classes are countable. In his exposition, he replaces ℓ^1 with the equivalence relation given by the *summable ideal* $\mathcal{I}_{1/n}$ on the power set of ω: If $A \subseteq \omega$, then $A \in \mathcal{I}_{1/n}$ just in case $\sum_{n\in A} 1/(n+1) < \infty$. Since $\ell^1 \leq_B \mathcal{I}_{1/n}$ and $\mathcal{I}_{1/n} \leq_B \ell^1$, this substitution is legitimate. Here we use the following.

Convention 1.1. We write $\mathcal{I}_{1/n}$ for the equivalence relation on $\mathcal{P}(\omega)$ given by $A \sim B \leftrightarrow A \triangle B \in \mathcal{I}_{1/n}$. Similarly we write $\mathcal{Z}_0$ for the equivalence relation on $\mathcal{P}(\omega)$ induced by $\mathcal{Z}_0$ (see Definition 1.2), and c_0 and ℓ^1 for the equivalence relations on $\mathbb{R}^\omega$ induced by the actions of those groups by coordinatewise addition.

Kechris had suggested that, as ℓ^1 is mutually Borel reducible with $\mathcal{I}_{1/n}$, c_0 might similarly be equivalent to the *ideal of density* $\mathcal{Z}_0$:

Definition 1.2. For $A \subseteq \omega$,

$$A \in \mathcal{Z}_0 \iff \frac{|A \cap n|}{n} \to 0 \text{ as } n \to \omega.$$

Received by the editors April 5, 2004 and, in revised form, October 25, 2004 and February 18, 2005.

2000 *Mathematics Subject Classification.* Primary 03E15; Secondary 37A20.

Key words and phrases. Borel equivalence relations.

Definition 1.3. As usual, for X and Y Polish spaces, E and F Borel equivalence relations on X and Y respectively, we write $E \leq_B F$, and say E *is Borel reducible to* F, just in case there exists a Borel function $\theta : X \to Y$ such that for all $x_0, x_1 \in X$, $x_0 E x_1$ if and only if $\theta(x_0) F \theta(x_1)$. In this case we say that F *reduces* E.

In this section we shall demonstrate that, as Kechris had conjectured, c_0 and $\mathcal{Z}_0$ are mutually Borel reducible.

1.2. **Easy direction.** We want to see that c_0 reduces $\mathcal{Z}_0$ (i.e. $\mathcal{Z}_0 \leq_B c_0$); that is, there is a Borel function θ: $\mathcal{P}(\omega) \to \mathbb{R}^\omega$ such that for any $X, Y \in \mathcal{P}(\omega)$,

$$X \triangle Y \in \mathcal{Z}_0$$

if and only if

$$\theta(X)_n - \theta(Y)_n \to 0 \qquad \text{as} \qquad n \to \omega.$$

1.2.1. *c_0-equalities.* The first thing to note is that $\mathcal{Z}_0$ is what Farah ([Far01]) calls a *c_0-equality*:

Definition 1.4 (Farah)**.** Given finite sets $\langle X_i | i < \omega \rangle$, an equivalence relation E on $\prod_{i<\omega} X_i$ is a *c_0-equality* if there are metrics d_i on the X_i such that, for $x, y \in \prod_{i<\omega} X_i$,

$$xEy \iff \lim_{i \to \omega} d_i(x(i), y(i)) = 0.$$

To see that $\mathcal{Z}_0$ is a c_0-equality, let $X_i = [2^i - 1, 2^{i+1} - 1)$ and, for $X, Y \in \prod_{i<\omega} X_i$, let $d_i(x, y) = \dfrac{|X(i) \triangle Y(i)|}{|X_i|}$. Now, identifying $\prod_{i<\omega} X_i$ with $\mathcal{P}(\omega)$ via the obvious isomorphism, one easily checks that

$$X \triangle Y \in \mathcal{Z}_0 \iff \lim_{i \to \omega} d_i(X(i), Y(i)) = 0.$$

Now let us show

Theorem 1.5. *Let $\langle X_i | i < \omega \rangle$ be finite sets, and let E be a c_0-equality on $\prod_{i<\omega} X_i$. Then $E \leq_B c_0$.*

Proof. For each $i < \omega$ let d_i be a metric on X_i such that the sequence $\langle d_i | i < \omega \rangle$ witnesses that E is a c_0-equality. Now let $h : \omega \to \{\langle j, x \rangle | j < \omega \wedge x \in X_j\}$ be a bijection, and write j_i and x_i, respectively, for the first and second coordinates of $h(i)$. Now define $\Theta : \prod_{i<\omega} X_i \to \mathbb{R}^\omega$ by

$$\Theta(f)(i) = d_{j_i}(f(j_i), x_i)$$

where $f \in \prod_{i<\omega} X_i$.

