SELF DELTA-EQUIVALENCE OF COBORDANT LINKS

YASUTAKA NAKANISHI, TETSUO SHIBUYA, AND AKIRA YASUHARA

(Communicated by Ronald A. Fintushel)

Abstract. Self \( \Delta \)-equivalence is an equivalence relation for links, which is stronger than the link-homotopy defined by J. Milnor. It is known that cobordant links are link-homotopic and that they are not necessarily self \( \Delta \)-equivalent. In this paper, we will give a sufficient condition for cobordant links to be self \( \Delta \)-equivalent.

1. Introduction

In this paper, all links will be assumed to be ordered and oriented, and they will be considered up to ambient isotopy.

A \( \Delta \)-move \([6]\) is a local move on links as illustrated in Figure 1. If the three strands in Figure 1 belong to the same component of a link, we call it a self \( \Delta \)-move \([11]\). Two links are said to be self \( \Delta \)-equivalent if one can be deformed into the other by a finite sequence of self \( \Delta \)-moves.

Two links are said to be link-homotopic if one can be deformed into the other by a finite sequence of self-crossing changes \([5]\). Note that self \( \Delta \)-equivalence implies link-homotopy, i.e., if two links are self \( \Delta \)-equivalent, then they are link-homotopic.

Let \( L_i = K_1 \cup \cdots \cup K_n \) \((i = 0, 1)\) be \( n \)-component links. Two links \( L_0 \) and \( L_1 \) are cobordant if there is a disjoint union \( \mathcal{A} = A_1 \cup \cdots \cup A_n \) of \( n \) annuli in \( S^3 \times [0, 1] \) with \((\partial(S^3 \times \{0\}), \partial A_j) = (S^3 \times \{0\}, K_{0j}) \cup (-S^3 \times \{1\}, -K_{1j}) (j = 1, \ldots, n)\), where \( -X \) denotes \( X \) with the opposite orientation. Then \( \mathcal{A} \) is called a concordance between \( L_0 \) and \( L_1 \).

It is known that cobordism implies link-homotopy \([11, 3]\). In \([11]\), it is shown that every ribbon link is self \( \Delta \)-equivalent to a trivial link, and it is conjectured that if two links are cobordant, then they are self \( \Delta \)-equivalent. Nakanishi and Shibuya \([9]\) give a counterexample for this conjecture. After that Nakanishi and Ohyama...
give a classification for $2$-component links up to self $\Delta$-equivalence \cite{7}. By using their classification theorem, we have the following: (1) If $2$-component links with linking numbers zero are cobordant, then they are self $\Delta$-equivalent. (2) The links illustrated in Figure 2 are cobordant and are not self $\Delta$-equivalent if $|p| \neq 0$. This implies that for any integer $p \neq 0$, there are two links with linking number $p$ such that they are cobordant and not self $\Delta$-equivalent. The case that $|p| = 1$ is due to Nakanishi and Shibuya \cite{9}. The linking number is an obstruction for links which are cobordant to be self $\Delta$-equivalent. Our purpose is to find a sufficient condition for links which are cobordant to be self $\Delta$-equivalent.

Figure 2.

Since any knot is (self) $\Delta$-equivalent to a trivial knot \cite{6}, any link is self $\Delta$-equivalent to a link each component of which is a trivial knot. For a $2$-component link with trivial components, the following conditions are equivalent:

1. The linking number is 0.
2. It is link-homotopic to a trivial link.
3. Each component is null-homotopic in the complement of the other component.

For links with $n$ trivial components ($n \geq 3$), the conditions above are not equivalent. Note that $(3) \Rightarrow (2) \Rightarrow (1)$. So we have the following questions.

Questions. Let $L_0$ and $L_1$ be cobordant and their components all trivial.

1. If the linking numbers of all $2$-component sublinks of $L_i$ ($i = 0, 1$) are 0, then are they self $\Delta$-equivalent?
2. If $L_0$ and $L_1$ are link-homotopic to a trivial link, then are they self $\Delta$-equivalent?
3. If each component $K_i$ of $L_i$ is null-homotopic in $S^3 \setminus (L_i - K_i)$ ($i = 0, 1$), then are they self $\Delta$-equivalent?

In this paper, we give a negative answer to Question (1) and an affirmative answer to Question (3). Question (2) is still open (likely negative).

