A FAMILY OF SCHOTTKY GROUPS
ARISING FROM THE HYPERGEOMETRIC EQUATION

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ABSTRACT. We study a complex 3-dimensional family of classical Schottky
groups of genus 2 as monodromy groups of the hypergeometric equation. We
find non-trivial loops in the deformation space; these correspond to continuous
integer-shifts of the parameters of the equation.

1. INTRODUCTION

When the three exponents of the hypergeometric differential equation are purely
imaginary, its monodromy group is a classical Schottky group of genus 2; such
groups form a real 3-dimensional family ([IY]). Since, under a small deformation,
a Schottky group remains to be a Schottky group, by deforming the parameters of
the hypergeometric equation, we have a complex 3-dimensional family of Schottky
groups. We introduce (in §4) a complex 3-dimensional family S of classical Schottky
groups, containing the above ones coming from pure-imaginary-exponents cases,
equipped with some additional structure. We study a structure of S, and construct
loops (real 1-dimensional families of classical Schottky groups) in S generating the
fundamental group of S. These loops correspond to continuous integer-shifts of the
parameters of the hypergeometric equation.

Here we give a few examples of general background references. The hypergeo-
metric function: [IKSY], Schottky groups: Chapter 4 of [MSW] and Chapter 5 of
[BBEIM], history: [Gray].

2. THE HYPERGEOMETRIC EQUATION

We consider the hypergeometric differential equation
\[ E(a, b, c) : x(1-x)\frac{d^2u}{dx^2} + \{c - (a + b + 1)x\} \frac{du}{dx} - abu = 0. \]

For (any) two linearly independent solutions \( u_1 \) and \( u_2 \), the (multi-valued) map
\[ s : X := \mathbb{C} - \{0, 1\} \ni x \mapsto z = u_1(x) : u_2(x) \in \mathbb{P}^1 := \mathbb{C} \cup \{\infty\} \]
is called a Schwarz map (or Schwarz’s \( s \)-map). If we choose as solutions \( u_1 \) and
\( u_2 \), near \( x = 0 \), a holomorphic one and \( x^{1-c} \) times a holomorphic one, respectively,
then the circuit matrices around $x = 0$ and $1$ are given by

$$
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i (1-c)} \end{pmatrix} \quad \text{and} \quad \gamma_2 = P^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i (c-a-b)} \end{pmatrix} P,
$$

respectively, where $P$ is a connection matrix given by

$$
P = \begin{pmatrix}
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\
\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(c-a)\Gamma(a+b-c)} & \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(2-c)\Gamma(1-c+1)}
\end{pmatrix},
$$

here $\Gamma$ denotes the Gamma function. Note that the matrices act on the row vector $(u_1, u_2)$ from the right. These generate the monodromy group $M(a, b, c)$ of the equation $E(a, b, c)$.

3. Hypergeometric equations with purely imaginary exponent-differences ([SY], [IY])

If the exponent-differences

$$
\lambda = 1 - c, \quad \mu = c - a - b, \quad \nu = b - a
$$

(at the singular points $x = 0, 1$ and $\infty$, respectively) are purely imaginary, the image of the upper-half part

$$
X^+ := \{x \in X \mid \exists x \geq 0\}
$$

is bounded by the three circles (in the $s$-plane) which are images of the three intervals $(-\infty, 0), (0, 1)$ and $(1, +\infty)$; put

$$
C_1 = s((-\infty, 0)), \quad C_2 = s((0, 1)), \quad C_3 = s((0, +\infty)).
$$

Note that if we continue $s$ analytically through the interval, say $(0, 1)$, to the lower-half part $X^- := \{x \in X \mid \exists x \leq 0\}$, then the image of $X^-$ is the mirror image of that of $X^+$ under the reflection with center $C_3$.