Now given $f, g \in \prod_{i<\omega} X_i$, suppose $f \not\!E g$. Then there are $\epsilon > 0$ and arbitrarily large j such that $d_j(f(j), g(j)) > \epsilon$. Since each j is equal to j_i for only finitely many i, there are infinitely many i such that $d_{j_i}(f(j_i), g(j_i)) > \epsilon$. For each i, let k_i be the unique natural number such that $j_{k_i} = j_i$ and $x_{k_i} = g(j_i)$. Then $\Theta(g)(k_i) = d_{j_i}(g(j_i), g(j_i)) = 0$, but $\Theta(f)(k_i) = d_{j_i}(f(j_i), g(j_i))$, which is greater than ϵ for infinitely many values of i and therefore for infinitely many values of k_i. Thus $\Theta(f) - \Theta(g) \notin c_0$.

Conversely, suppose fEg. Then

$$\begin{aligned} |\Theta(f)(i) - \Theta(g)(i)| &= |d_{j_i}(f(j_i), x_i) - d_{j_i}(g(j_i), x_i)| \\ &\leq d_{j_i}(f(j_i), g(j_i)) \end{aligned}$$

and this last quantity approaches zero as $i \to \omega$, by the assumption that fEg and by the finite-to-one character of the mapping $i \mapsto j_i$. □

We have established

Claim 1.6. $\mathcal{Z}_0 \leq_B c_0$. □

1.3. **Harder direction.**

Claim 1.7. $c_0 \leq_B \mathcal{Z}_0$, *i.e. there is a Borel function* $\theta\colon \mathbb{R}^\omega \to \mathcal{P}(\omega)$ *such that for any* $\vec{x}, \vec{y} \in \mathbb{R}^\omega$,

$$|x_n - y_n| \to 0 \qquad \textit{as} \qquad n \to \omega$$

if and only if

$$\theta(\vec{x}) \mathbin{\triangle} \theta(\vec{y}) \in \mathcal{Z}_0.$$

Proof. It is enough to get such a function from $[0,1]^\omega \to \mathcal{P}(\omega)$. For, suppose we have a Borel function $\theta : [0,1]^\omega \to \mathcal{P}(\omega)$ satisfying the above criterion; we can recover a reduction $\theta' : \mathbb{R}^\omega \to \mathcal{P}(\omega)$ as follows: Given $f \in \mathbb{R}^\omega$ and a bijective pairing function $\langle , \rangle : \omega \times \omega \to \omega$, define $\hat{f} : [0,1]^\omega \to \mathcal{P}(\omega)$ by

$$\hat{f}(\langle n, k\rangle) \triangleq \begin{cases} 0 & \text{if } f(k) < n, \\ 1 & \text{if } f(k) > n+1, \\ f(k) - n & \text{otherwise.} \end{cases}$$

Now take $\theta'(f) \triangleq \theta(\hat{f})$.

Definition 1.8. A rational is i-*dyadic* for a natural number i if it equals $j/2^i$, for some integer j.

Definition 1.9. For a real number x, we write

$$[\![x]\!]_i \triangleq \lfloor x \cdot 2^i + \tfrac{1}{2} \rfloor / 2^i$$

where $\lfloor \cdot \rfloor$ is the greatest-integer function. That is, $[\![x]\!]_i$ is the nearest i-dyadic to x.

For $W \in \mathcal{P}(\omega)$, we let $\mu_n(W) \triangleq |W \cap n|/n$, the density of W up to n. We want to fix in advance a sequence $k_0, k_1, \ldots$ such that for all i,

(1) k_i is a multiple of 2^i.
(2) We always have $k_i \ll k_{i+1}$ in the sense that if $W \in \mathcal{P}(\omega)$ and

$$|W \cap [k_i, k_{i+1})| = j \cdot (k_{i+1} - k_i)/2^i,$$

then $[\![\mu_{k_{i+1}}(W)]\!]_i = j/2^i$. That is, if you fill in the space from k_i to k_{i+1} with density $j/2^i$, then the part up to k_i is negligible up to finding the nearest i-dyadic. More simply, what we need is $k_{i+1} > 2 \cdot 2^i \cdot k_i$.
(3) We also have $2^i \ll k_i$. Here what we want is that, beyond k_i, anything you do to 2^i or fewer coordinates cannot change the density up to that point by more than 2^{-i}. For this it is enough that $k_i > 2^{2i}$.