The following theorem gives us an affirmative answer to Question (3).

Theorem 1. Let $L_0$ and $L_1$ be links which are cobordant. If each component $K_i$ of $L_i$ is null-homotopic in $S^3 \setminus (L_i - K_i)$ ($i = 0, 1$), then $L_0$ and $L_1$ are self $\Delta$-equivalent.

The following proposition gives a negative answer to Question (1).

Proposition 1. There exists a $3$-component link with trivial components such that it is cobordant to Borromean rings and it is not self $\Delta$-equivalent to Borromean rings.
2. Proof of Theorem 1

A concordance $\mathcal{A}$ in $S^3 \times [0, 1]$ between $L_0 \subset S^3 \times \{0\}$ and $L_1 \subset S^3 \times \{1\}$ is called a **ribbon concordance** [2] if the restriction to $\mathcal{A}$ of the projection $S^3 \times [0, 1] \to [0, 1]$ is a Morse function with no minimal points. In this case we write $L_0 \cong L_1$.

For a concordance $\mathcal{A}$ between $L_0$ and $L_1$, there is a concordance $\mathcal{A}'$ between $L_0$ and $L_1$ such that $\mathcal{A}'$ is ambient isotopic to $\mathcal{A}$ in $S^3 \times [0, 1]$ and $\mathcal{A}' \cap (S^3 \times [0, 1/2])$ (resp. $\mathcal{A}' \cap (S^3 \times [1/2, 1])$) is a ribbon concordance for $\mathcal{A}' \cap (S^3 \times [1/2]) \cong K_0$ (resp. $\mathcal{A}' \cap (S^3 \times \{1/2\}) \cong K_1$). Therefore, Theorem 1 follows directly the following theorem.

**Theorem 2.** Let $L_0$ and $L_1$ be links. If $L_0 \cong L_1$ and each component $K$ of $L_1$ is null-homotopic in $S^3 \setminus (L_1 - K)$, then $L_0$ and $L_1$ are self $\Delta$-equivalent.

**Proof.** Suppose $L_0 = K_{01} \cup K_{02} \cup \cdots \cup K_{0n}$ and $L_1 = K_{11} \cup K_{12} \cup \cdots \cup K_{1n}$ are not self $\Delta$-equivalent. We choose $L_0$ up to self $\Delta$-equivalence so that the number of the maximal points for ribbon concordance $A_1 \cup A_2 \cup \cdots \cup A_n$ between $L_0$ and $L_1$ is minimum. It is known in [4] that $L_0 \cong L_1$ if and only if $L_0$ is a band sum of $L_1$ and a trivial link. Hence, there exist a disjoint union $\bigcup_{i,j} D_{ij}$ of 2-disks $D_{11}, \ldots, D_{1m}, \ldots, D_{nm}, \ldots$ and a disjoint union $\bigcup_{i,j} b_{ij}$ of 2-disks, bands $b_{11}, \ldots, b_{1m}, \ldots, b_{n1}, \ldots, b_{nm}$ such that

1. $L_1 \cap (\bigcup_{i,j} D_{ij}) = \emptyset$,
2. $b_{kl} \cap (L_1 \cup \bigcup_{i,j} \partial D_{ij}) = b_{kl} \cap (K_{1k} \cup \bigcup_{i} \partial D_{k})$ consists of disjoint two-arcs in $\partial b_{kl}$,
3. $b_{kl} \cap (\bigcup_{i,j} \partial D_{ij})$ consists of proper arcs in $b_{kl}$, which are called ribbon singularities (see Figure 3),
4. $L_0 = L_1 \cup \bigcup_{j}(\partial D_{ij} \cup \partial b_{ij}) - \operatorname{int}(\bigcup_{i,j} \partial b_{ij} \cap (L_1 \cup \bigcup_{i,j} \partial D_{ij}))$, and
5. $K_{1kl} = K_{1kl} \cup \bigcup_{j}(\partial D_{kij} \cup \partial b_{kij}) - \operatorname{int}(\bigcup_{j} \partial b_{kij} \cap (K_{11} \cup \bigcup_{i,j} \partial D_{k}))$.