The reflection with center (mirror) $C_j$ will be denoted by $\rho_j$ ($j = 1, 2, 3$). The monodromy group $M(a, b, c)$ is the group consisting of even words of $\rho_j$ ($j = 1, 2, 3$). This group is a Schottky group of genus 2. The domain of discontinuity modulo $M(a, b, c)$ is a curve of genus 2 defined over the reals.

Choosing the solutions $u_1$ and $u_2$ suitably, we can assume that the centers of the three circles are on the real axis, and that the circles $C_2$ and $C_3$ are inside the circle $C_1$ (see Figure 1). Let $C'_1$ and $C'_2$ be the mirror images with respect to $C_3$ of the circles $C_1$ and $C_2$, respectively. Then $\gamma_1 := \rho_3 \rho_1$ maps the inside of $C_1$ onto the inside of $C'_1$; and $\gamma_2 := \rho_3 p_2$ maps the inside of $C_2$ onto the outside of $C'_2$. Note that one of the two fixed points of $\gamma_1$ is inside $C'_1$, and the other one is outside $C_1$; one of the two fixed points of $\gamma_2$ is inside $C'_2$, and the other one is inside $C_2$. The transformations $\gamma_1$ and $\gamma_2$ are conjugate to those given in §2.

We thus have four disjoint disks $D_1, D'_1, D_2,$ and $D'_2$, whose centers are on the real axis, and two loxodromic transformations $\gamma_1$ and $\gamma_2$ taking $D_1$ and $D_2$ to the complementary disks of $D'_1$ and $D'_2$, respectively. Note that one of the two fixed points of $\gamma_1$ is in $D'_1$ and the other one in $D_1$; one of the two fixed points of $\gamma_2$ is in $D'_2$ and the other one in $D_2$. 
4. The moduli space $S$

Two loxodromic transformations $\gamma_1$ and $\gamma_2$ generate a (classical) Schottky group if and only if there are four disjoint closed disks $D_1, D'_1, D_2,$ and $D'_2,$ such that $\gamma_1$ and $\gamma_2$ take $D_1$ and $D_2$ to the complementary disks of $D'_1$ and $D'_2$, respectively. Throughout the paper, disks are always assumed to be closed; we simply call them disks. The closure of the complement of a disk in $\mathbb{P}^1$ will simply be called the complementary disk of the disk.

**Definition.** Let $S$ be the space of two loxodromic transformations $\gamma_1$ and $\gamma_2$ equipped with four disjoint disks $D_1, D'_1, D_2,$ and $D'_2,$ such that $\gamma_1$ and $\gamma_2$ take $D_1$ and $D_2$ to the complementary disks of $D'_1$ and $D'_2$, respectively.

The space has a natural structure of a complex 3-dimensional manifold. We are interested in its homotopic property.

Note that if a loxodromic transformation $\gamma$ takes a disk $D$ onto the complementary disk of a disk $D'$ ($D \cap D' = \emptyset$), then $\gamma$ has a fixed point in $D$ and the other fixed point in $D'$.

A loxodromic transformation is determined by the two fixed points and the multiplier.

**Lemma 1.** For two given disjoint disks $D$ and $D'$, there is a unique point $F \in D$ (resp. $F' \in D'$) such that by any linear fractional transformation taking $F$ (resp. $F'$) to $\infty$, the two circles $\partial D$ and $\partial D'$ are transformed into concentric circles.

**Proof.** Let the two disks be given as

$$D : |z - a| \leq r, \quad D' : |z - a'| \leq r'. $$

By the transformation $z \rightarrow w$ defined by

$$z = \frac{1}{w} + \zeta$$

taking $\zeta$ to $\infty$, the circles $C = \partial D$ and $C' = \partial D'$ are transformed into circles with centers

$$c := \frac{\bar{\zeta} - \bar{a}}{r^2 - |\zeta - a|^2}, \quad c' := \frac{\bar{\zeta} - \bar{a'}}{r'^2 - |\zeta - a'|^2}. $$
respectively. Equating $c$ and $c'$, we have the quadric equation

$$\zeta^2 + \left(-a - a' + \frac{r^2 - r'^2}{a - a'}\right) \zeta + aa' + \frac{ar^2 - a'r'^2}{a - a'} = 0.$$ 