Clearly $k_i = 2^{2^{i+47}}$ works.

Now we are ready to define θ, as follows: Given $\vec{x} \in [0,1]^\omega$ and a particular i, let $j = [\![x_i]\!]_i \cdot 2^i$, and then write

$$B_{\vec{x},i} \triangleq \left\{ k \in [k_i, k_{i+1}) \;\middle|\; k \bmod 2^i \geq 2^i - j \right\}$$

(note that the j on the right-hand side depends on $\vec{x}$ and i). Then we take

$$\theta(\vec{x}) \triangleq \bigcup_{i \in \omega} B_{\vec{x},i}.$$

That is, we break the interval $[k_i, k_{i+1})$ into blocks of length 2^i, from each of which we accept the final j numbers. Thus the density in the interval $[k_i, k_{i+1})$ is the nearest i-dyadic to x_i, and in fact by (2) the density up to k_{i+1} is near x_i.

If $x_n - y_n$ does *not* approach zero, then there will be some ℓ for which, for infinitely many $n > \ell$, x_n and y_n differ by more than $2/2^\ell$. For such n, supposing $x_n > y_n$, note:

$$\begin{aligned}
\left|\theta(\vec{x}) \triangle \theta(\vec{y}) \cap [k_n, k_{n+1})\right| &= |B_{\vec{x},n} \triangle B_{\vec{y},n}| \\
&= |B_{\vec{x},n}| - |B_{\vec{y},n}| \\
&= ([\![x_n]\!]_n - [\![y_n]\!]_n)(k_{n+1} - k_n) \\
&\geq (x_n - y_n - 1/2^n)(k_{n+1} - k_n) \\
&\geq (1/2^\ell) \cdot k_{n+1} \cdot (1 - 2^{-(n+1)}) \\
&\geq (1/2^{\ell+1}) \cdot k_{n+1}.
\end{aligned}$$

Thus for infinitely many n, $\mu_{k_{n+1}}(\theta(\vec{x}) \triangle \theta(\vec{y})) \geq 1/2^{\ell+1}$ where ℓ is fixed. Therefore $\theta(\vec{x}) \triangle \theta(\vec{y})$ is *not* in $\mathcal{Z}_0$.

On the other hand, suppose $x_n - y_n$ does approach zero, and let

$$\rho_n \triangleq \mu_n(\theta(\vec{x}) \triangle \theta(\vec{y})).$$

We need to see that $\rho_n \to 0$ as $n \to \omega$.

First note that for any i, we will have

$$[\![\rho_{k_{i+1}}]\!]_i = \left| \; [\![x_i]\!]_i - [\![y_i]\!]_i \; \right|$$

by construction and by property (2) of the k_i. This value clearly goes to zero as i goes to infinity, so all we need to do is make sure that the value of ρ_n does not get too large for values of n between the values of k_i.

If n between k_i and k_{i+1} is a multiple of 2^i, writing $\rho' \triangleq |\; [\![x_i]\!]_i - [\![y_i]\!]_i \;|$, then ρ_n is a weighted average of ρ_{k_i} and ρ', because all the blocks of length 2^i starting at k_i have density ρ'. Thus, for such n, $\rho_n \leq \max(\rho_{k_i}, \rho')$. But now by property (3) of the k_i, it is true for *any* n between k_i and k_{i+1} that

$$\rho_n \leq \max(\rho_{k_i}, \rho') + 2^{-i}.$$

But all terms above go to zero, so ρ_n goes to zero. Thus the reduction is established. $\square$

2. Borel upper bounds for the Louveau–Veličkovič and Mazur towers

2.1. **Introduction and nomenclature.**

2.1.1. *Basic definitions.*

2.1.2. *Background.* In [LV94], it is shown that the $\leq_B$ ordering is very rich, that in particular the partial order of almost–inclusion on sets of naturals may be embedded into $\leq_B$ restricted to the $\boldsymbol{\Pi}^0_3$ equivalence relations. In [Maz00] this result is improved to $\boldsymbol{\Sigma}^0_2$; however, Mazur's equivalence relations, unlike Louveau's and Veličkovič's, may not be regarded as induced by the action of a Polish group on a Polish space. (However, Adams and Kechris, in [AK00, Theorem 4.1], obtain a large family of countable equivalence relations–therefore $\boldsymbol{\Sigma}^0_2$ and induced by Polish group actions–that are pairwise Borel irreducible.)