Note that the 2-disks $D_{ij}$’s correspond to the maximal points for the ribbon concordance. Set $D_k = \bigcup_{j} D_{kij}, \ D_k = \bigcup_{j} b_{kij}, \ B_k = \bigcup_{j} b_{kij}$. We may suppose that $D_1 \neq \emptyset$. Since $K_{11}$ is null-homotopic in $S^3 \setminus (L_1 - K_{11})$, $K_{11}$ bounds a singular 2-disc $D_0$ in $L_1 - K_{11}$ each singularity of which is a clasp [13] (see Figure 4). We may assume that $D_0 \cap D = \emptyset, B$ is disjoint from the clasp singularities of $D_0$, and $(D_0 - K_{11}) \cap B$ consists of ribbon singularities.

Moreover, by **sliding bands** in $B_1$ suitably [4], we may assume that $b_{11}$ connects $\partial D_0$ and $\partial D_{11}$, and that $b_{11}$ connects $\partial D_{11}$ and $\partial D_{11}$, without changing the number of 2-disks in $D$.

We deform $D_0 \cup D_1 \cup B_1$ into $D'_0 \cup D'_1 \cup B'_1$ as illustrated in Figure 5 (a), (b) so that $D'_0 \cap B'_k = \emptyset$ and that each 2-disc in $D'_k$ contains at most a single ribbon singularity of $D'_1 \cap B'_k$. Here we temporarily ignore $D'_1 \cap B_k$ ($k \geq 2$). Set $D'_1 = \cdots$
2468 YASUTAKA NAKANISHI, TETSUO SHIBUYA, AND AKIRA YASUHARA

clasp singularity

**Figure 4.**

Figure 5.

\[ D_1 \cup D_2 \cup \cdots \cup D_m, \ B'_1 = b_1 \cup b_2 \cup \cdots \cup b_m, \text{ and assume that } b_1 \text{ connects } D'_0 \text{ and } D_1, \text{ and that } b_j \text{ connects } D_{j-1} \text{ and } D_j \ (j \geq 2). \]

\[ \square \]

**Claim.** The deformations as illustrated in Figure 6 (a), (b) and (c) are realized by \( \Delta \)-moves.

**Proof of Claim.** Since the local move as illustrated in Figure 7 is realized by a single \( \Delta \)-move and ambient isotopies (for example, see [12]), the claim above follows Figure 8 (a), (b) and (c).

\[ \square \]

By combining the deformations as illustrated in Figure 6 (a), (b) and (c), we can change \( D_m \cup b_m \) so that \( D_m \cap B'_1 = \emptyset \). Then we shrink \( D_m \cup b_m \) into a part of \( D_{m-1} \). In these deformations, the ambient isotopy class of \((L_1 - K_{11}) \cup \bigcup_{k \geq 2} (D_k \cup B_k)\) is preserved, although \( B_k \) might be trailed by \( D_{m-1} \). Thus we have a new band sum of \( L_1 \) and \( \partial(D'_1 - D_m) \cup \bigcup_{k \geq 2} \partial D_k \) with bands \((B'_1 - b_m) \cup \bigcup_{k \geq 2} B'_k \). Repeating these deformations, we have a band sum \( L'_0 = K_{11} \cup K_{12} \cup \cdots \cup K_{1n} \) of \( L_1 \) and \( \bigcup_{k \geq 2} \partial D_k \) with bands \( \bigcup_{k \geq 2} B'_k \) such that \((L_1 - K_{11}) \cup \bigcup_{k \geq 2} (D_k \cup B'_k)\) is ambient isotopic to \((L_1 - K_{11}) \cup \bigcup_{k \geq 2} (D_k \cup B_k)\). We note that \( L'_0 \) is self \( \Delta \)-equivalent to \( L_0 \), and that \( L'_0 \) and \( L_1 \) bound a ribbon concordance \((K_{11} \times I) \cup A_2 \cup \cdots \cup A_n\). This contradicts the minimality of the number of maximal points for the ribbon concordance.

\[ \square \]

3. **Proof of Proposition 1**

Let \( L_0 \) be Borromean rings, and let \( L_1 \) be a link as illustrated in Figure 9. Since \( L_0 \) and \( L_1 \) are cobordant, we will show that \( L_0 \) and \( L_1 \) are not self \( \Delta \)-equivalent.
In order to prove this, we need the following proposition given in [8].