Put $\zeta = (a - a') \eta + a'$. Note that $\zeta = a'$ and $a$ correspond to $\eta = 0$ and 1, respectively. Then the equation above for $\zeta$ reduces to

$$\eta^2 + \left(-1 + \frac{r^2 - r'^2}{|a - a'|^2}\right) \eta + \frac{r'^2}{|a - a'|^2} = 0.$$ 

It is a high school mathematics problem to see that each of the two roots of this equation is in each of the two intervals

$$\left(0, \frac{r'}{|a - a'|}\right) \text{ and } \left(1 - \frac{r}{|a - a'|}, 1\right).$$

Since concentric circles are mapped to concentric circles under linear transformations, this completes the proof. \hfill \Box

Remark 1. The circles $\partial D$ and $\partial D'$ are Apollonius circles with respect to the two centers $F$ and $F'$.

Lemma 2. For two given disjoint disks $D$ and $D'$ and a point $f'$ in the interior of $D'$, there is a 1-parameter family (parametrized by a circle) of loxodromic transformations $\gamma$ which take $D$ onto the complementary disk of $D'$, and fix $f'$. The absolute value $|m|$ of the multiplier $m$ of $\gamma$ is determined by the given data.

1. If $f' \neq F'$, then the other fixed point $f \in D$ of $\gamma$ must be on the Apollonius circle $A = A(f')$ in $D$ determined by the two centers of the circles $\partial D$ and $\partial D'$, and proportion $|m|$. arg $m \in \mathbb{R}/2\pi \mathbb{Z}$ determines $f$, and vice versa.

2. If $f' = F'$, then the other fixed point is $F \in D$. arg $m \in \mathbb{R}/2\pi \mathbb{Z}$ remains free.

Proof. We can assume that $f = 0$ and $f' = \infty$, so that the transformation in question can be presented by $z \mapsto mz$. Let $c$ and $r$ be the center and the radius of the disk $D$, and $c'$ and $r'$ those of the complementary disk of $D'$. Then we have $c' = mc$ and $r' = |m|r$.

1. If $c \neq c'$, $f = 0$ is on the Apollonius circle

$$A : |f - c| = |m||f - c|$$

with centers $c$ and $c'$, and proportion $|m|$. It is easy to see that this circle is in $D$. (See Figure 2.)

2. If $c = c' = 0$, then the Apollonius circle reduces to a point. \hfill \Box

In case (2), we blow up the point $F$ to be the circle $\mathbb{R}/2\pi \mathbb{Z}$, and call this circle also the Apollonius circle $A$. Under this convention, two disjoint disks $D$ and $D'$, an interior point $f' \in D'$, and a point $f$ on the Apollonius circle $A$ uniquely determine a loxodromic transformation.

Thus an element of $S$ can be determined by four disjoint disks $D_1, D'_1, D_2, D'_2$, two interior points $f'_1 \in D'_1, f'_2 \in D'_2$, and two points $f_1$ on the Apollonius circle $A_1$ (determined by $D_1, D'_1, f'_1$) in $D_1$, and $f_2$ on the Apollonius circle $A_2$ (determined by $D_2, D'_2, f'_2$) in $D_2$. Since disks are contractible, we have

**Proposition 1.** The fundamental group of $S$ can be generated by the moves of the four disjoint disks $D_1, D'_1, D_2, D'_2$, and the moves of $f_1$ in $A_1$, and $f_2$ in $A_2$. 
In the next section we explicitly construct loops (real 1-dimensional families of classical Schottky groups) in $S$.