For each subset X of ω, the authors (in effect) define an equivalence relation E_X such that $E_X \leq_B E_Y$ if and only if X is almost included in Y (i.e. $X \setminus Y$ is finite).

in this section we observe that all these relations E_X are Borel reducible to c_0. This refutes a conjecture from [HK97].

Similarly, we show that all the equivalence relations defined by Mazur are reducible to ℓ^∞.

2.1.3. *The equivalence relations in question.* Louveau and Veličkovič fix two sequences $\{a_n | n \in \omega\}$ and $\{b_n | n \in \omega\}$, where the exact values are not important except that the a_n grow very fast and the b_n grow much faster than that. They then divide the natural numbers up into intervals $I_n = [m_n, m_{n+1})$ where $m_n = \sum_{k<n} b_k$. Then for $A \subseteq \omega$, $X, Y \in \mathcal{P}(\omega)$, they define $X E_A Y$ if and only if

$$\frac{\log(|(X \triangle Y) \cap I_n| + 1)}{a_n} \to 0 \quad \text{as } n \to \omega, n \in A.$$

It should be clear that given any $A \subseteq \omega$, we can obtain the equivalence relation E_A (up to mutual Borel reducibility) by altering the sequence of a_n's and b_n's and then looking at E_ω. Therefore we shall suppress the dependence on A and simply define, for $X, Y \in \mathcal{P}(\omega)$, the following.

Definition 2.1.

$$X \sim_{LV} Y \iff \frac{\log(|(X \triangle Y) \cap I_n| + 1)}{a_n} \to 0 \text{ as } n \to \omega.$$

Mazur, on the other hand, defines equivalence relations of exactly the same form except that instead of asking whether the sequence $\frac{\log(|(X \triangle Y) \cap I_n| + 1)}{a_n}$ goes to zero, one asks whether it is bounded. Thus we define:

Definition 2.2.

$$X \sim_M Y \iff \exists M \forall n \left(\frac{\log(|(X \triangle Y) \cap I_n| + 1)}{a_n} < M \right).$$

Now we can state that the purpose of this section is to show that

$$\sim_{LV} \leq_B c_0$$

and

$$\sim_M \leq_B \ell^\infty.$$

Note that $\sim_M$ *cannot* be Borel reducible to c_0, because it is easily seen that E_1 (the equivalence relation of eventual equality on $\mathbb{R}^\omega$) is Borel reducible to $\sim_M$, and E_1 is not reducible to the orbit equivalence relation of any Polish group action (see [KL97, Theorem 4.2]).

2.2. **Reduction for the Louveau–Veličkovič tower.**

Theorem 2.3. $\sim_{LV} \leq_B c_0$.

Proof. By Theorem 1.5, this is immediate once we know $\sim_{LV}$ is a c_0-equality, which is noted in [Far01] (and also easy to check directly). □

In [Hjo00, Theorem 5.3], Hjorth showed that if E is a Borel equivalence relation Borel reducible to ℓ^1, then either $\ell^1 \leq_B E$, or E is Borel reducible to some countable equivalence relation (that is, every equivalence class is countable). He had asked the current author to attempt to prove an analogous result for c_0; namely, that if $E \leq_B c_0$, then either $c_0 \leq_B E$, or else E is Borel reducible to the isomorphism relation on countable structures in some language. Theorem 2.3 eliminates this possibility, because the Louveau-Veličkovič equivalence relations are induced by *turbulent* Polish group actions (see [Hjo00] or [HK97] for definitions).

In [HK97, p. 338], the authors propose a "perhaps overly optimistic" Conjecture 7, which states that if G is a Polish group acting continuously on a Polish G-space X in a turbulent manner, then either c_0 or ℓ^1 is Borel reducible to the induced orbit equivalence relation of the action. Theorem 2.3 refutes this conjecture as well (note that if ℓ^1 were Borel reducible to the Louveau-Veličkovič relations, it would also be Borel reducible to c_0, which contradicts [Hjo00, Theorem 6.1]).