**Proposition 2** ([8, Lemma 3.1]). If two \( n \)-component links are self \( \Delta \)-equivalent, then they have the same Alexander matrices modulo \( ((1-t_1)^2, \ldots, (1-t_n)^2) \). Further, we have similar statements for the elementary ideals of deficiency greater than \( 0 \).

**Proof of Proposition 1.** We take a diagram of \( L_0 \) as illustrated in Figure 10. Then we have

\[
\pi_1(S^3 \setminus L_0) = \left\langle a, b, c, p, x, y \mid y = axa^{-1}, x = b^{-1}yb, p = b^{-1}cp^{-1}b, b = papa^{-1}, c = y^{-1}by, a = yp^{-1}cpy^{-1} \right\rangle.
\]

This can be shown to be isomorphic to \( \langle a, p, x \mid S_1, S_2 \rangle \), where

\[
S_1 = papa^{-1}axa^{-1}papa^{-1}x^{-1},
S_2 = a^{-1}p^{-1}axa^{-1}apa^{-1}axa^{-1}paxa^{-1}pa^{-1}p^{-1}axa^{-1}papa^{-1}.
\]

Hence the Alexander matrix is equivalent to the following matrix modulo \( ((a-1)^2, (p-1)^2, (x-1)^2) \):

\[
\begin{pmatrix}
(p-1)(x-1) & -(a-1)(x-1) & 0 \\
(p-1)(x-1) & -a^{-1}(a-1)(p-1)(x-1) & (a-1)(p-1)
\end{pmatrix}.
\]

So we have

\[
E_1 \equiv (0) \mod ((a-1)^2, (p-1)^2, (x-1)^2).
\]
On the other hand, we take a diagram of $L_1$ as illustrated in Figure 11. Then we have

$$
\pi_1(S^3 \setminus L_1) = \left\{ (x, y, z, w, u, a, b, p, q, r, s) \mid \begin{array}{l}
    y = r^{-1}ar, z = x^{-1}yx, w = szs^{-1}, u = bwv^{-1}, \\
    x = a^{-1}aa, b = q^{-1}aq, a = pbb^{-1}, q = upu^{-1}, \\
    r = yyy^{-1}, s = w^{-1}rw, p = z^{-1}sz
\end{array} \right\}.
$$

This can be shown to be isomorphic to $\langle y, w, a, q \rangle R_1, R_2, R_3$, where

$$
R_1 = q^{-1}y^{-1}ayqy^{-1},
R_2 = yyy^{-1}wa^{-1}q^{-1}aqw^{-1}a^{-1}qava^{-1}q^{-1}aqwq^{-1}a^{-1}qaw^{-1}y^{-1}w^{-1},
R_3 = qaqw^{-1}a^{-1}qa^{-1}q^{-1}aqw^{-1}a^{-1}qava^{-1}q^{-1}aqwq^{-1}a^{-1}qaw^{-1}yq.
$$

The Alexander matrix is modulo $((a - 1)^2, (q - 1)^2, (y - 1)^2)$ equivalent to

$$
\begin{pmatrix}
-a(q - 1) - 1 & 0 & 1 & y(a - 1) \\
y(q - 1) + 1 & 0 & -1 & 0 \\
0 & -yaq(q - 1)(a - 1) & ya(q - 1)(y - 1) & y(q - 1)(a - 1)(q - 1)
\end{pmatrix}
$$

By fundamental deformations of presentation matrices up to modulo $((a - 1)^2, (q - 1)^2, (y - 1)^2)$, we have

$$
\begin{pmatrix}
(y - 1)(q - 1)(y - a) \\
(y - 1)(a - 1)(q - 1)
\end{pmatrix}.
$$

Hence we have

$$E_1 \equiv ((q - 1)(y - a)) \mod ((a - 1)^2, (q - 1)^2, (y - 1)^2).
$$

By Proposition 2, we have the conclusion. \qed

### References


Department of Mathematics, Kobe University, Nada, Kobe 657-8501, Japan
E-mail address: nakanishi@math.kobe-u.ac.jp

Department of Mathematics, Osaka Institute of Technology, Omiya 5-16-1, Asahi, Osaka 535-8585, Japan
E-mail address: shibuya@ge.oit.ac.jp

Department of Mathematics, Tokyo Gakugei University, Nukuikita 4-1-1, Koganei, Tokyo 184-8501, Japan
E-mail address: yasuhara@u-gakugei.ac.jp