5. Loops in $S$

Let $\gamma_1$ and $\gamma_2$ be as in §2. The fixed points of $\gamma_1$ are $\{0, \infty\}$, and those of $\gamma_2$ are

$$f_2 = \frac{\Gamma(c)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(2-c)\Gamma(a)\Gamma(b)} \quad \text{and} \quad f'_2 = \frac{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}.$$

We change the coordinate $z$ by multiplying by $1/f_2$. Then the fixed points of $\gamma_1$ remain to be $\{0, \infty\}$, and those of $\gamma_2$ become 1 and

$$\alpha = g(a)g(b), \quad \text{where} \quad g(x) = \frac{\sin(\pi c - \pi x)}{\sin(\pi x)}.$$

Indeed we have

$$\frac{f'_2}{f_2} = \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot \frac{\Gamma(a)\Gamma(b)}{\Gamma(a-c+1)\Gamma(b-c+1)} \quad \text{and} \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Definition. When the three exponent-differences are purely imaginary:

$$\lambda = i\theta_0, \quad \mu = i\theta_1, \quad \nu = i\theta_2, \quad \theta_0, \theta_1, \theta_2 > 0,$$

the generators $\{\gamma_1, \gamma_2\}$ of the monodromy group of $E(a, b, c)$ given in §2 (take the four disks given in §3) form a simply connected real 3-dimensional submanifold of $S$; this submanifold is called $S_0$.

In this section, we construct loops in $S$, with base in $S_0$, which generate the fundamental group of $S$. When the exponent-differences are as above, note that the parameters can be expressed as

$$a = \frac{1}{2} - \frac{i}{2}(\theta_0 + \theta_1 + \theta_2), \quad b = \frac{1}{2} - \frac{i}{2}(\theta_0 + \theta_1 - \theta_2), \quad c = 1 - i\theta_0.$$

For clarity, we denote the four disks $D_1, D'_1, D_2, D'_2$ by

$$D_0 (\ni 0), \quad D_\infty (\ni \infty), \quad D_1 (\ni 1), \quad D_\alpha (\ni \alpha).$$
5.1. **The disk $D_\alpha$ travels around the disk $D_0$.** We construct a loop in $S$ (with base in $S_0$), which is represented by a travel of the disk $D_\alpha$ around the disk $D_0$ with a change of argument (by $2\pi$) of the multiplier of $\gamma_2$. We fix $c$ and the real part of $a$ (as $1/2$), and let the real part of $b$ move from $1/2$ to $3/2$; the imaginary parts of $a$ and $b$ are so chosen that the monodromy group $M(a,b,c)$ remains to be a Schottky group along the move. Putting $c = 1 - i\theta_0$, we have

$$g(x) = \frac{e^{-x} - e^{2\pi ix} + e^{-x} - e^{2\pi ix}}{1 - e^{2\pi ix}} = e + e^{2\pi i}, \quad \text{where} \quad e = e^{-\pi \theta_0} < 1.$$ 

Set

$$r = g\left(\frac{1}{2} - \frac{i}{2}(\theta_0 + 2)\right), \quad R = g\left(\frac{1}{2} - \frac{i}{2}(\theta_0 + 1)\right).$$

Then we have

$$0 < \varepsilon < r = e^{2\pi i} + e^{-1} < R = e^{2\pi i} + e^{-1} < 1.$$ 

Let $D_0$ be the disk with center at 0 and with radius $\varepsilon r$, and $D_\infty$ the complementary disk of the disk with center at 0 and with radius $e^{-1}r$. Note that we have

$$0 < \varepsilon r < \varepsilon R < R < 1 < e^{-1}r,$$

and that $\gamma_1$ maps $D_0$ onto the outside disk of $D_\infty$.