2.3. **The Mazur tower.** Thanks to the anonymous referee for suggesting that the notion of a c_0-equality might be generalized:

Definition 2.4. Given countable sets $\langle X_i | i < \omega\rangle$, an equivalence relation E on $\prod_{i<\omega} X_i$ is an ℓ^∞-*equality* if there are metrics d_i on the X_i such that, for $x, y \in \prod_{i<\omega} X_i$,

$$xEy \iff (\exists M)(\forall i < \omega)\, d_i(x(i), y(i)) < M.$$

Note that, unlike in the definition of c_0-equality, we do not require the sets X_i to be finite.

Corollary 2.5 (to the proof of Theorem 1.5)**.** *Let $\langle X_i | i < \omega\rangle$ be countable sets, and let E be an ℓ^∞-equality on $\prod_{i<\omega} X_i$. Then $E \leq_B \ell^\infty$.*

Proof. For each $i < \omega$ let d_i be a metric on X_i such that the sequence $\langle d_i | i < \omega\rangle$ witnesses that E is an ℓ^∞-equality. Now let $h : \omega \to \{\langle j, x\rangle | j < \omega \wedge x \in X_j\}$ be a bijection, and write j_i and x_i, respectively, for the first and second coordinates of $h(i)$. Now define $\Theta : \prod_{i<\omega} X_i \to \mathbb{R}^\omega$ by

$$\Theta(f)(i) = d_{j_i}(f(j_i), x_i)$$

where $f \in \prod_{i<\omega} X_i$.

Now given $f, g \in \prod_{i<\omega} X_i$, suppose $f \not\!E g$. Then for any M there is a j such that $d_j(f(j), g(j)) > M$. For each i, let k_i be the unique natural number such that $j_{k_i} = j_i$ and $x_{k_i} = g(j_i)$. Then $\Theta(g)(k_i) = d_{j_i}(g(j_i), g(j_i)) = 0$, but $\Theta(f)(k_i) = d_{j_i}(f(j_i), g(j_i))$, and for any M there is some value of i (and therefore some value of k_i) such that $d_{j_i}(f(j_i), g(j_i)) > M$. Thus $\Theta(f) - \Theta(g) \notin \ell^\infty$.

Conversely, suppose fEg. Then

$$\begin{aligned} |\Theta(f)(i) - \Theta(g)(i)| &= |d_{j_i}(f(j_i), x_i) - d_{j_i}(g(j_i), x_i)| \\ &\leq d_{j_i}(f(j_i), g(j_i)) \end{aligned}$$

and this last quantity is bounded, by the assumption that fEg. □

Corollary 2.6. $\sim_M \leq_B \ell^\infty$. □

References

[AK00] Scot Adams and Alexander S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), no. 4, 909–943. MR1775739 (2001g:03086)

[Far01] Ilijas Farah, *Basis problem for turbulent actions. II. c_0-equalities*, Proc. London Math. Soc. (3) **82** (2001), 1–30. MR1794255 (2002c:03075)

[Hjo00] Greg Hjorth, *Actions by the classical Banach spaces*, J. Symb. Logic **65** (2000), no. 1, 392–420. MR1782128 (2001h:03088)

[HK97] Greg Hjorth and Alexander S. Kechris, *New dichotomies for Borel equivalence relations*, Bull. Symb. Logic **3** (1997), no. 3, 329–346. MR1476761 (98m:03101)

[JK84] Winfried Just and Adam Krawczyk, *On certain Boolean algebras $\mathcal{P}\omega/I$*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 411–429. MR0748847 (86f:04003)

[KL97] Alexander S. Kechris and Alain Louveau, *The classification of hypersmooth Borel equivalence relations*, J. Amer. Math. Soc. **10** (1997), no. 1, 215–242. MR1396895 (97e:03067)

[LV94] Alain Louveau and Boban Veličković, *A note on Borel equivalence relations*, Proc. Amer. Math. Soc. **120** (1994), no. 1, 255–259. MR1169042 (94f:54076)

[Maz00] Krzysztof Mazur, *A modification of Louveau and Veličković's construction for F_σ-ideals*, Proc. Amer. Math. Soc. **128** (2000), no. 5, 1475–1479. MR1626442 (2000j:03067)

[Sol96] Sławomir Solecki, *Analytic ideals*, Bull. Symb. Logic **2** (1996), no. 3, 339–348. MR1416872 (97i:04002)

Department of Mathematics, University of California, Box 951555, Los Angeles, California 90095–1555

E-mail address: `oliver@cs.ucla.edu`

Current address: Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3

URL: `http://www.math.unt.edu/~moliver`