Define real continuous functions $\phi(t)$ and $\psi(t)$ for $0 \leq t \leq 1$ by

$$Re^{2\pi it} = g\left(\frac{1}{2} + \phi(t) - \frac{i}{2}(\theta_0 + \psi(t))\right).$$

Since

$$2\pi i\left(\frac{1}{2} + \phi(t) - \frac{i}{2}(\theta_0 + \psi(t))\right) = \log(Re^{2\pi it} - e^{-1}) - \log(Re^{2\pi it} - e),$$

the function $\phi(t)$ is monotone increasing with $\phi(0) = 0$, $\phi(1) = 1$, and $\psi(t)$ satisfies $\psi(0) = \psi(1) = 1$. Choose $\theta_1$ so large that

- $\theta_1 > \max\{\psi(t) \mid 0 \leq t \leq 1\}$, and that
- for any $z$ in the ring $\{\varepsilon R \leq |z| \leq R\}$, there are two disjoint disks $D_1 (\ni 1)$ and $D_z (\ni z)$, and a fractional linear transformation with multiplier $e^{-2\pi \theta_1}$ mapping $D_1$ onto the complementary disk of $D_z$. 

Now set \( \theta_2(t) = \theta_1 - \psi(t) > 0 \) and deform the parameters \( a \) and \( b \) as
\[
  a(t) = \frac{1}{2} - i \frac{1}{2} (\theta_0 + \theta_1 + \theta_2(t)), \\
  b(t) = \frac{1}{2} + \phi(t) - i \frac{1}{2} (\theta_0 + \theta_1 - \theta_2(t)).
\]
Then we have \( \varepsilon < g(a(t)) < 1, |g(b(t))| = R \), and so \( \alpha(t) = g(a(t))g(b(t)) \) satisfies
\[
  \varepsilon R < |\alpha(t)| = g(a(t))|g(b(t))| < R.
\]
Thus \( \alpha(t) \) travels around the disk \( D_0 \) in the ring \( \{ \varepsilon R \leq |z| \leq R \} \), and there are two disjoint disks \( D_1 (\ni 1) \) and \( D_{2 \alpha(t)} (\ni \alpha) \) in the ring \( \{ \varepsilon r < |z| < \varepsilon^{-1} r \} \), and a transformation \( \gamma(t) \) with multiplier \( e^{2\pi i \mu} = e^{-2\pi \theta_1} \) which maps \( D_1 \) onto the complementary disk of \( D_{2 \alpha(t)} \).

5.2. The disk \( D_{\alpha} \) travels around the disk \( D_1 \). We construct a loop in \( S \) (with base in \( S_0 \)), which is represented by a travel of the disk \( D_{\alpha} \) around the disk \( D_1 \) with a change of argument (by \( 2\pi i \)) of the multiplier of \( \gamma_2 \). We fix \( c, a \) and the imaginary part of \( b \), and let the real part of \( b \) move from \( 1/2 \) to \( 3/2 \). Take \( \theta_0 > 0 \) and \( \theta' < 0 \) satisfying
\[
  \frac{e^{\pi \theta'} (1 + e^{\pi \theta'})}{1 - e^{\pi \theta'}} < \varepsilon^{-2}, \quad \varepsilon = e^{-\pi \theta_0}.
\]
Set
\[
  r = \varepsilon + \frac{e^{-1} - \varepsilon}{1 + e^{\pi \theta'}}, \quad R = \varepsilon + \frac{e^{-1} - \varepsilon}{1 - e^{\pi \theta'}}.
\]
Since we have
\[
  \frac{1 - \varepsilon r}{r} = \frac{1 - \varepsilon^2}{r} \left( 1 - \frac{1}{1 + e^{\pi \theta'}} \right) > 0
\]
and
\[
  \varepsilon^{-1} r - \varepsilon R = (1 - \varepsilon^2) \left( 1 + \frac{e^{-2}}{1 + e^{\pi \theta'}} - \frac{1}{1 - e^{\pi \theta'}} \right) > 0,
\]
where the last inequality holds thanks to (1), there is a positive number \( s \) satisfying
\[
  0 < s < \min \left\{ r, \frac{1 - \varepsilon r}{r}, \frac{\varepsilon^{-1} r - \varepsilon R}{\varepsilon^{-1} + R} \right\},
\]
which implies
\[
  (\varepsilon + s) r < 1, \quad (\varepsilon + s) R < \varepsilon^{-1} (r - s).
\]
Take \( \theta_1 \) so large that
\[
  \varepsilon + \frac{e^{-1} - \varepsilon}{1 + e^{\pi (\theta_0 + \theta_1)}} < \varepsilon + s.
\]
Now set \( \theta_2 := \theta_0 + \theta_1 - \theta' \) and
\[
  a = \frac{1}{2} - i \frac{1}{2} (\theta_0 + \theta_1 + \theta_2),
\]
and deform (the real part of) \( b \) as
\[
  b(t) = \frac{1}{2} + t - i \frac{1}{2} (\theta_0 + \theta_1 - \theta_2) = \frac{1}{2} + t - i \frac{1}{2} \theta'.
\]
Note that
\[
  \varepsilon < g(a) = \varepsilon + \frac{e^{-1} - \varepsilon}{1 + e^{\pi (\theta_0 + \theta_1 + \theta_2)}} < \varepsilon + s \quad \text{(here we used (3))},
\]
These inequalities together with (2) imply that
\[ \alpha(t) = g(a)g(b(t)) \] is in the ring
\[ \{ \varepsilon r \leq |z| \leq (\varepsilon + s)R \} \]
a nd that \( \alpha(0) = \alpha(1) \leq (\varepsilon + s)r < \varepsilon R \leq \alpha(1/2) \).

This shows that \( \alpha(t) \) travels around 1. Let \( D_0 \) be the disk with center at 0 and with radius \( \varepsilon(r - s) \), \( D_\infty \) the complementary disk of the disk with center at 0 and with radius \( \varepsilon^{-1}(r - s) \). Then \( \gamma_1 \) maps \( D_0 \) onto the complementary disk of \( D_\infty \), and by the right side of (2), \( D_0 \cup D_\infty \) is disjoint from the ring
\[ \{ \varepsilon r \leq |z| \leq (\varepsilon + s)R \} \ni 1, \alpha(t) \).

Thus by taking \( \theta_1 > 0 \) sufficiently large, for any \( 0 \leq t \leq 1 \), there are two disjoint disks \( D_1 (\ni 1), D_\alpha(t) (\ni \alpha(t)) \) in the complement of \( D_0 \cup D_\infty \) and a transformation \( \gamma_2(t) \) with multiplier \( e^{2\pi i \mu} = e^{-2\pi \theta_1}e^{-2\pi it} \) which maps \( D_1 \) onto the complementary disk of the \( D_\alpha(t) \) (see Figure 4).

5.3. The multiplier \( \gamma_2 \) travels around 0. We construct a loop in \( S \) (with base in \( S_0 \)), which is represented by the change of argument (by \( 2\pi \)) of the multiplier of \( \gamma_2 \). We fix \( b \) and \( c \), and the imaginary part of \( a \), and let the real part of \( a \) move from \( 1/2 \) to \( 3/2 \). Set
\[ \theta := \theta_0 + \theta_1 + \theta_2, \quad \varepsilon := e^{-\pi \theta_0}, \quad r := \frac{\varepsilon^{-1} - \varepsilon}{e^{\pi \theta} - 1}. \]
Choose and fix \( \theta_0, \theta_1 \) and \( \theta_2 \) so that \( \theta_1 = \theta_2 \) and
\[ r < \min \left\{ 1 - \varepsilon, \frac{1}{1 + \varepsilon^{-1}} \right\}. \]
Then since
\[ b = \frac{1}{2} - \frac{i}{2}(\theta_0 + \theta_1 - \theta_2) = \frac{1}{2} - \frac{i}{2} \theta_0, \]
we have \( e^{2\pi ib} = -\varepsilon^{-1} \), and so \( g(b) = 1 \).
Now we deform the parameter \( a \) as
\[ a(t) = \frac{1}{2} + t - \frac{i}{2} \theta, \quad 0 \leq t \leq 1. \]
Figure 5. The fixed point $\alpha$ of $\gamma_2$ travels along the Apollonius circle.

Since we have

$$\alpha(t) = g(a(t))g(b) = g(a(t)) = \varepsilon + \frac{\varepsilon^{-1} - \varepsilon}{1 + e^{2\pi i t}},$$

the point $\alpha$ is in the disk with center at $\varepsilon$ and with radius $r$. We thus have

$$\varepsilon^2(1 + r) < \varepsilon - r \leq |\alpha(t)| \leq \varepsilon + r < 1 < 1 + r.$$  

Let $D_0$ be the disk with center at 0 and with radius $\varepsilon^2(1+r)$, $D_\infty$ the complementary disk of the disk with center at 0 and with radius $1+r$, $D_{\alpha(t)} (\ni \alpha(t))$ the disk with center at $\varepsilon$ and with radius $r$. Then $\gamma_1$ maps $D_0$ onto the complementary disk of $D_\infty$, and $\alpha(t)$ belongs to the disk $\{|z - \varepsilon| \leq r\}$ which is disjoint from $D_0 \cup D_\infty \cup \{1\}$. Thus by taking $\theta_1 = \theta_2 > 0$ sufficiently large, for any $0 \leq t \leq 1$, there are two disjoint disks $D_1 (\ni 1)$, $D_{\alpha(t)} (\ni \alpha(t))$ in the complement of $D_0 \cup D_\infty$ and a transformation $\gamma_2(t)$ with multiplier $e^{2\pi i t} = e^{-2\pi \theta_1} e^{-2\pi i t}$ which maps $D_1$ onto the complementary disk of $D_{\alpha(t)}$ (see Figure 5).

5.4. The multiplier of $\gamma_1$ travels around 0. We construct a loop in $S$ (with base in $S_0$), which is represented by the change of argument (by $2\pi$) of the multiplier of $\gamma_1$. Note that by the change of variable $x \to 1 - x$, the equation $E(a, b, c)$ changes into $E(a, b, a + b + 1 - c)$. So we have only to literally follow §5.3 exchanging $c$ and $a + b + 1 - c$, and $\theta_0$ and $\theta_1$. We thus fix $b$ and the imaginary parts of $a$ and $c$, and let the real part of $c$ move from 1 to 2, and let the real part of $a$ move from $1/2$ to $3/2$ keeping $a + b + 1 - c$ constant.

6. MISCELLANEA

For a Schottky group $\Gamma$ of genus 2, the quotient of the domain of discontinuity modulo $\Gamma$ is a curve of genus 2. A curve of genus 2 is a double cover of $\mathbb{P}^1$ branching at six points, which are uniquely determined modulo automorphisms of $\mathbb{P}^1$ by the curve. Thus a Schottky group determines a point of the configuration space $X\{6\}$ of six-point sets on $\mathbb{P}^1$. (The fundamental group of $X\{6\}$ is the Braid group with five strings.) When all the exponent-differences of the hypergeometric equation are purely imaginary, its monodromy group is a Schottky group, which determines six colored points on a line. Thus they determine a point of the configuration space $X\{6\}$ of six colored points on $\mathbb{P}^1$. (The fundamental group of $X\{6\}$ is the colored Braid group with five strings.) The space $X\{6\}$ is well studied.
Remark 2. When all the exponent-differences are purely imaginary, there is a Schottky automorphic function defined by an absolutely convergent infinite product, which induces a holomorphic map of the genus 2 curve onto $\mathbb{P}^1$ ([IY]). This infinite product remains to be convergent for groups represented by loops above if we take the multipliers of $\gamma_1$ and $\gamma_2$ sufficiently small. This is because there is a circle separating two disks among the four ([BBEIM, Chapter 5]).

Problems. The four loops constructed in §5 induce those in $X\{6\}$. Do they generate the fundamental group of $X\{6\}$? Same problem for $X(6)$.

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